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Solution of a class of mixed boundary-value problems for Laplace's equation arising in water-wave scattering

A. CHAKRABARTI[1], B. N. MANDAL[2], S. BANERJEA[3] AND T. SAHOO[1]
[1]Department of Mathematics, Indian Institute of Science, Bangalore 560 012, India
[2]PAMU, Indian Statistical Institute, Calcutta 700 035, India
[3]Department of Mathematics, Jadavpur University, Calcutta 700 032, India

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Abstract

A class of mixed boundary-value problems, for the two-dimensional Laplace's equation, occurring in the study of scattering of surface water waves by vertical barriers, is attacked for their solutions with the aid of multiple integral equations involving trigonometric kernels. These multiple integral equations are solved by utilizing the natural singular behaviour of one of the integrals at the turning points and the final solutions are observed to depend on the solutions of certain simple Abel-type integral equations of the first kind. Known results are recovered and are presented in concise form.

Keywords: Mixed boundary-value problems, scattering, multiple integral equations, Abel integral equation.

1. Introduction

The problems of scattering of surface water waves, in the linearised theory, by vertical barriers[1], have attracted the attention of many workers[2–6] to develop increasingly interesting mathematical methods to handle a class of mixed boundary-value problems associated with Laplace's equation, in two dimensions, having the special features that the conditions at infinity are not known fully (the so-called 'reflected' and 'transmitted' waves[1]). The solutions of the three basic problems, involving (*i*) a fully submerged barrier, (*ii*) a partially immersed barrier, and (*iii*) a submerged finite plate, have been obtained by a number of researchers[2–6] by using methods all of which, in some sense, depend on the solution of singular integral equations with Cauchy-type kernels and therefore, on the details of the complex function theory and allied ideas inherent in the basic structure of such singular integral equations.

In the present paper we have shown that the solutions of these three basic problems can be determined just by the use of the inversion of some Abel-type integral equations, and hence it is believed that complicated analysis can be avoided in handling these problems. Other general problems involving many vertical scatterers are also expected to give rise to such Abel-type integral equations or their generalisations.

Following is the plan of the paper. In Section 2, we give a brief mathematical statement of the three boundary-value problems under consideration here and reduce these problems to multiple integral equations (dual or triple, as the case may be). In Section 3, we present the methods of solution of these multiple integral equations, wherein we utilize the well-known property on the mixed boundary conditions for problems involving Laplace's equation that the normal derivatives of a harmonic function possesses square root singularities at the turning points (*i.e.*, at the points on either side of which different boundary conditions are prescribed) on a straight boundary. The problems then get reduced, in a natural manner, to those of solving certain Abel-type integral equations of the first kind and solutions of these Abel integral equations are derived for each of the problems separately. The full solutions of all the problems are presented in Section 4.

2. The statement of the problems

Mathematically speaking, the three basic problems of scattering of surface water waves by (*i*) a fully submerged vertical barrier, (*ii*) a partially immersed vertical barrier, and (*iii*) a fully submerged vertical plate, referred to hereafter as P_1, P_2 and P_3, respectively, are the following[1]:

Determine three harmonic functions ϕ_1, ϕ_2 and ϕ_3 of two variables x and y (representing rectangular Cartesian co-ordinates of a point in two dimensions), with $y > 0$, in the forms:

$$\phi_j(x,y)=\begin{cases} T_j e^{-Ky+iKx}+\int_0^\infty A_j(k)L(k,y)e^{-kx}dk, & x>0 \\ e^{-Ky+iKx}+R_j e^{Ky-iKx}+\int_0^\infty B_j(k)L(k,y)e^{kx}dk, & x<0 \end{cases} \tag{1}$$

with $i^2=-1$, $(j=1, 2, 3)$ and

$$L(k, y) = k\cos ky - K\sin ky, \tag{2}$$

where $A_j(k)$, $B_j(k)$ are unknown functions, T_j and R_j are unknown constants to be determined by utilizing the following requirements:

$$\left.\begin{aligned} &(a)\ \frac{\partial\phi_j}{\partial x}\ \text{is continuous on } x=0, \text{ for all } y,\\ &(b)\ \frac{\partial\phi_j}{\partial x}=0, \text{ on } x=0\pm, \text{ for } y\in L_j\\ &\text{and}\\ &(c)\ \phi_j \text{ is continuous on } x=0, \text{ for } y\in G_j, \end{aligned}\right\} \tag{3}$$

where L_j represents the interval $a_j < y < b_j$ and $G_j = (0, \infty) - L_j$, with $a_1 = a$, $b_1 = \infty$ (corresponding to P_1), $a_2 = 0$, $b_2 = b$ (corresponding to P_2) and $a_3 = c$, $b_3 = d$, $(c > 0, d > 0, d > c)$ (corresponding to P_3). It is rather natural that the functions ϕ_j will have the properties that the derivatives $\partial\phi_j/\partial x$ on $x = 0$ will have square-root singularities at the turning points $(x = 0, y = a)$ for P_1, $(x = 0, y = b)$ for P_2 and $[(x = 0, y = c)$ and $(x = 0, y = d)]$ for P_3. Using the conditions (3a) along with Havelock's expansion theorem[1], we find that we must have that

$$A_j(k) = -B_j(k) \tag{4}$$

and

$$T_j + R_j = 1, \; (j = 1, 2, 3). \tag{5}$$

Then, conditions (3b) and (3c) give rise to the following multiple integral equations for the determination of the remaining unknown quantities:

and

$$\left.\begin{aligned} &\int_0^\infty A_j(k)L(k,y)dk = R_j e^{-Ky}, && y \in G_j \\ &\int_0^\infty kA_j(k)L(k,y)dk = iK(1-R_j)e^{-Ky} && y \in L_j \quad (j=1,2,3) \end{aligned}\right\}. \tag{6}$$

Keeping in mind the singular behaviour of the integrals on the left of the second relations (6), at the turning points, we integrate these relations with respect to y and recast them into the form

$$\int_0^\infty A_j(k)(k \sin ky + K \cos ky)dk = -i(1-R_j)e^{-Ky} - D_j, \quad y \in L_j, \tag{7}$$

where $D_j s$ are arbitrary constants of integration. The relations (7) also can be represented as

$$\frac{d}{dy}\int_0^\infty \frac{A_j(k)}{k} L(k,y)dk = i(1-R_j)e^{-Ky} + D_j, \quad \text{for} \quad y \in L_j. \tag{8}$$

Thus, by using the representations (8), the multiple integral equations (6) of our concern take up the following forms:

and

$$\left.\begin{aligned} &\int_0^\infty A_j(k)L(k,y)dk = R_j e^{-Ky}, && y \in G_j \\ &\frac{d}{dy}\int_0^\infty \frac{A_j(k)}{k} L(k,y)dk = i(1-R_j)e^{-Ky} + D_j, && y \in L_j \end{aligned}\right\}. \tag{9}$$

These equations will be solved for the unknowns A_j, R_j and D_j by utilizing methods to be described in the next section.

We finally observe, in this section, that equations (9) can further be expressed, after using the operator $(d/dy + K)$ on both sides, formally, in the forms:

and

$$\left.\begin{aligned} &\int_0^\infty F_j(k) \sin kydk = 0, && y \in G_j \\ &\frac{d}{dy}\int_0^\infty \frac{F_j(k)}{k} \sin kydk = C_j, && y \in L_j \quad (j=1,2,3) \end{aligned}\right\} \tag{10}$$

with

$$F_j(k) = (k^2 + K^2)A_j(k), \tag{11}$$

where $C_j s$ are arbitrary constants.

We also note, while passing, that because of the Riemann Lebesgue lemma[7], we must have that $C_1 = 0$, for the class of functions A_1, for the problem P_1, for which our solution becomes acceptable. The constants C_2 and C_3 associated with the problems P_2 and P_3 remain arbitrary still and we shall determine them fully in the next section.

3. Reduction to Abel-type integral equations and their solutions

The solutions of the multiple equations (10) can be determined by making the following observations:

(I) The functions

$$\psi_j(x,y) = \int_0^\infty \frac{F_j(k)}{k} e^{-kx} \sin kydk \quad (j = 1, 2, 3), \tag{12}$$

represent harmonic functions in the x–y plane, and

(II) The normal derivative $\partial\psi_j/\partial x$ has the form, on the boundary $x = 0$, as given by

$$\frac{\partial\psi_j}{\partial x} = -\int_0^\infty F_j(k) \sin kydk \quad (j = 1,2,3). \tag{13}$$

The boundary conditions (10) suggest that the integrals in the relation (13) must have square-root singularities at the turning points $(a_1 = a)$, $(a_2 = b)$ and $(a_3 = c,\ b_3 = d)$, corresponding to the problems P_1, P_2 and P_3, respectively. These observations immediately suggest that we may use the following representations for the integrals in (13), for the three different problems at hand[8]:

$$\int_0^\infty F_j(k)\sin kydk = \begin{cases} \dfrac{1}{y}\dfrac{d}{dy}\displaystyle\int_a^y \frac{tS_1(t)dt}{(y^2-t^2)^{1/2}}, & \text{for } j=1,\ (a<y<\infty), \\ \dfrac{d}{dy}\displaystyle\int_y^b \frac{tS_2(t)dt}{(t^2-y^2)^{1/2}}, & \text{for } j=2,\ (0<y<b), \\ \dfrac{1}{y(y^2-c^2)^{1/2}}\dfrac{d}{dy}\displaystyle\int_y^d \frac{tS_3(t)dt}{(t^2-y^2)^{1/2}}, & \text{for } j=3,\ (c<y<d), \end{cases} \tag{14}$$

where S_1, S_2 and S_3 are differentiable functions having the properties that $S_1(a) \neq 0$, $S_2(b) \neq 0$ and $S_3(d) \neq 0$, ensuring the desired square-root singularities of the integrals $\int_0^\infty F_j(k) \sin kydk$ at the turning points described earlier.

Using first of the two relations in (10) and the above representations, the relation (14) for the integrals $\int_0^\infty F_j(k) \sin kydk$ in the two complementary ranges $y \in G_j$. and $y \in L_j$, respectively, and utilizing the standard Fourier sine inversion formula, we find after some simple manipulations, in each of the cases $j = 1$, 2 and 3 that the functions S_1, S_2, S_3 must satisfy the following equations:

$$\frac{d}{dy}\left[\int_a^{\infty}\left[\frac{S_1(a)}{(u^2-a^2)^{1/2}}+\int_a^u\frac{S_1'(t)dt}{(u^2-t^2)^{1/2}}\right]\ \ln\ \left|\frac{y-u}{y+u}\right|du\right]=0,\quad (a<y<\infty), \tag{15}$$

$$\frac{d}{dy}\int_0^b\frac{\partial}{\partial u}\left[\int_u^b\frac{tS_2(t)dt}{(t^2-u^2)^{1/2}}\right]\ \ln\ \left|\frac{y-u}{y+u}\right|du=\pi C_2\quad (0<y<b) \tag{16}$$

and

$$\frac{d}{dy}\left[\int_c^d\frac{1}{(u^2-c^2)^{1/2}}\left[-\frac{S_3(d)}{(d^2-u^2)^{1/2}}+\int_u^d\frac{S_3'(t)dt}{(t^2-u^2)^{1/2}}\right]\ \ln\ \left|\frac{y-u}{y+u}\right|du\right]=\pi C_3,\quad (c<y<d), \tag{17}$$

(dash denoting derivative w.r.t. the argument).

In deriving the above equations, we have made use of the well-known result that

$$\int_0^{\infty}\frac{\sin ku\sin ky}{k}dk=-\frac{1}{2}\ \ln\ \left|\frac{y-u}{y+u}\right|,\quad (0<y,u<\infty). \tag{18}$$

We shall next present the solutions of equations (15)–(17) *via* Abel integral equations, one by one. For eqn (15), we use the following easily derivable results:

$$(i)\ \int_a^{\infty}\frac{udu}{(u^2-a^2)^{1/2}(y^2-u^2)}=0,\quad \text{for}\quad y>a,$$

$$(ii)\ \int_t^{\infty}\frac{udu}{(u^2-t^2)^{1/2}(y^2-u^2)}=\begin{cases}0 & \text{for}\quad y>t\\ -\dfrac{\pi}{2(t^2-y^2)^{1/2}}, & \text{for}\quad t>y.\end{cases}$$

Equation (15), then simplifies to the most simple Abel integral equation as given by

$$\int_y^{\infty}\frac{S_1'(t)dt}{(t^2-y^2)^{1/2}}=0\quad \text{for}\quad y>a, \tag{19}$$

whose solution decides that[9]

$$S_1(t)=\lambda_1,\ \text{a constant (say)}. \tag{20}$$

Next, for eqn (16), we use the following result:

$$(iii)\ \int_0^{t}\frac{du}{(t^2-u^2)^{1/2}(y^2-u^2)}=\begin{cases}0 & \text{for}\quad t>y\\ \dfrac{\pi}{2y(y^2-t^2)^{1/2}}, & \text{for}\quad t<y\end{cases}$$

and simplify the equation (16) to give rise to another simple Abel integral equation as given by

$$\int_0^y \frac{tS_2(t)dt}{(y^2-t^2)^{1/2}} = C_2 y \quad \text{for} \quad 0<y<b, \tag{21}$$

whose solution decides that[9]

$$S_2(t) = C_2. \tag{22}$$

Finally, for eqn (17), we use the following results:

$$(iv) \int_c^d \frac{udu}{[(u^2-c^2)(d^2-u^2)]^{1/2}(y^2-u^2)} = 0 \quad \text{for} \quad c<y<d$$

and

$$(v) \int_c^t \frac{udu}{[(u^2-c^2)(t^2-u^2)]^{1/2}(y^2-u^2)} = \begin{cases} 0 & \text{for} \quad y<t \\ \dfrac{\pi}{2}\dfrac{1}{[(y^2-t^2)(y^2-c^2)]^{1/2}} & \text{for} \quad t<y. \end{cases}$$

Equation (17) then simplifies to the third Abel integral equation as given by

$$\int_c^y \frac{S_3'(t)dt}{(y^2-t^2)^{1/2}} = C_3(y^2-c^2)^{1/2}, \quad \text{for} \quad c<y<d. \tag{23}$$

The solution of eqn (23) is also straightforward, and we find that the function $S_3(t)$ is given by[9]

$$S_3(t) = C_3(t^2-c^2) + \lambda_2, \tag{24}$$

where λ_2 is another arbitrary constant. The alternative form of S_3, that is convenient for further calculations, is taken as

$$S_3(t) = \lambda_3(d_0^2 - t^2), \tag{25}$$

in which λ_3 and d_0 are new arbitrary constants.

4. The full solutions

Substituting the functions S_1, S_2 and S_3 from the relations (20), (22) and (25), respectively, into the expressions in the relations (14) and using the first of the conditions (10), we obtain, after employing the Fourier sine inversion formula and some straightforward calculations that

$$F_1(k) = \frac{2\lambda_1}{\pi}\int_a^\infty \frac{\sin kudu}{(u^2-a^2)^{1/2}} = \lambda_1 J_0(ka), \tag{26}$$

$$F_2(k) = -\frac{2C_2}{\pi}\int_0^b \frac{u\sin kudu}{(b^2-u^2)^{1/2}} = -C_2 bJ_1(kb) \tag{27}$$

and

$$F_3(k)=\frac{2\lambda_3}{\pi}\int_c^d \frac{1}{u(u^2-c^2)^{1/2}}\frac{d}{du}\left[\int_u^d \frac{t(d_0^2-t^2)dt}{(t^2-u^2)^{1/2}}\right]\sin kudu=-\frac{2\lambda_3}{\pi}J(k),\quad \text{(say)} \tag{28}$$

where

$$J(k)=\int_c^d \frac{(d_0^2-u^2)\sin ku}{[(u^2-c^2)(d^2-u^2)]^{1/2}}du \tag{29}$$

and $J_\nu(x)$ is the well-known Bessel function of the first kind[10]. The complete solutions of the three problems under consideration can be determined by using the relations (11), once the arbitrary constants λ_1, C_2, λ_3 and d_0 and also the unknown constants R_j and T_j are fully determined. To achieve this, we have taken recourse to the original multiple integral equations (6) to which we have applied, rather formally, the operator $(d/dy+K)$ and make use of the relations $T_j+R_j=1$, $(j=1,2,3)$.

The following standard results[11] become extremely important for this purpose for the problems P_1 and P_2:

$$(i)\ \int_0^\infty \frac{J_0(ka)\sin kydk}{(k^2+K^2)}=\frac{\sinh Ky}{K}K_0(Ka),\quad (K>0, 0<y<a),$$

$$(ii)\ \int_0^\infty \frac{kJ_0(ka)\sin kydk}{(K^2+k^2)}=\frac{\pi}{2}e^{-Ky}I_0(Ka),\quad (K>0, a<y<\infty),$$

$$(iii)\ \int_0^\infty \frac{J_1(kb)\sin kydk}{(k^2+K^2)}=\frac{\pi}{2K}e^{-Ky}I_1(Kb),\quad (K>0, b<y<\infty),\ \text{and}$$

$$(iv)\ \int_0^\infty \frac{kJ_1(kb)\sin kydk}{(K^2+k^2)}=K_1(Kb)\sinh Ky,\quad (K>0, 0<y<b),$$

where $I_\nu(x)$ and $K_\nu(x)$ are the well-known modified Bessel functions[11]. We find after some straightforward manipulations, that

$$R_1=\lambda_1 aK_0(Ka),$$

$$\lambda_1 a=\frac{1}{K_0(Ka)+i\pi I_0(Ka)},$$

$$R_2=C_2 b\pi I_1(Kb)$$

$$\text{and } C_2 b=\frac{1}{\pi I_1(Kb)+iK_1(Kb)}. \tag{30}$$

These results agree completely with the ones obtained by earlier workers[2].

For the problem P_3, the manipulations are a bit more involved, but it is not difficult to arrive at the following results, which also agree with the known ones[1,4];

$$R_3 = -\lambda_3 \int_c^d \frac{\left(d_0^2 - u^2\right)e^{-Ku}du}{\left[\left(u^2 - c^2\right)\left(d^2 - u^2\right)\right]^{1/2}}, \tag{31}$$

$$d_0^2 = \int_c^d \frac{u^2 e^{Ku}du}{\left[\left(u^2 - c^2\right)\left(d^2 - u^2\right)\right]^{1/2}} \Big/ \int_c^d \frac{e^{Ku}du}{\left[\left(u^2 - c^2\right)\left(d^2 - u^2\right)\right]^{1/2}} \tag{32}$$

and

$$\lambda_3 = \frac{1}{i(\beta - \alpha) - \gamma}, \tag{33}$$

where

$$\left.\begin{aligned} \alpha &= \int_{-c}^{c} \frac{\left(d_0^2 - u^2\right)e^{-Ku}du}{\left[\left(c^2 - u^2\right)\left(d^2 - u^2\right)\right]^{1/2}}, \\ \beta &= \int_d^{\infty} \frac{\left(d_0^2 - u^2\right)e^{-Ku}du}{\left[\left(u^2 - c^2\right)\left(u^2 - d^2\right)\right]^{1/2}}, \\ \text{and} \quad \gamma &= \int_c^d \frac{\left(d_0^2 - u^2\right)e^{-Ku}du}{\left[\left(u^2 - c^2\right)\left(d^2 - u^2\right)\right]^{1/2}} \end{aligned}\right\} \tag{34}$$

5. Conclusion

Known solutions for three basic mixed boundary-value problems arising in the study of scattering of surface water waves by vertical barriers are handled completely by utilizing simplified integral equations of the Abel type. The final solutions agree fully with the known ones. Further generalisations of the present method are expected to work similarly and will be the subject of our next study.

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