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Comparison of conjugate gradient methods and strongly implicit procedure for groundwater flow simulation

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Abstract

A comparative study of modified strongly implicit procedure (MSIP) with a host of conjugate gradient methods as applicable to groundwater flow problems in nonrectangular flow domains is made. Among the methods—Hestenes and Steifel's conjugate gradient methods (CGHS), diagonal scaling conjugate gradient method (DSCG) and incomplete Cholesky conjugate gradient method (ICCG) and MSIP—it was found that DSCG and MSIP performed equally well. DSCG was found to be easily adoptable for nonrectangular flow domains unlike MSIP which needed derivation of special equations.

Keywords: Comparison, conjugate gradient methods, strongly implicit procedure, groundwater flows.

1. Introduction

Groundwater modelling is a powerful tool that can help in analysing many groundwater problems. Modelling is useful in understanding the flow behaviour and the response of the aquifer under a given stress. Since the development of digital computer, tremendous progress has been made in the numerical modelling of groundwater systems covering both well field and regional situations. To identify efficient methods of analysis of the groundwater problems several comparative studies involving finite-difference techniques have been made earlier. Notable in this are Tresscott and Larson[1], Mohan Kumar[2] and Mohan Kumar *et al.*[3] All the above-mentioned papers consider only methods like alternating direction implicit procedure (ADI), successive over relaxation procedure (SOR), and strongly implicit procedure (SIP) to solve the linear equations arising out of approximating the governing equations.

One of the early attempts to compare the incomplete Cholesky conjugate gradient (ICCG) method along with SIP was made by Kuiper[4]. He concluded that ICCG is superior to SIP for confined aquifer problems. A comparative study of a family of conjugate gradient methods as applied to regional groundwater flow problems has not been made.

Most of the groundwater basins will be nonrectangular in shape. Thus the existing finite-difference methods either have to be checked for their suitability for such situations or modified accordingly. Mohan Kumar *et al.*[3] have modified the SIP, originally

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developed by Stone[5] , for nonrectangular flow domains. It has also been proved that SIP is one of the powerful methods for groundwater flow analysis[1, 3, 6].

Of late, conjugate gradient methods have gained a lot of importance in groundwater flow analysis. A new method called the incomplete Cholesky conjugate gradient method is proposed by Meijerink and Van der Vorst[7] for groundwater flows. The conjugate gradient method with various preconditioners was applied to groundwater flow problems and compared with other methods by Kuiper[8] and Meyer *et al.*[9] In this paper, a comparison of the three types of conjugate gradient (CG) methods and strongly implicit procedure is made with specific reference to nonrectangular flow domains. It has been found that CG methods can handle nonrectangular boundary domains naturally. The comparative study is made on three test problems depicting regional groundwater flow problems and the results are discussed in terms of computational work and accuracy.

2. Mathematical formulation and methodology

2.1. *Governing equation*

The partial differential equation of groundwater flow in a nonhomogeneous anisotropic aquifer in two dimensions is given by[10]

$$\frac{\partial}{\partial x}\left[T_x \frac{\partial h}{\partial x}\right]+\frac{\partial}{\partial y}\left[T_y \frac{\partial h}{\partial y}\right]=S\frac{\partial h}{\partial t}+Q_p-R_r-R_s-Q_l \tag{1}$$

where h is the piezometric head, T_x and T_y are the transmissivities in the x and y directions, respectively, S, the storage coefficient or specific yield, Q_p, the net rated groundwater pumping per unit area, R_r and R_s, the recharge due to rainfall and surface irrigation per unit area, respectively, Q_l, the rate of leakage per unit area if present, x and y are the cartesian co-ordinates and t, the time. The above equation is in general form and it represents flows in different types of aquifers depending on the terms in it.

For a confined system, R_r, R_s and Q_l will be absent and $S = S_t$, which is the storage coefficient. For an unconfined linear system, $Q_l = 0$ and $S = S_y$, which is the specific yield. For an unconfined nonlinear system $Q_l = 0$ and $S = S_y$, along with $T_x = K_x(h–B_l)$ and $T_y = K_y\,(h–B_l)$ where K_x and K_y are the hydraulic conductivities in the x and y directions, respectively, and B_l is the bed rock level. For a mixed confined–unconfined system, the confined system equation is used for that part of the aquifer which is under confined state while the unconfined linear system equation is used for the part of aquifer which is conceptualised as an unconfined system. In the leaky system, which has a water table aquitard, the leakage from the aquitard to the aquifer is given by

$$Q_l = C_l(h' - h) \tag{2}$$

where $C_l = K'/b'$ is the leakage coefficient, K', the vertical hydraulic conductivity of the aquitard, h, the piezometric head of the aquifer and h', the water table head in the aquitard. Equation (1) being a parabolic type of equation, needs an open boundary, along with the initial boundary conditions to get the solution. Boundary conditions along the

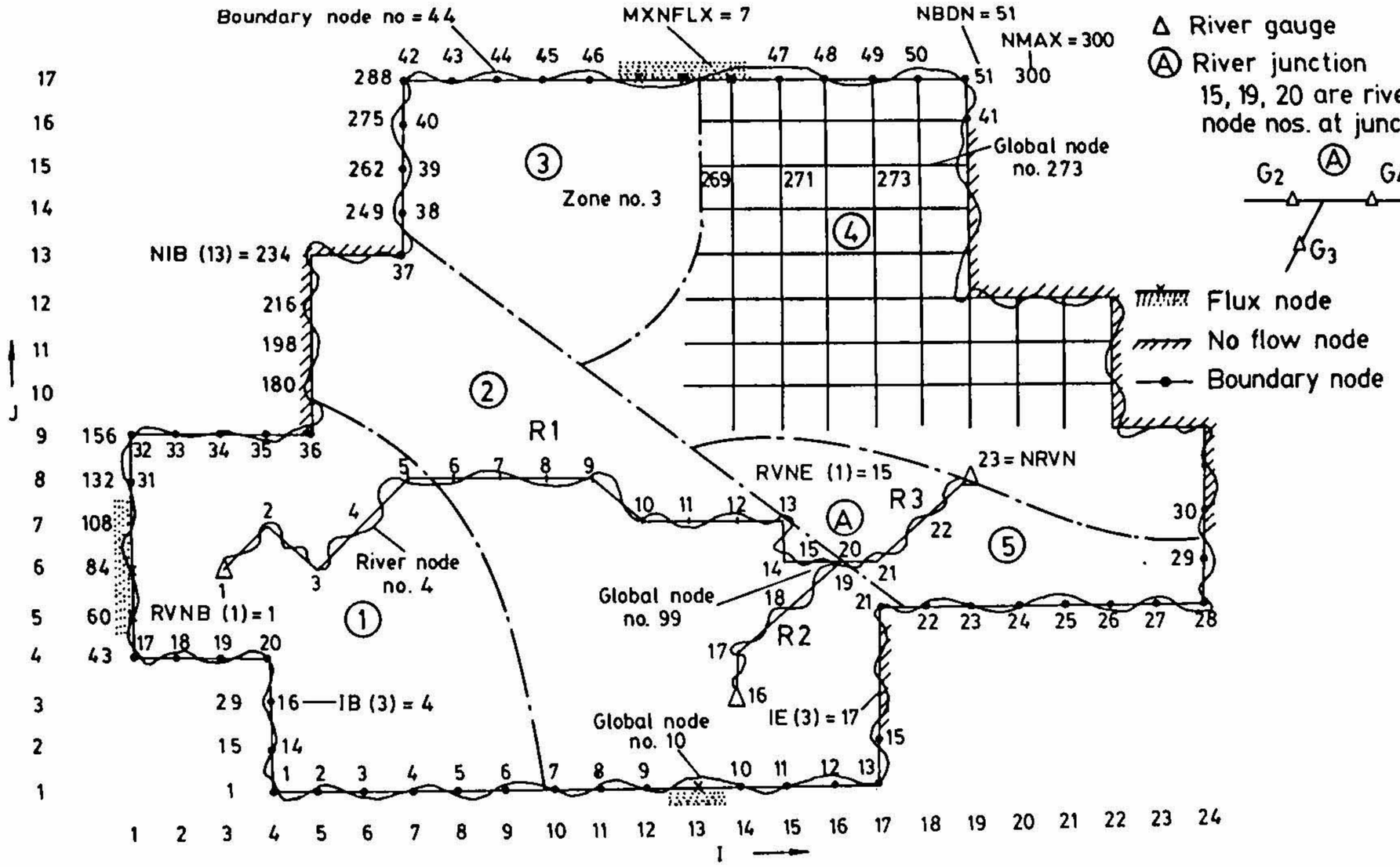

FIG. 1. Schematic diagram of finite-difference model.

external and internal boundaries such as rivers or dykes have to be specified. Since this equation cannot be solved using analytical techniques, a numerical technique such as the finite-difference method has been used.

2.2. *Finite-difference schematisation*

Finite-difference methods are widely used to solve the groundwater flow problems[2, 11–13]. In view of the special nature of the coefficient matrix arising from the use of the five-point formula, finite-difference methods are most suitable for computation. To get the solution using finite-difference technique, a finite-difference rectangular grid is superposed over the aquifer region. The grid-point approximation of the boundary and rivers, which are internal boundaries, is made (Fig. 1). The block-centered finite-difference scheme (Fig. 2) has been used to approximate the governing equation (eqn 1). Figure 1 also shows the typical data input for a regional model, details of which are presented in Sridharan *et al.*[10]

2.3. *Solution algorithm*

The governing equation represents the transient flow of groundwater in the aquifer system. To obtain the solution for hydraulic head at any given time step, it is necessary that the initial conditions should be specified in addition to the boundary conditions.

Knowing the distribution of $h_{i,j}$, at time $t = 0$, the solution can be marched in time, which is in an open domain. Knowing the solution at time t, the solution at time $(t + \Delta t)$ can be got. Because of the nature of the finite-difference approximations used, the system of equations for each time step is implicit in nature. The structure of the coefficient matrix is such that it is advantageous to solve these equations by iterative technique. The finite-difference approximations of eqn (1) are written at all nodes of the aquifer. This set of equations has to be solved simultaneously at any given time step, knowing the solution at the previous time step. Following these steps, leads to the matrix equation.

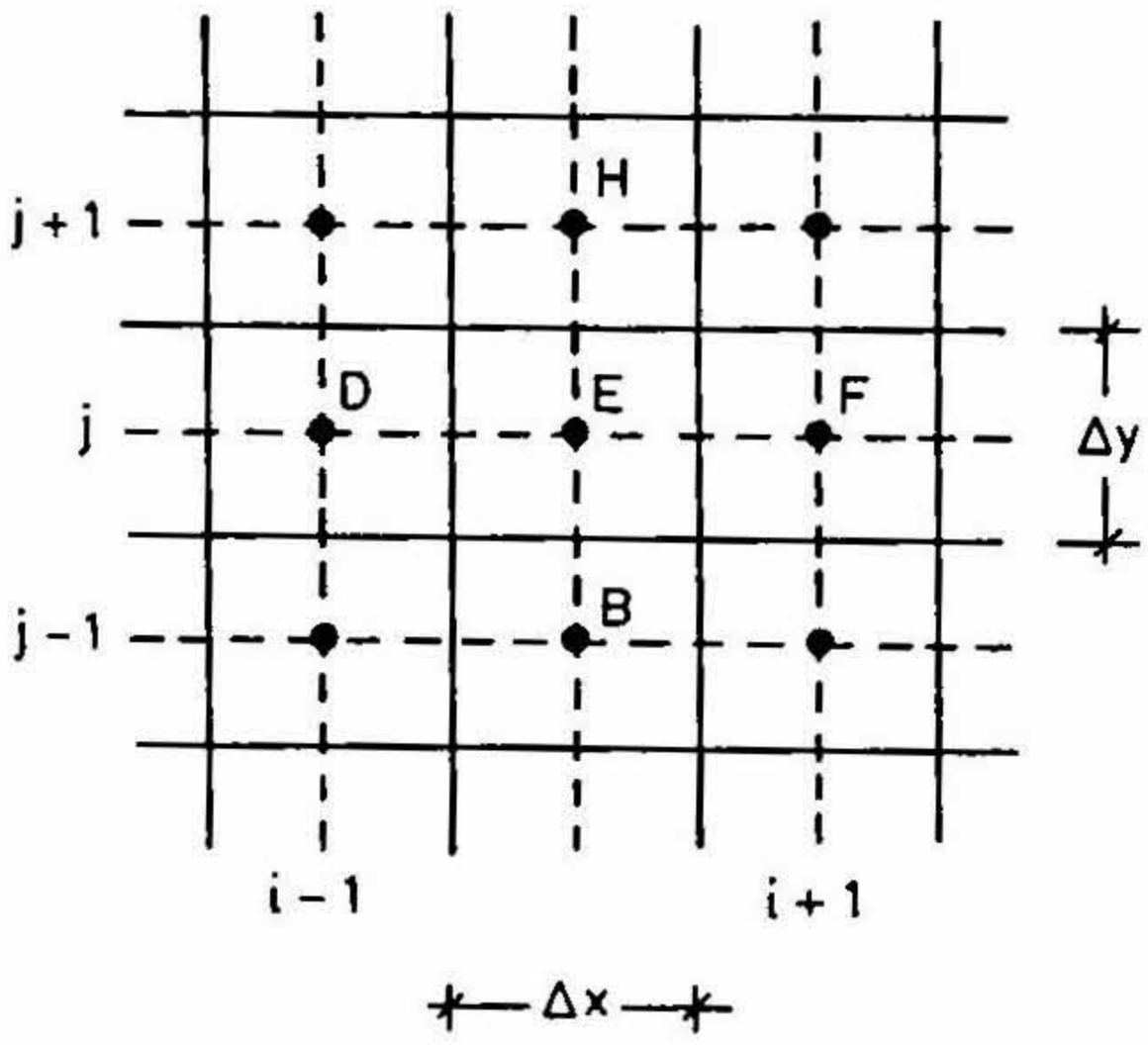

FIG. 2. Block-centered finite-difference scheme.

$$\overline{A}\,\overline{H} = \overline{Y} \tag{3}$$

where $\overline{A}$ is the coefficient matrix, $\overline{Y}$, the column vector of known quantities and $\overline{H}$, the column vector for unknown piezometric head, h. The details of generation of $\overline{A}$ are available in Sridharan *et al.*[10] There are various iterative methods for solving eqn (3) of which SIP and CG methods are known to be the best. Hence only a comparison of the SIP and a family of CG methods is made and the methods are discussed here.

3. Strongly implicit procedure

The SIP was developed by Stone[5] to solve a system of simultaneous equations involving a five diagonal matrix equations arising out of the finite-difference approximations of the governing equation of the groundwater system. The basic methodology of the procedure is as follows:

The finite-difference approximation of the governing equation at all the $I.J$ nodes in a rectangular flow region (J rows and I columns) will lead to a set of $I.\ J$ linear equations in as many unknowns, $h_{i,\,j}$ ($i = 1,\ 2,\ldots\ I$ and $j = 1,\ 2,\ldots\ J$). In writing eqn (1) at boundary nodes, the coefficients of terms falling outside the computational region are made zero as in Trescott[13] . The $I.\ J$ equations can be written in the matrix notation as given in eqn (3). The procedure for solving this equation is as follows.

$\overline{A}$ matrix cannot be decomposed into a product of lower triangular matrix $\overline{L}$ and upper triangular matrix $\overline{U}$. Suppose we assume, matrices $\overline{L}$ and $\overline{U}$ when $\overline{L}$ and $\overline{U}$ are multiplied, a matrix $A+B$, which is similar to A but not identical with it is obtained. This $\overline{A+B}$ matrix will have two more nonzero elements than $\overline{A}$ matrix on each row. As $\overline{A+B}$ is decomposable into $\overline{L}$ and $\overline{U}$, this matrix can be used in an iteration process to get the solution. The following iterative procedure can be used to obtain the solution

$$\overline{A+B}\,\overline{H}^{(n)} - \overline{A+B}\,\overline{H}^{(n-1)} = \overline{Y} - \overline{A}\,\overline{H}^{(n-1)} \tag{4}$$

where n is the iteration index. Using the definition of $\overline{\xi}^{(n)} = \overline{H}^{(n)} - \overline{H}^{(n-1)}$ and the residue $\overline{Y}^{(n-1)} = \overline{Y} - \overline{A}\,\overline{H}^{(n-1)}$, we get

$$\overline{A+B}\,\overline{\xi}^{(n+1)} = \overline{Y}^{(n+1)}. \tag{5}$$

As $\overline{A+B}$ can decomposed easily, it can be solved for $\overline{\xi}^{(n)}$ and hence for $\overline{H}^{(n)}$ values. Recurrence relations needed to generate elements of $\overline{A+B}$ and $\overline{L}$ and $\overline{U}$ are not presented here. Given the elements of $\overline{A}$, these could be generated and the iteration can be proceeded with. For a nonrectangular flow domain, the banded nature of $\overline{A+B}$ is retained and hence the similar recurrence relations are derived except at some special boundary nodes where they are modified[3]. This is again improved (ISIP) by Sridharan *et al.*[10] in which no special equations are used at these special boundary nodes.

4. Conjugate gradient (CG) methods

The CG method was originally proposed by Hestenes and Stiefel[14]. A gradient method solves $N \times N$ nonsingular system of equations by iteration process, which is as follows :

The finite-difference approximations of the governing eqn (1) at all the nodes of the computational domain lead to a set of linear equations (eqn (3)). Given an arbitrary initial guess $\overline{H}_i$ for the solution vector of $\overline{H}$, one can write

$$\overline{H}_{i+1} = \overline{H}_i + \overline{d}_i \tag{6}$$

where $\overline{d}_i$ is a gradient, $\overline{H}_i$, the approximation to the solution vector $\overline{H}$ at the ith iterative step. A CG method chooses $\overline{d}_i$ such that at each iteration the B norm of the error vector is minimized, which is defined as

$$\|e_{i+1}\|_{\overline{B}} = \langle \overline{B} e_{i+1}, e_{i+1} \rangle^{0.5} \tag{7}$$

in which e_{i+1} is the error at the $(i + 1)$th iteration. In eqn (7), the angle bracket denotes the Euclidean inner product, which is defined as

$$\langle x, y \rangle = \sum_{j=1}^{n} x_i y_i \,. \tag{8}$$

In eqn (7), $\overline{B}$ is a symmetric positive definite (spd) inner product matrix. In the case of symmetric positive definite matrix $\overline{A}$, such as that arising from the finite-difference approximation of the groundwater flow equation, the usual choice for the inner product matrix is

$$\overline{B} = \overline{A}. \tag{9}$$

The convergence of CG method depends upon the distribution of eigenvalues of $\overline{A}$ matrix and to a lesser extent upon the condition number $[k(\overline{A})]$ of the matrix. The condition number of a symmetric positive matrix is defined as

$$k(\overline{A}) = \lambda_{\max} / \lambda_{\min} \tag{10}$$

where $\lambda_{\max}$ and $\lambda_{\min}$ are the largest and smallest eigenvalues of $\overline{A}$, respectively. When $k\ (\overline{A})$ is large, the matrix is said to be ill-conditioned; in this case, CG method may converge slowly. The condition number may be reduced by multiplying the system by a preconditioning matrix $\overline{C}$ to increase the convergence rate. The given system is modified as

$$\overline{C}\,\overline{A}\,\overline{H} = \overline{C}\,\overline{Y}. \tag{11}$$

With a proper choice of $\overline{C}$, the resulting preconditioned conjugate gradient method can be quite efficient. A general algorithm for this method is as follows[7]:

Initialize

$$\overline{H}_0 = \text{arbitrary initial guess}$$
$$\overline{r}_0 = \overline{Y} - \overline{A}\ \overline{H}_0$$
$$\overline{s}_0 = \overline{C}\ \overline{r}_0$$
$$\overline{p}_0 = \overline{s}_0$$
$$i = 0$$

Do while (the stopping criterion is not satisfied)

$$a_i = <\overline{s}_i, \overline{r}_i> / <\overline{A}\overline{p}_i, \overline{p}_i>$$
$$\overline{H}_{i+1} = \overline{H}_i + a_i\ \overline{p}_i$$
$$\overline{r}_{i+1} = \overline{r}_i - a_i\ \overline{A}\ \overline{p}_i$$
$$\overline{s}_{i+1} = \overline{C}\ \overline{r}_{i+1}$$
$$b_i = <\overline{s}_{i+1}, \overline{r}_{i+1}> / <\overline{s}_i, \overline{r}_i>$$
$$\overline{p}_{i+1} = \overline{s}_{i+1} + b_i\ \overline{p}_i$$
$$i = i + 1$$

Enddo

where $\overline{r}_0$ is the initial residue vector, $\overline{s}_0$, a vector, $\overline{p}_0$, the initial conjugate direction vector, $\overline{r}_{i+1}$, $\overline{s}_{i+1}$ and $\overline{p}_{i+1}$ are the corresponding vectors at $(i+1)$th iterative step, $\overline{C}$ is the preconditioning matrix and $\overline{A}$, the given matrix.

There are three types of operations that are performed by the CG method: inner products, linear combination of vectors and matrix-vector multiplications. The computational characteristics of these operations have an impact on the efficiency of the different CG methods.

4.1. *Types of CG preconditioning methods*

(i) *Hestenes and Steifel's conjugate gradient (CGHS) method*

This method is based on the original algorithm proposed by Hestenes and Stiefel[14] and is called CGHS. This algorithm is the base for all types of preconditioning methods. In this method the preconditioning matrix is taken as identity matrix, *i.e.*,

$$\overline{C} = \overline{I}. \tag{12}$$

The CGHS method is very simple to implement but it may not be efficient.

(ii) *Diagonal scaling conjugate gradient (DSCG) method*

In this method, the choice for $\overline{C}$ is a diagonal matrix ($\overline{D}$)

$$\overline{C} = (\overline{D})^{-1} \tag{13}$$

where element $D_{ii} = A_{ii}$.

The diagonal elements of matrix $\overline{A}$ are taken as elements of diagonal matrix $\overline{D}$. This preconditioner is easy to implement and can be quite effective. By diagonally scaling the system, eqn 3 can be written as

$$\overline{A}'\,\overline{H}' = \overline{Y}' \tag{14}$$

where $\overline{A}' = \left[(\overline{D})^{-1/2}\,\overline{A}(\overline{D})^{1/2}\right]$, $\overline{H}' = (\overline{D})^{-1/2}\,\overline{H}$ and $\overline{Y}' = (\overline{D})^{-1/2}\,\overline{Y}$.

This diagonally scaled method will be called DSCG.

(iii) *Incomplete Cholesky conjugate gradient method* (ICCG)

A popular choice for preconditioning matrix $\overline{C}$ is the incomplete Cholesky factorization of the matrix $\overline{A}$[7]. In this method, the choice for $\overline{C}$ is written as

$$\overline{C} = \left(\overline{LL^T}\right)^{-1} \tag{15}$$

where $\overline{L}$ is the lower triangular matrix resulting from a Cholesky factorization of $\overline{A}$ except that fill-in is limited to maintain sparsity and is called incomplete Cholesky decomposition.

5. Basic differences between CG and SIP methods

The methodology for CG and SIP methods is already discussed in previous sections. Here the basic difference between the two methods as applicable to regional groundwater flows is discussed.

5.1. *Calculation of finite-difference coefficients*

Finite-difference coefficients have to be calculated before proceeding for any solution technique. These are calculated at all nodes of the computational domain and stored in matrix form ($\overline{A}$) in CG methods. For linear problems this is a constant. Hence it is calculated and stored before entering into time stepwise computation. In case of nonlinear problems this is calculated at every time step and iteration.

In SIP method, these coefficients are calculated at every iteration thus avoiding storing of elements. The procedure is the same for both linear and nonlinear problems.

5.2. *Applicability to nonrectangular flow domains*

When a finite-difference grid is superimposed on a groundwater basin, normally this will result in computational nodes outside the region of interest being considered in the computation. To avoid such external computation, using SIP, Mohan Kumar *et al.*[3] have derived modified SIP equation as applicable to nonrectangular flow domains. In SIP,

while formulating the equations for $\bar{A}$ matrix, even though the nodes outside the region of interest are avoided, due to the nature of the algorithm (splitting of $\overline{A+B}$ into $\bar{L}$ and $\bar{U}$), some spurious coupling will take place near the inverted corners. To avoid this, MSIP has been derived by Mohan Kumar *et al.*[3]

In conjugate gradient methods, such a situation will not arise as only the nodes within the zone of interest are considered and no spurious coupling of nodes will take place at any of the nodes. It has been found, after careful examination, that conjugate gradient methods are directly applicable to nonrectangular, anisotropic and nonhomogeneous regions without any modifications in the original algorithm. This is a tremendous advantage compared to the SIP method where modifications in the SIP equations are needed and special equations have to be derived for various types of boundary nodes. This procedure has been derived by Mohan Kumar *et al.*[3] Further this has been improved upon by Sridharan *et al.*[10] resulting in improved strongly implicit procedure (ISIP).

In the SIP, the identification of these special boundary nodes is normally handled through a code designation which is a cumbersome procedure. Thus the SIP requires all these extra work to handle nonrectangular boundary, whereas the CG methods are directly applicable.

5.3. *Iteration parameters*

The CG method does not require a priori estimation of iteration parameters to achieve optimal convergence. The optimum parameters are calculated by the algorithm itself. But in the SIP method, the iteration parameters are needed for faster convergence and these have to be calculated.

6. Convergence criteria

There are various methods or norms to check the convergence of results in an iterative method. The three most commonly used norms for V are (Remson *et al.*[11])

$$\|V\|_1 = |V_1| + |V_2| + |V_3| + \ldots$$
$$\|V\|_2 = \sqrt{|V_1|^2 + |V_2|^2 + |V_3|^2 + \ldots} \tag{16}$$
$$\|V\|_3 = \text{Max}\,|V_i|$$

where $\| \|$ indicates the norm of correction. The use of average norm and maximum norm are quite common. Here the maximum norm of the nodal residue is used for convergence check in SIP as well as CG methods.

7. Extrapolation

The head values at all the nodes for the previous time step are known at any computational time step. If the same values of head are used as the starting solution for both SIP and CG iterations to compute the head values for the next time step, the convergence may be slow. To improve the convergence rate, a damped linear extrapolation is made

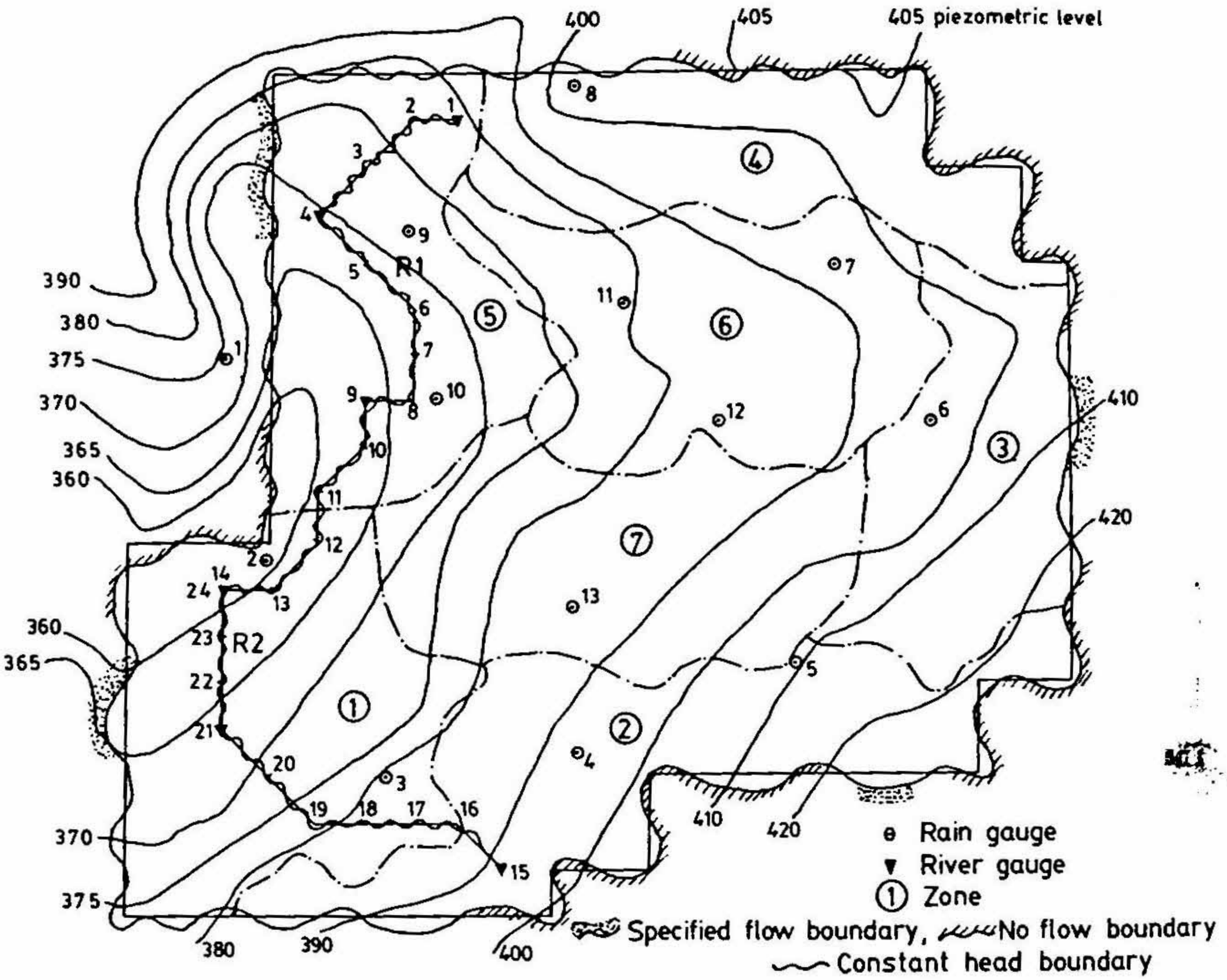

FIG. 3. Aquifer model for Test problems 1 and 3.

for the head values. These extrapolated heads obtained at any time step are used to compute the heads at the next time step. The extrapolation is done using the following equation

$$h_{ij}^{e} = h_{ij} + \frac{h_{ij} - ho_{ij}}{2} \tag{17}$$

where $h_{i,j}^{e}$ is the extrapolated head value at a node (i, j) at time $(t + \Delta t)$, $h_{i,j}$, the head at node (i, j) at time t and $ho_{i,j}$ is the head at node (i, j) at time $(t-\Delta t)$.

8. Test problems

The CG methods described in the previous section and SIP are applied to study groundwater flow in different types of aquifers. The SIP used here for comparison is the MSIP[3] developed specially for nonrectangular flow domains. All the test problems have nonrectangular computational regions with nonhomogenous properties. Both Dirichlet and

Neumann boundary conditions are considered in all problems. Due to the lack of exhaustive filed data, hypothetical problems have been chosen for the study. However, the input data for these problems have been chosen to depict different practical situations. Test problems 1 and 3 are chosen from Raghunath[15] and Test problem 2 is chosen from Sridharan *et al.*[6] The CG methods are built around the regional groundwater flow model already developed by Sridharan *et al.*[10] which can handle various types of aquifers and internal and external boundaries.

8.1. *Test problem* 1—*Unconfined linear model*

Test problem 1 depicts unconfined aquifer (linear). The study region with discretisation is shown in Fig. 3. The aquifer extends over an area of about 2100 km^2. The initial piezometric levels vary from 356 to 422 m in the basin. The aquifer is discretised into 7 hydrogeological zones and 328 nodes. The groundwater flow in this aquifer is simulated for a total period of 210 days. There are 2 rivers, 8 flux nodes and 30 Dirichlet boundary nodes in the study region. The average draft for the study region varies from 1500 to 3000 m^3/day/node. The unconfined linear model is most commonly used in practical situations.

8.2. *Test problem* 2—*Unconfined nonlinear model*

Test problem 2 considers flow in an unconfined aquifer (nonlinear). Transmissivity is taken to vary with variation in piezometric head and is written as product of hydraulic conductivity and saturated thickness at a node as obtained from the previous iteration. The study region consisting of 3 rivers (Fig. 4) has 5 zones and 172 global nodes. Dirichlet and Neumann boundary nodes are also considered. The hydraulic conductivity values as zonal parameters and bed rock levels at all 172 nodes are specified. Simulation is done for a period of one year, divided into rainy season of 5 months (150 days) and nonrainy season of 7 months (210 days). A time step of 15 days is used in the computations. Monthly rainfall data are provided. The annual average pumping rates vary from 172.5 to 275 m^3/day/node. At Dirichlet boundary node, details of seasonal water level are given. For the river nodes, water level data are given every two months.

8.3. *Test problem* 3—*Mixed confined–unconfined aquifer*

This test problem deals with a regional groundwater simulation in a mixed confined–unconfined aquifer. The study region discretisation is the same as in Fig. 3. The data are the same as given for Test problem 1. The zones 1, 2, 3, 4 and 5 are under unconfined conditions while zones 6 and 7 are under confined zones. Zones 6 and 7 are recharged from the surrounding zones through lateral flows.

9. Results and discussion

A comparative study of various CG methods and SIP is presented in this section. All the computational results for the test problems are obtained for different CG methods such as DSCG, CGHS, ICCG and for modified SIP (MSIP) method. These results are com-

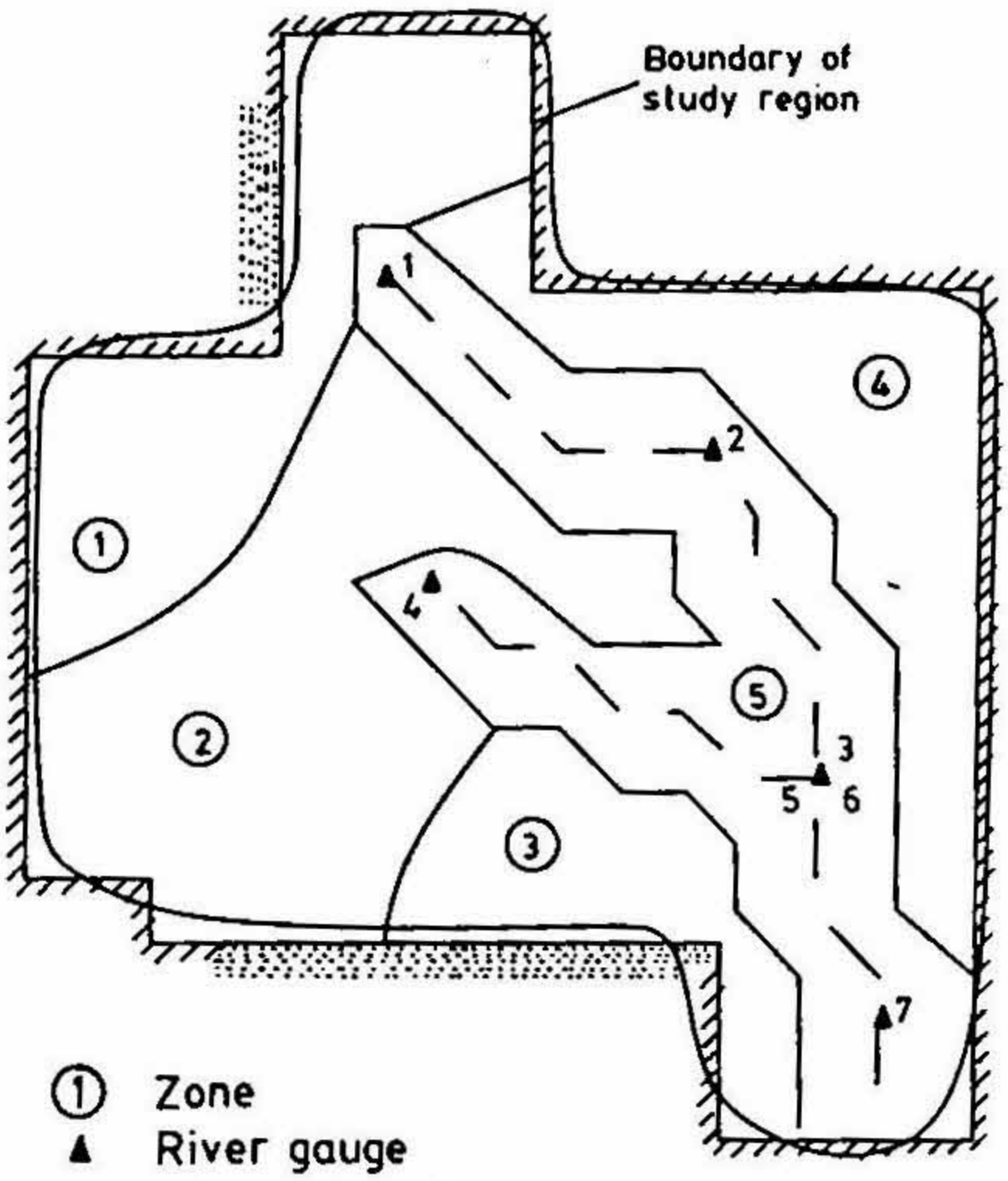

FIG. 4. Aquifer model for Test problem 2.

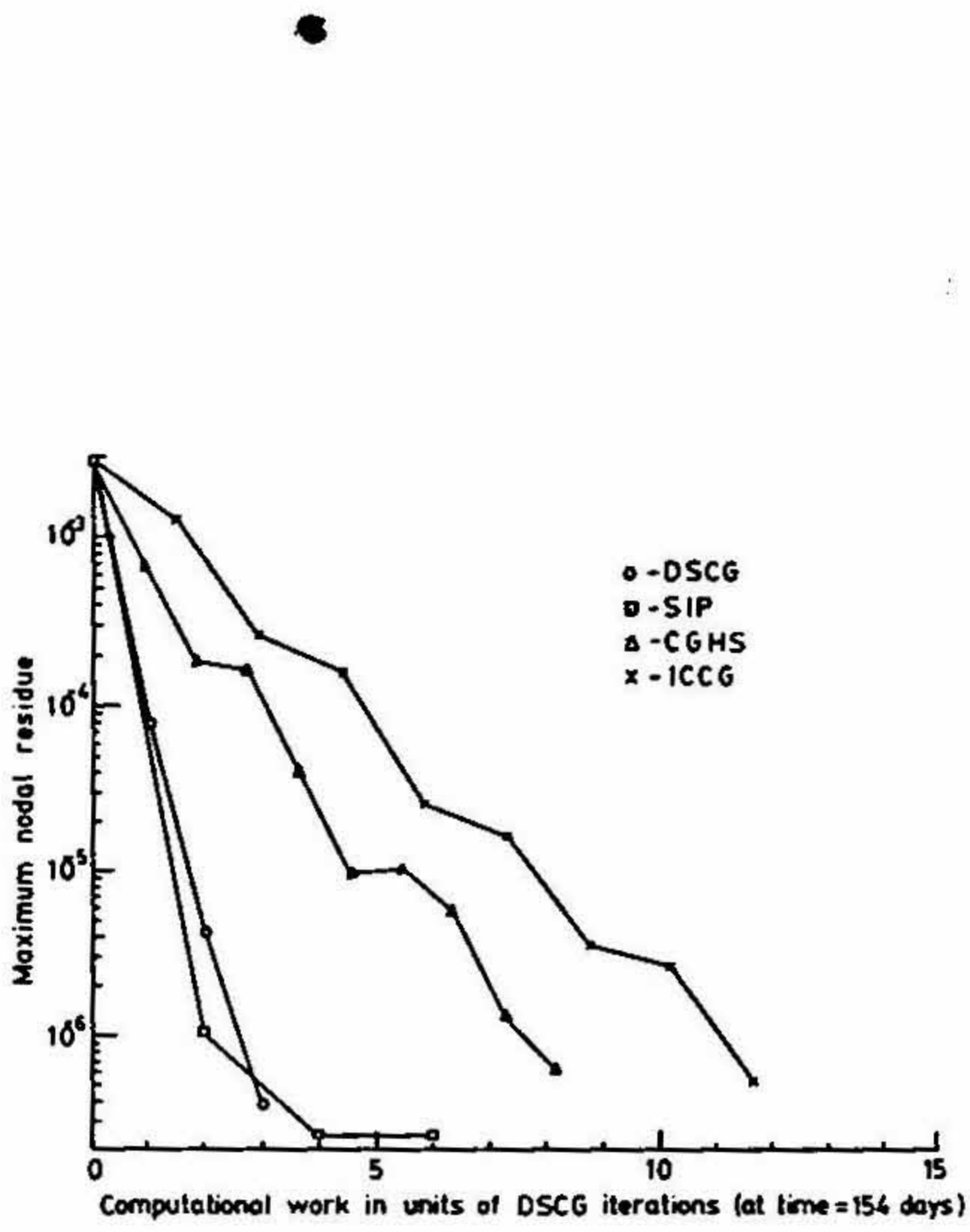

FIG. 5. Comparison of convergence rate–Test problem 1.

pared in terms of convergence rate, global residue, CPU time, zonal head variation and head value at selected nodes. For further discussion, MSIP method is referred to as SIP. Three types of criteria are chosen to compare the methods, namely, computational work, CPU time and accuracy. First two criteria are chosen to compare the efficiency of the methods and the last one is to check the correctness of the results.

Computational work involved per one iteration is chosen to check the convergence rate. To compare all the methods one common scale is taken. Diagonal scaling conjugate gradient (DSCG) method uses 11 N multiplications per iteration where N is the number of equations to be solved. This is chosen as the base to measure computational work (*i.e.*, 11 N multiplications equal to 1 computational work). Using this procedure, equivalent work is calculated for the remaining methods. Computational work *vs* maximum residue is plotted at selected time steps for all the test problems. The residue in the finite-difference equation is calculated at all the nodes of the computational domain, the maximum of which is taken as the maximum nodal residue.

For Test problem 1, Fig. 5 represents the plots of computational work *vs* maximum nodal residue for different methods. This plot is at time $t = 154$ days. From the plot it is clear that both DSCG and SIP are doing well for this test problem. ICCG and CGHS methods are somewhat slower than the other two methods.

It is seen from Fig. 6 that for nonlinear unconfined aquifer problem (Test problem 2) also both DSCG and SIP methods perform equally well when compared to other

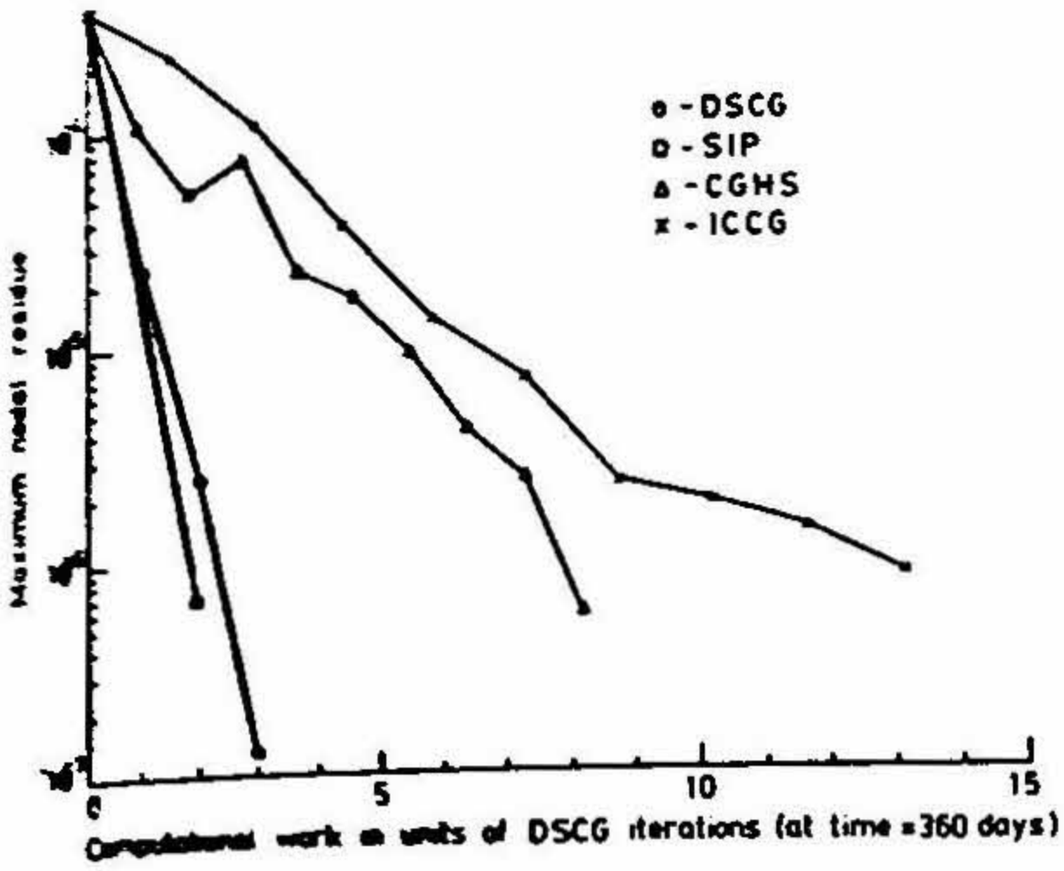

FIG. 6. Comparison of convergence rate–Test problem 2.

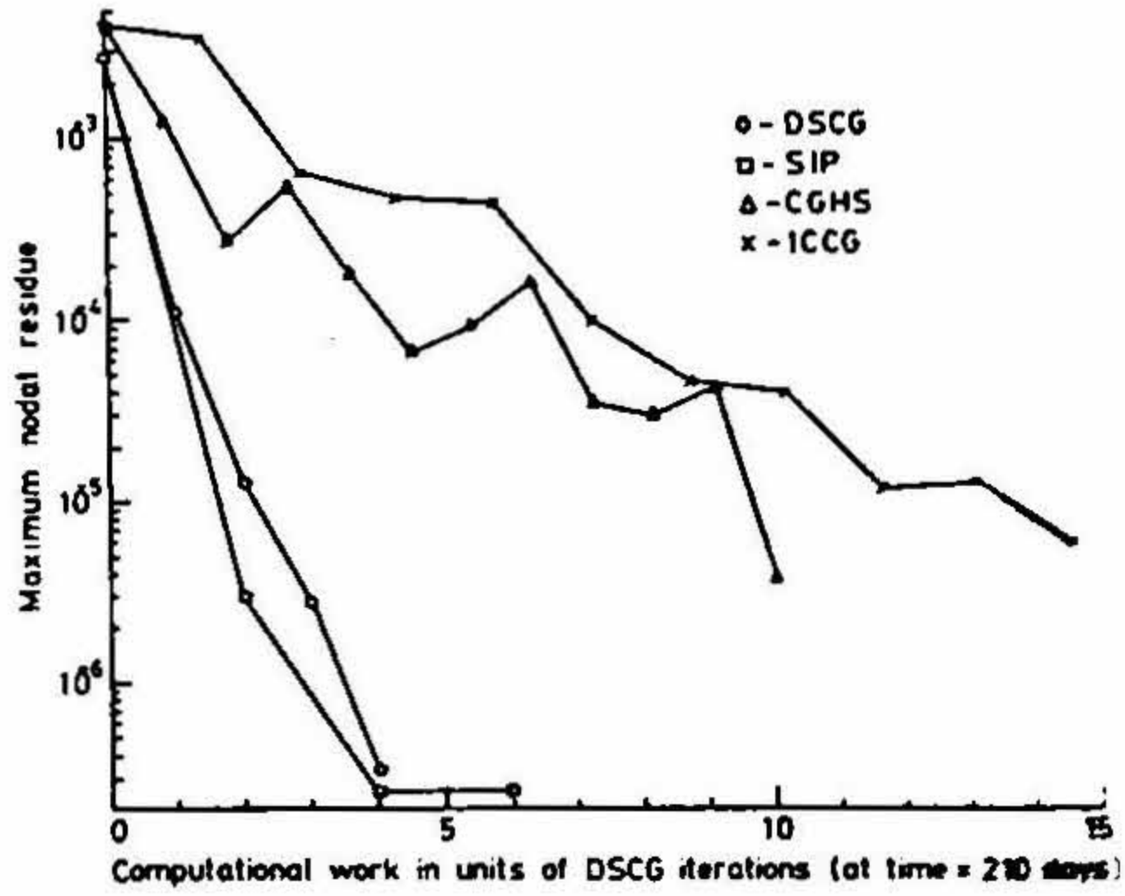

FIG. 7. Comparison of convergence rate–Test problem 3.

methods. In DSCG method the finite-difference coefficients are calculated only once for given time step. But in SIP these are calculated for every iteration. This work is not accounted for calculation of computational work. ICCG and CGHS converge slowly as seen in Fig. 6.

For Test problem 3, the computational work is the same for both DSCG and SIP (Fig. 7). However, the SIP convergence curve is at lower level. In this problem, ICCG and CGHS also converge very slowly.

From these plots it is clear that DSCG and SIP perform equally well for unconfined aquifer (linear), unconfined aquifer (nonlinear) and mixed confined-unconfined problems. Even though both the methods perform equally well, the implementation of DSCG for nonrectangular domains is much simpler.

9.2. *CPU time*

All the test problems were tested on VAX 8810 (VMS 5.1) and the CPU time taken for complete test problem is noted down. Three trial runs are made and the average of these is given in Table I. The CPU time is the time taken to complete the test problem, excluding the compilation time. The DSCG method takes the least time compared to other methods, and the ICCG method the highest, the latter because of decomposition of matrix and other matrix operations.

Table I
CPU time taken on VAX 8810 (VMS V 5.1)
(time in seconds)

	SIP	*DSCG*	*ICCG*	*CGHS*
Test problem 1	5.08	2.98	7.07	7.84
Test problem 2	2.41	1.56	9.01	2.75
Test problem 3	5.63	3.24	9.74	9.72

9.3. *Accuracy*

To check the accuracy of the various conjugate gradient methods, heads at selected nodes are compared for all the test problems.

Table II
Head values at selected nodes

Selected nodes	*SIP*	*DSCG*	*ICCG*	*CGHS*
Test problem 1 (at time = 210 days)				
12	0.3807E + 03	0.3807E + 03	0.3807E + 03	0.3807E + 03
43	0.3892E + 03	0.3892E + 03	0.3892E + 03	0.3892E + 03
100	0.3558E + 03	0.3558E + 03	0.3558E + 03	0.3558E + 03
111	0.4196E + 03	0.4196E + 03	0.4196E + 03	0.4196E + 03
113	0.4223E + 03	0.4223E + 03	0.4223E + 03	0.4223E + 03
119	0.3646E + 03	0.3646E + 03	0.3646E + 03	0.3646E + 03
176	0.3820E + 03	0.3820E + 03	0.3820E + 03	0.3820E + 03
184	0.3346E + 03	0.3346E + 03	0.3346E + 03	0.3346E + 03
245	0.4040E + 03	0.4040E + 03	0.4040E + 03	0.4040E + 03
277	0.3928E + 03	0.3928E + 03	0.3928E + 03	0.3928E + 03
281	0.4129E + 03	0.4129E + 03	0.4129E + 03	0.4129E + 03
286	0.3954E + 03	0.3954E + 03	0.3954E + 03	0.3954E + 03
301	0.3940E + 03	0.3940E + 03	0.3940E + 03	0.3940E + 03
328	0.4161E + 03	0.4161E + 03	0.4161E + 03	0.4161E + 03
Test problem 2 (at time = 360 days)				
6	0.1009E + 03	0.1009E + 03	0.1009E + 03	0.1009E + 03
35	0.1097E + 03	0.1097E + 03	0.1097E + 03	0.1097E + 03
45	0.1170E + 03	0.1170E + 03	0.1170E + 03	0.1170E + 03
87	0.1235E + 03	0.1235E + 03	0.1235E + 03	0.1235E + 03
90	0.1204E + 03	0.1204E + 03	0.1204E + 03	0.1204E + 03
96	0.1124E + 03	0.1124E + 03	0.1124E + 03	0.1124E + 03
133	0.1257E + 03	0.1257E + 03	0.1257E + 03	0.1257E + 03
144	0.1143E + 03	0.1143E + 03	0.1143E + 03	0.1143E + 03
167	0.1271E + 03	0.1271E + 03	0.1271E + 03	0.1271E + 03
Test problem 3 (at time = 210 days)				
12	0.3807E + 03	0.3807E + 03	0.3807E + 03	0.3807E + 03
43	0.3892E + 03	0.3892E + 03	0.3892E + 03	0.3892E + 03
100	0.3609E + 03	0.3609E + 03	0.3609E + 03	0.3609E + 03
111	0.4196E + 03	0.4196E + 03	0.4196E + 03	0.4196E + 03
113	0.4223E + 03	0.4223E + 03	0.4223E + 03	0.4223E + 03
119	0.3646E + 03	0.3646E + 03	0.3646E + 03	0.3646E + 03
176	0.3820E + 03	0.3820E + 03	0.3820E + 03	0.3820E + 03
184	0.3644E + 03	0.3644E + 03	0.3644E + 03	0.3644E + 03
245	0.4040E + 03	0.4040E + 03	0.4040E + 03	0.4040E + 03
277	0.3941E + 03	0.3941E + 03	0.3941E + 03	0.3941E + 03
281	0.4129E + 03	0.4129E + 03	0.4129E + 03	0.4129E + 03
286	0.3947E + 03	0.3947E + 03	0.3947E + 03	0.3947E + 03
301	0.3940E + 03	0.3940E + 03	0.3940E + 03	0.3940E + 03
328	0.4161E + 03	0.4161E + 03	0.4161E + 03	0.4161E + 03

All the methods, namely, SIP, DSCG, CGHS and ICCG give exactly the same results at all the time steps of simulation. This shows that accuracy is good equally with all the methods. Since it is difficult to present results at all the nodes, the head values at select nodes are presented at the end of the simulation period (Table II). The values of heads are practically the same at the selected nodes for all the methods tested.

10. Conclusions

A comparative study of various types of conjugate gradient methods with SIP for different types of regional groundwater flow problems with nonrectangular boundaries has been made and the following conclusions are drawn from the present study.

- CG methods handle nonrectangular boundary regions naturally and no special setting of equations is needed for some special boundary nodes. Hence the identification of such nodes as in SIP is not needed.
- The DSCG and SIP perform equally well for all the test problems.
- Among the CG methods tested, DSCG is the best.
- In terms of CPU time, DSCG scores over the other methods.
- DSCG method does not require any relaxation parameters as in the SIP.
- DSCG method is found to be easy to implement and is easily programmable.

Hence from the above-mentioned points, it can be concluded that the DSCG is a very good method for regional groundwater flow analysis, especially, in nonrectangular flow domains.

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