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Short Communication

Hypersurfaces of a Riemannian manifold with semi-symmetric non-metric connection

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Abstract

Extending the work of Agashe and Chafle on semi-symmetric non-metric connection on a Riemannian manifold, we study the properties of hypersurfaces of a Riemannian manifold with a semi-symmetric non-metric connection.

Keywords: Riemannian manifolds, hypersurfaces, semi-symmetric non-metric connection.

1. Introduction

Let M^n be an n-dimensional differentiable manifold immersed in an $(n+1)$-dimensional Riemannian manifold $(M^{n+1}\,\bar{g})$ with a differentiable immersion $i: M^n \to M^{n+1}$. We identify the image $i(M^n)$ with M^n and M^n is then called a hypersurface of M^{n+1}. The differential di of the immersion i will be in the sequel, denoted by B, so that, to a vector field X on M^n, corresponds a vector field BX on $i(M^N)$. Suppose that the metric tensor $\bar{g}$ of the manifold M^{n+1} induces a metric tensor g defined by $g(X, Y) = \bar{g}(BX, BY)$, X and Y being arbitrary vector fields in M^n. If the Riemannian manifold M^{n+1} and M^n are both orientable we can choose a unique vector field N defined along M^n such that $\bar{g}(BX, N) = 0$, $\bar{g}(N, N) = 1$, X being an arbitrary vector field in M^n. We call this vector field the unit normal vector field to the hypersurface M^n.

Let M^{n+1} be an $(n+1)$-dimensional differentiable manifold of Class C^∞ with a metric tensor $\bar{g}$. A linear connection $\bar{\nabla}$ is said to be a semi-symmetric non-metric connection[1], if its torsion tensor $\bar{T}$ satisfies

$$\bar{T}(\bar{X},\bar{Y}) = \bar{\pi}(\bar{Y})\bar{X} - \bar{\pi}(\bar{X})\bar{Y} \tag{1}$$

and

$$\left(\bar{\nabla}_{\bar{X}}\bar{g}\right)\left(\bar{Y},\bar{Z}\right) = -\bar{\pi}\left(\bar{Y}\right)\bar{g}\left(\bar{X},\bar{Z}\right) - \bar{\pi}\left(\bar{Z}\right)\bar{g}\left(\bar{X},\bar{Y}\right) \tag{2}$$

for all fields $\bar{X}$ and $\bar{Y}$ in M^{n+1}, where $\bar{\pi}$ is the 1-form associated with a non-zero vector field ρ by $\bar{g}\left(\bar{X},\rho\right) = \bar{\pi}\left(\bar{X}\right)$.

It is now assumed that the Riemannian manifold $(M^{n+1}, \bar{g})$ admits a semi-symmetric non-metric connection[1] given by

$$\overline{\nabla}_{\overline{X}}\overline{Y} = \nabla_{\overline{X}}\overline{Y} + \overline{\pi}(\overline{Y})\overline{X} \tag{3}$$

where ∇ denotes the Levi–Civita connection with respect to the Riemannian metric g.

In the present paper it is shown that if a Riemannian manifold M^{n+1} admits a semi-symmetric non-metric connection then its hypersurface M^n also admits a semi-symmetric nonmetric connection. Also it is shown that a hypersurface is totally geodesic with respect to the Riemannian connection $\mathring{\nabla}$, if and only if it is totally geodesic with respect to the semi-symmetric nonmetric connection. We have derived equations of Gauss and those of Codazzi for this hypersurface with respect to the semi-symmetric non-metric connection and prove a theorem.

2. Main results

Denoting by $\mathring{\nabla}$ the connection induced on the hypersurface from ∇ with respect to the unit normal N, we have[2]

$$\nabla_{BX} BY = B\left(\mathring{\nabla}_X Y\right) + h(X,Y)N \tag{4}$$

for arbitrary vector fields X and Y of M^n, where h is the second fundamental tensor of the hypersurface M^n.

Denoting by $\mathring{\overline{\nabla}}$ the connection induced on the hypersurface from $\overline{\nabla}$ with respect to the unit normal N, we have

$$\overline{\nabla}_{BX} BY = B\left(\mathring{\overline{\nabla}}_X Y\right) + m(X,Y)N \tag{5}$$

for arbitrary vector fields X and Y of M^n, where m is a tensor field of type (0, 2) of the hysuperface M^n.

From eqn (3), we obtain

$$\overline{\nabla}_{BX} BY = \nabla_{BX} BY + \overline{\pi}(BY)BX. \tag{6}$$

Now from eqns (4–6) we get

$$B\left(\mathring{\overline{\nabla}}_X Y\right) + m(X,Y)N = B\left(\mathring{\nabla}_X Y\right) + h(X,Y)N + \pi(Y)BX, \tag{7}$$

where $\overline{\pi}(BX) = \pi(X)$.

By taking tangent and normal components from both the sides of eqn (7) one gets

$$\mathring{\overline{\nabla}}_X Y = \mathring{\nabla}_X Y + \pi(Y)X \tag{7}$$

and

$$m(X,Y) = h(X,Y). \tag{9}$$

Let $X_1, X_2, \ldots, X_n$ be n orthonormal local vector fields in M^n. Then the function $\frac{1}{n}\sum_{i=1}^{n} h(X_i, X_i)$ is the mean curvature of M^n with respect to $\overset{*}{\nabla}$ and $\frac{1}{n}\sum_{i=1}^{n} m(X_i, X_i)$ is called the mean curvature of M^n with respect to $\overset{\circ}{\overline{\nabla}}$.

From eqns (8–9) we get the following theorems:

Theorem 1. The connection induced on a hypersurface of a Riemannian manifold with semi-symmetric nonmetric connection with respect to the unit normal is also a semi-symmetric non-metric connection provided the associated vector field is non-null on the sub-manifold. Also the mean curvature of M^n with respect to $\overset{*}{\nabla}$ coincides with that of M^n with respect to $\overset{\circ}{\overline{\nabla}}$.

Theorem 2. A hypersurface is totally geodesic with respect to the Riemannian connection $\overset{*}{\nabla}$ if and only if it is totally geodesic with respect to the semi-symmetric non-metric connection $\overset{\circ}{\overline{\nabla}}$.

It is known that the equation of Weingarten with respect to the Riemannian connection ∇ is

$$\nabla_{BX} N = -BHX \tag{10}$$

for every vector field X in M^n, where H is a tensor field of type (1–1) of M^n given by $g(HX,Y) = h(X,Y)$.

From eqn (3) we have

$$\overline{\nabla}_{BX} N = \nabla_{BX} N + \lambda BX, \tag{11}$$

where $\pi(N) = \lambda$.

Now from eqns (10–11) we get

$$\overline{\nabla}_{BX} N = -MBX \tag{12}$$

for any vector field X in M^n, where $M = H - \lambda I$, I being the unit matrix.

Denoting the curvature tensor of M^{n+1} with respect to the semi-symmetric non-metric connection $\overline{\nabla}$ by $\overline{R}$ one gets, using eqns (5) and (12),

$$\begin{aligned}\overline{R}(BX,BY)BZ &= B(R(X,Y)Z + m(X,Z)MY - m(Y,Z)MX) \\ &+\left\{\left(\overset{\circ}{\overline{\nabla}}_X m\right)(Y,Z) - \left(\overset{\circ}{\overline{\nabla}}_Y m\right)(X,Z) + m(\pi(Y)X - \pi(X)Y, Z)\right\}N,\end{aligned} \tag{13}$$

where R is the curvature tensor of the semi-symmetric non-metric connection $\overset{\circ}{\overline{\nabla}}$.

Then by taking tangent and normal components from both the sides of (13) one gets

$$'\overline{R}(BX,BY,BZ,BU)BX = 'R(X,Y,Z,U) + m(X,Z)m(Y,U) - m(Y,Z)m(X,U) \tag{14}$$

$$'\overline{R}(BX,BY,BZ,N) = \left(\overset{\circ}{\overline{\nabla}}_X m\right)(Y,Z) - \left(\overset{\circ}{\overline{\nabla}}_Y m\right)(X,Z) + m(\pi(Y)X - \pi(X)Y, Z), \tag{15}$$

where $'\overline{R}(BX,BY,BZ,BU) = \overline{g}(\overline{R}(BX,BY)BZ,BU)$ and $'R(X,Y,Z,U) = g(R(X,Y)Z,U)$.

Equations (14–15) are, respectively, the equations of Gauss and those of Codazzi with respect to the semi-symmetric non-metric connection.

Now if we put $\overline{R} = 0$ and $m = kg$ in eqn (14), we get

$$'R(X,Y,Z,U) = k^2[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)]. \tag{16}$$

From (16) we get the following:

Theorem 3. A totally umbilical hypersurface M^n of M^{n+1} with vanishing curvature tensor with respect to the semi-symmetric non-metric connection is of constant curvature.

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