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On the solution of the partially immersed vertical wave-maker problem in surface water waves

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Abstract

The mixed boundary-value problem for Laplace's equation associated with the partially immersed vertical wave-maker problem is handled for solution by utilising Abel integral equations and their inversions. The present method appears to be sufficiently general being usable in more general problems of this class. Particular cases are observed to yield the known results.

Keywords: Surface water waves, wave-maker, dual integral equations, Abel integral equations.

1. Introduction

The problem of generation of surface water waves by small oscillation of vertical plates in the linearised theory of water waves has attracted the attention of many researchers[1-3] to develop newer and newer mathematical methods to solve a class of mixed boundary-value problems, associated with Laplace's equation, in two dimensions. The speciality of this class of problems lies in the fact that the conditions at infinity are not known fully. One such plate problem is the problem of generation of surface water waves due to small oscillation of a partially submerged vertical plate in water of infinite depth, which was first studied by Ursell[1]. Ursell converted the boundary-value problem associated with the physical problem at hand to two Cauchy-type integral equations and obtained full solution of the problem along with the wave amplitude at large distances in closed form. Later on, Evans[2] gave an alternative method for a similar problem which involves the application of Green's integral theorem to obtain the wave amplitude at large distance. Mandal and Banerjea[3] considered the nearly vertical partially submerged plate problem and obtained, using a perturbation approach, the velocity potential along with the wave amplitude at large distance by exploiting the idea of Evans.

In the present paper we have reinvestigated the partially submerged vertical wave-maker problem under sufficiently general boundary data (for detailed description of the physical problem, see Evans[2]) by utilising the well-known property of the mixed boundary conditions for problems involving Laplace's equation, that the normal derivative of a harmonic function possesses square-root singularities at the turning points (*i.e.*, at the points on either side of which different boundary conditions are prescribed) on a straight boundary. The problem then gets reduced to a standard Abel-type integral equation, the solution of which is well known.

Section 2 deals with the detailed mathematical formulation of the boundary-value problem under consideration and in Sections 3 and 4 the derivation of its solution is discussed along with a particular known case of the general problem undertaken.

2. The statement of the problem

The mathematical problem under consideration is to determine a harmonic function $\phi(x, y)$ of the variables (x, y) in the two-dimensional Cartesian coordinate system, in the form:

$$\phi(x,y)=\begin{cases} Te^{-Ky+iKx}+\int_0^\infty A(k)L(k,y)e^{-kx}dk, & x>0 \\ Re^{-Ky-iKx}+\int_0^\infty B(k)L(k,y)e^{kx}dk, & x<0 \end{cases} \tag{1}$$

in the half plane $y > 0$, with $i^2 = -1$ and

$$L(k, y) = k \cos ky - K \sin ky, \tag{2}$$

where T and R are two unknown constants, and $A(k)$ and $B(k)$ are unknown functions which have to be determined by utilising the following conditions:

$$\frac{\partial\phi}{\partial x} \text{ is continuous on } x = 0, \text{ for all } y > 0, \tag{3}$$

$$\frac{\partial\phi}{\partial x} = f(y) \text{ on } x = 0 \pm y \in L, \tag{4}$$

$$\phi \text{ is continuous on } x = 0, \text{ for } y \in G \tag{5}$$

$$\frac{\partial\phi}{\partial x} \approx O((r-a)^{-1/2}) \text{ as } r = (x^2+y^2)^{1/2} \to a, \tag{6}$$

where $L = (0, a)$, $G = (0, \infty) - (0, a)$ and $f(y)$ is a function whose first-order derivative exists and integrable in $(0, a)$. Condition (6) represents the physical requirement that velocity has to be singular like $O(r^{-1/2})$, as $r \to 0$, where $r = 0$ represents a sharp edge.

3. Reduction to dual integral equations

Using the continuity of $\partial\phi/\partial x$ along $x = 0$ as given in condition (3) along with Havelock's expansion theorem[4] we must have

$$\left.\begin{aligned} A(k) &= -B(k) \\ \text{and}\quad T &= -R \end{aligned}\right\}. \tag{7}$$

Then utilising conditions (4) and (5) along with the relation (7), from the expression (1) for $\phi(x, y)$ we obtain a set of integral relations for the determination of the function $A(k)$, as given by

$$-\int_0^\infty A(k)L(k,y)dk = Te^{-Ky}, \quad a<y<\infty \tag{8}$$

and

$$iKTe^{-Ky} - \int_0^\infty kA(k)L(k,y)dk = f(y), \quad 0 < y < a, \tag{9}$$

in which the constant T is also an unknown quantity.

Integrating with respect to y the relation (9) can be rewritten as

$$-iTe^{-Ky} - \int_0^\infty A(k)(k\sin ky + K\cos ky)dk = \int f(y)dy + C \text{ for } 0 < y < a, \tag{10}$$

where C is an arbitrary constant. The relation (10) can again be rewritten as

$$-iTe^{-Ky} + \frac{d}{dy}\int_0^\infty \frac{A(k)}{k} L(k,y)dk = \int f(y)dy + C \text{ for } 0 < y < a, \tag{11}$$

Operating by $(d/dy + K)$, both sides of eqns (8) and (11), we obtain the derived set of dual integral equations for the function $F(k)$ as given by

$$\int_0^\infty F(k)\sin kydk = 0, \quad a \le y < \infty, \tag{12}$$

and

$$\frac{d}{dy}\int_0^\infty \frac{F(k)}{k} \sin kydk = H(y) + D \quad 0 < y < a, \tag{13}$$

with,

$$F(k) = (k^2 + K^2)A(k) \tag{14}$$

and

$$H(y) = -(\frac{d}{dy} + K)\int f(y)dy, \tag{15}$$

where $D = KC$ is an arbitrary constant.

4. Method of solution of the dual integral equations

The solution of the set of dual integral equations as given by relations (12) and (13) can be obtained by making use of the following observations:

(a) The function

$$\psi(x,y) = \int_0^\infty \frac{F(k)}{k} e^{-kx} \sin kydk, \quad x > 0,\ y > 0 \tag{16}$$

represents a harmonic function in the (x, y) plane.

(b) The normal derivative $\partial\psi/\partial x$ has the form on the boundary $x = 0$ as given by

$$\frac{\partial\psi}{\partial x} = -\int_0^\infty F(k)\sin kydk, \quad \text{on} \quad x = 0. \tag{17}$$

So, the boundary conditions (12) and (13) suggest that the integral in (17) will have a square-root singularity at the point $y = a$.

These observations suggest that we may use the following representation for the integral in (17)[5]:

$$\int_0^\infty F(k)\sin kydk = \frac{d}{dy}\int_y^a \frac{ts(t)dt}{(t^2-y^2)^{1/2}}, \quad y>0, \tag{18}$$

where $s(t)$ is a bounded and differentiable function of t in $(0, a)$ having the property that $s(a) \neq 0$. Utilising the relation (12) and the above representation as in relation (18) in the two complementary ranges $(0, a)$ and (a, ∞) after using the Fourier sine inversion formula, we obtain

$$F(k) = \frac{2}{\pi}\int_0^a \frac{\partial}{\partial u}\left(\int_u^a \frac{ts(t)dt}{\left(t^2-u^2\right)^{1/2}}\right)\sin kudu,\ k>0. \tag{19}$$

Now substituting for $F(k)$ in eqn (13) we derive that

$$\frac{2}{\pi}\frac{d}{dy}\int_0^a \frac{\sin ky}{k}\left[\int_0^a \frac{\partial}{\partial u}\left(\int_u^a \frac{ts(t)dt}{\left(t^2-u^2\right)^{1/2}}\right)\sin kudu\right]dk = H(y)+D,\ 0<y<a, \tag{20}$$

which gives, after simplification,

$$\frac{1}{\pi}\frac{d}{dy}\int_0^a \frac{\partial}{\partial u}\left[\left(\int_u^a \frac{ts(t)dt}{\left(t^2-u^2\right)^{1/2}}\right)\right]\ln\left|\frac{y+u}{y-u}\right|du = H(y)+D, \text{ for } 0<y<a. \tag{21}$$

Integrating the relation (21) with respect to y in $(0, y)$ we obtain

$$\frac{1}{\pi}\int_0^a \frac{\partial}{\partial u}\left[\left(\int_u^a \frac{ts(t)dt}{\left(t^2-u^2\right)^{1/2}}\right)\right]\ln\left|\frac{y+u}{y-u}\right|du = \int_0^y H(t)dt + Dy, \text{ for } 0<y<a. \tag{22}$$

Using the result

$$\frac{2}{\pi}\int_0^t \frac{du}{\left(t^2-u^2\right)^{1/2}\left(y^2-u^2\right)} = \begin{cases} \dfrac{1}{y(y^2-t^2)^{1/2}} & \text{for } y>t \\ 0, & \text{for } t>y \end{cases}$$

and simplifying the left-hand side of the relation (22) we arrive at the Abel-type integral equation for $s(t)$ as given by

$$\int_0^y \frac{ts(t)dt}{\left(y^2-t^2\right)^{1/2}} = -\left(\int_0^y H(t)dt + Dy\right),\quad 0<y<a. \tag{23}$$

The solution of eqn (23) is standard and is given by[6]

$$ts(t) = -\frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{y[Dy + \int_0^y H(y)du]\,dy}{\left(t^2 - y^2\right)^{1/2}}, \quad 0 < t < a. \tag{24}$$

Once $s(t)$ is determined, $F(k)$ can be obtained by substituting for $s(t)$ in the relation (19). Then $A(k)$ can be obtained from the relation (14). The two unknown constants T and D appearing in the picture are finally determined by utilizing the identities (8) and (9) in appropriate manner, the details of which are presented in the next section, by considering a well-known particular case of the function $f(y)$.

5. Particular case

In the case when

$$f(y) = i\sigma\theta_0(y - s), \tag{25}$$

in which $i^2 = -1$, θ_0, σ and s are positive constants, σ representing the angular frequency, for the case of an oscillatory wave-maker, as considered by Evans[2], we proceed as follows[2-3].

Using the relation (24), the expression for $s(t)$ is obtained in this particular case in the form

$$s(t) = A_1 t^2 + B_1 t + C_1 \tag{26}$$

where

$$A_1 = \frac{i\sigma\theta_0 a}{4}, \ B_1 = \frac{2i\sigma\theta_0(1 - Ks)}{\pi},$$

and C_1 is an arbitrary constant.

Then substituting for $s(t)$ from the relation (26) into the relation (19) and using the relation (14) we obtain that

$$A(k) = -\frac{\sigma\theta_0 a}{k^2 + K^2}\left[C_2 J_1(ka) - \frac{iaJ_2(ka)}{2k} + \frac{i(Ks-1)}{k}\left\{J_1(ka)H_0(ka) - H_1(ka)J_0(ka)\right\}\right], \tag{27}$$

where C_2 is an unknown constant in terms of C_1 to be determined and use has been made of the following results[7]:

$$\int_y^a \frac{t^3 dt}{\left(t^2 - y^2\right)^{1/2}} = y^2\left(a^2 - y^2\right)^{1/2} + \frac{\left(a^2 - y^2\right)^{3/2}}{3}, 0 < y < a, \tag{28}$$

$$\int_y^a \frac{t^3 dt}{\left(t^2 - y^2\right)^{1/2}} = \frac{a}{2}\left(a^2 - y^2\right)^{1/2} + \frac{y^2}{2}\cosh^{-1}\left(\tfrac{a}{y}\right), \ 0 < y < a, \tag{29}$$

$$\int_0^a y\left(a^2-y^2\right)^{1/2}\sin ky\,dy=\frac{\pi a^2 J_2(ak)}{2k},\quad 0<y<a,\ k>0, \tag{30}$$

$$\int_0^a \frac{y\sin ky}{\left(a^2-y^2\right)^{1/2}}dy=\frac{\pi a J_1(ak)}{2},\quad 0<y<a,\ k>0, \tag{32}$$

and

$$\int_0^a y\cosh^{-1}(a/y)\sin ky\,dy=\left(\frac{\pi}{2}\right)^2\frac{a}{k}\left[J_1(ak)H_0(ak)-J_0(ak)H_1(ak)\right]\quad k>0,0<y<a, \tag{32}$$

where $J_\nu(x)$s are the well-known Bessel functions of the first kind and $H_\nu(x)$s, the well-known Struve functions[8].

Next, to obtain T and C_2 we substitute for $A(k)$ from the expression (27) in the identity (8) and obtain that

$$T=\sigma\theta_0 a\pi\left\{-C_2 I_1(Ka)+\frac{ia}{2}I_2(ka)-\frac{i(Ks-1)}{K}\left[I_1(Ka)L_0(Ka)-L_1(Ka)I_0(Ka)\right]\right\}, \tag{33}$$

where $I_\nu(x)$s are the modified Bessel functions of the first kind and $L_\nu(x)$s are the modified Struve functions. The following results[7] have been utilised in obtaining the above expression for T:

$$\int_0^\infty \frac{J_1(ka)\sin ky\,dk}{\left(k^2+K^2\right)}=\frac{\pi}{2K}e^{-Ky}I_1(Ka),\ (K>0,a<y<\infty), \tag{34}$$

$$\int_0^\infty \frac{J_2(ka)\sin ky\,dk}{k\left(k^2+K^2\right)}=\frac{\pi}{2K^2}e^{-Ky}I_2(Ka),\ (K>0,a<y<\infty), \tag{35}$$

and

$$\int_0^\infty \frac{\left[J_1(ka)H_0(ka)-H_1(ka)J_0(ka)\right]\sin ky\,dk}{k\left(k^2+K^2\right)}=$$

$$=\frac{\pi}{2K^2}e^{-Ky}\left[I_1(Ka)L_0(Ka)-L_1(Ka)I_0(Ka)\right],\ (K>0,a<y<\infty). \tag{36}$$

Again, substituting for $A(k)$ from the expression (27) in the relation (9) and considering the limiting case as $y\to 0$, we obtain that

$$T=\sigma\theta_0 a\left\{iC_2K_1(Ka)-\frac{a}{2}K_2(Ka)+\frac{(1-Ks)}{aK^2}\left[1+\frac{2}{\pi}\int_0^a tK_1(t)dt\right]\right\}, \tag{37}$$

where $K_\nu(x)$s are the modified Bessel functions of the second kind and use has been made of the following results[7] (see appendix):

$$\int_0^\infty \frac{k^2 J_1(ka)dk}{\left(k^2+K^2\right)} = K_1(Ka), \ a, K > 0, \tag{38}$$

$$\int_0^\infty \frac{kJ_2(ka)dk}{\left(k^2+K^2\right)} = \frac{2}{a^2K^2} - K_2(Ka), \ a, K > 0 \tag{39}$$

and

$$\int_0^\infty \frac{k\left[J_1(ka)H_0(ka) - H_1(ka)J_0(ka)\right] dk}{\left(k^2+K^2\right)} =$$

$$= \left[K_1(Ka)L_0(Ka) + L_1(Ka)K_0(Ka)\right], \ a, K > 0. \tag{40}$$

Finally, solving eqns (33) and (37) for the two unknowns T and C_2, we find that

$$T = -\frac{\sigma\theta_0 a\pi}{K\Delta}\left[\frac{1}{2} + \frac{(Ks-1)}{aK}\{I_1(Ka) + L_1(Ka)\}\right] \tag{41}$$

and

$$C_2 = \frac{a}{2}\left[\frac{\{\pi i I_2(Ka) + K_2(Ka)\}}{\Delta}\right.$$

$$\left. - \frac{(1-Ks)}{aK^2\Delta} + \frac{(1-Ks)}{K}\left[iL_0(Ka) - L_1(Ka)\left(\frac{\pi i I_0(Ka) + K_0(Ka)}{\Delta}\right)\right]\right], \tag{42}$$

with

$$\Delta = \pi i I_1(Ka) + iK_1(Ka), \tag{43}$$

after making use of the identities[1]:

$$I_1(Ka)K_2(Ka) + K_1(Ka)I_2(Ka) = \frac{1}{Ka}, \tag{44}$$

and

$$I_1(Ka)\int_0^{Ka} uK_1(u)du - K_1(Ka)\int_0^{Ka} uI_1(u)du = \frac{1}{2}\pi L_1(Ka). \tag{45}$$

Hence, we obtain the results for $A(k)$ and T as given by the relations (27) and (41) with C_2 as given by (42). This result for T coincides with the result as obtained by Evans[2]. Also the full solution coincides with the result obtained by Mandal and Banerjea[3] in case of the vertical wave-maker if one takes care of an algebraic error that is appearing in their final solution.

6. Conclusion

The problem of a partially immersed vertical wave-maker in the linearised theory of water waves in the case of water of infinite depth has been considered for solution, by converting the boundary-value problem to a set of dual integral equations. These equations have been converted into an Abel-type integral equation by using the behaviour of the harmonic functions at turning points, whose solution is well known. Particular case of the more general problem has been considered as a check. The problem of a nearly vertical partially submerged wave-maker problem can also be dealt with by the present method, as is the case with similar such mixed boundary-value problems.

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Appendix

To evaluate the integral

$$\int_0^\infty \frac{k}{k^2+K^2}\left[J_1(ka)H_0(ka)-H_1(ka)J_0(ka)\right]dk, \tag{A1}$$

Consider the integral

$$I=\int_\Gamma F(z)dz, \tag{A2}$$

where

$$F(z)=\frac{\left[H_0(za)H_1^1(za)-H_1(za)H_0^1(za)\right]}{z^2+K^2}z, \tag{A3}$$

Γ is a half circle in the complex z-plane with radius R very large. Then

$$I=\int_{\Gamma}F(z)dz=\int_{-R}^{R}F(k)dk+\int_{C}F(z)dz \quad with \quad z=k+i\eta. \tag{A4}$$

Along the curve C, $z=Re^{i\theta}$ with $0\leq\theta<\pi$.

From the behaviour of $F(z)$ it is clear that along the curve C,

$$\int_{C}F(z)dz\to 0 \quad \text{as} \quad R\to\infty. \tag{A5}$$

Hence

$$I=\int_{-\infty}^{\infty}F(k)dk=\int_{0}^{\infty}[F(k)+F(-k)]\,dk. \tag{A6}$$

Again

$$F(k)+F(-k)=\frac{2k}{k^2+K^2}[J_1(ka)H_0(ka)-H_1(ka)J_0(ka)]. \tag{A7}$$

Further,

$$\text{residue of } F(z) \text{ at } (z=ik)=\frac{-i}{\pi}[L_0(Ka)K_1(Ka)+L_1(Ka)K_0(Ka)]. \tag{A8}$$

So, using the Cauchy's residue theorem we obtain that

$$I=2[L_0(Ka)K_1(Ka)+L_1(Ka)K_0(Ka)], \tag{A9}$$

which gives

$$\int_0^{\infty}\frac{k}{k^2+K^2}[J_1(ka)H_0(ka)-H_1(ka)J_0(ka)]\,dk=[L_0(Ka)K_1(Ka)+L_1(Ka)K_0(Ka)]. \tag{A10}$$

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