J. Indian Inst. Sci., 66, July 1986, pp. 457–464

On points of *k*th absolute continuity

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Received on January 21, 1986; Revised on May 29, 1986.

Abstract

In this paper we define points of kth absolute continuity of a real function and study their properties including those induced by higher order (approximate) Riemann* derivatives. Also we investigate the structure of the class of all continuous functions having at least one point of kth absolute continuity on an interval [a, b].

Key words: AC_k functions, BV_k functions, AC_k points, (k) singular functions, (approximate) kth Riemann* derivative.

1. Introduction

Russell[1] introduced the concept of functions of bounded kth variation (BV_k functions) and, as a natural consequence, the definition of absolutely kth continuous functions (AC_k functions) was introduced by Das and Lahiri[2]. Extension of these notions on a linear set was made by De Sarkar and Das[3]. In the generalisations of many results of classical bounded variation and absolute continuity the concept of kth Riemann* derivative is seen to be essentially important. An approximate generalisation of this derivative is the outcome of the introduction of the definition of approximate kth Riemann* derivative by De Sarkar *et al*[4].

Chakrabarty and Bhakta[5] defined points of absolute continuity and studied their properties. Šalát[6] investigated the structure of the metric space $C(a, b)$ of all continuous functions on $[a, b]$ (with sup-metric) from the view point of points of absolute continuity and generalised some results of Chakrabarty and Bhakta[5] using the notions of approximate derivative, locally (strongly locally) Hölderian and locally (strongly locally) Lipschitzian functions (at a point).

The present authors introduced the definition of points of bounded kth variation in a recent communication[7]. In view of the developments of theories of bounded variation and related absolute continuity and the facts discussed above we propose, in this paper, to define points of kth absolute continuity of a real function and study some of their

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rather interesting properties along with those induced by higher order (approximate) Riemann* derivatives. Also we show that the class of all continuous functions having at least one point of kth absolute continuity on an interval [a, b] is an $F_{\sigma\delta\sigma}$-set of the first Baire category.

Let f be a real valued function defined on the real line R_1. Let a,b be two fixed real numbers with $a<b$ and let k be a positive integer greater than 1. The ordinary kth derivative of f at x, $x \varepsilon R_1$, denoted by $f^{(k)}(x)$: $f^{(0)}$, stands for f. Let $C(a, b)$ denote the class of all continuous functions on $[a,b]$.

Definition 1.1[1]: Let $x_0, x_1, \ldots, x_k$ be $(k+1)$ distinct points, not necessarily in the linear order, belonging to R_1. Then define the kth divided difference of f as

$$Q_k(f;x_0,x_1,\ldots,x_k) = \sum_{i=0}^{k} \{ f(x_1)/ \prod_{\substack{j=0\\ j\neq i}}^{k} (x_i - x_j) \}.$$

For the definitions of functions of bounded kth variation (BV_k functions) and kth absolute continuity (AC_k functions) we refer to De Sarkar and Das[3].

Definition 1.2[4]: Let x be any point in $[a,b]$ and let $x_1, x_2, \ldots, x_k$ be any set of k points in $[a,b]$ with the property $0 < |x_1 - x| < |x_2 - x| < \ldots < |x_k - x|$. If the iterated approximate limit

$$\lim_{x_k \to x} \text{ap} \ldots \lim_{x_1 \to x} \text{ap}\, k!\ Q_k(f;x,x_1,\ldots,x_k)$$

exists (possibly infinite), then this limit is called the approximate kth Riemann* derivative of f at x and is denoted by $AD^kf(x)$. Replacing the approximate limits by the ordinary limits in this definition we get the definition of kth Riemann* derivative of f at x, as in Russell[1], and we denote it by $D^kf(x)$.

Definition 1.3[8]: A function S which is BV_k on $[a, b]$ is said to be a (k) singular function if S is not a polynomial of degree less than k but $S^{(k)}$ vanishes almost everywhere on $[a,b]$.

We denote by $P^*(f)$, $P(f)$, $L^*(f)$ and $L(f)$, respectively, the set of all such points $p \in (a,b)$ at which f is strongly locally Hölderian, locally Hölderian, strongly locally Lipschitzian and locally Lipschitzian. (For definitions of such functions we refer the reader to Šalát[6], Beesley *et al*[9] and Belas and Šalát[10].)

2. AC_k points

Before going into the definition of AC_k points we prove a theorem from which useful inferences can be drawn.

Theorem 2.1: If $AD^{k-1}f$ is absolutely continuous on $[a,b]$, then f is AC_k on $[a,b]$.

Proof: Consider an elementary system

$I(x_{i,1}, \ldots, x_{i,k-1}) : (x_{i,0}, x_{i,k})$; $i=1,2, \ldots, n$ in $[a,b]$. Then in view of Lemma 4 of Russell[1], we have

$$\sigma|I| = \sum_{i=1}^{n} |Q_{k-1}(f;x_{i,1},\ldots, x_{i,k}) - Q_{k-1}(f;x_{i,0},\ldots, x_{i,k-1})|.$$

Now by Theorem 8 of De Sarkar *et al*[4], $AD^{k-1}f$ possesses the mean value property on $[a,b]$, namely, for any set of k points $x_1 < x_2 < \ldots < x_k$ in $[a,b]$ there exists at least one η in (x_i, x_k) such that

$$(k-1)!\ Q_{k-1}(f;x_1,x_2, \ldots, x_k) = AD^{k-1}f\ (\eta).$$

Hence it is easy to see that

$$\sigma\ |I| = \frac{1}{(k-1)!} \sum_{i=1}^{n} |AD^{k-1}f(\beta_i) - AD^{k-1}f(\alpha_i)|$$

where (α_i, β_i), $i=1,2 \ldots, n$ is a sequence of non-overlapping intervals in $[a,b]$. The rest of the proof is straightforward.

Definition 2.1: Let x be a point in $[a,b]$. We say that x is a point of kth absolute continuity of f if there exists a closed neighbourhood of x on which f is AC_k. On the other hand, if there exists no closed neighbourhood of x on which f is AC_k, then x is said to be a point of kth nonabsolute continuity of f.

For $k=1$, Definition 2.1 gives the definition of AC points as in Chakrabarty and Bhakta[5].

A point of kth absolute continuity and a point of kth nonabsolute continuity of f will be called an AC_k point and a NAC_k point of f respectively. We shall denote by $G(f)$ and $N(f)$, respectively, the set of all AC_k points and NAC_k points of f.

It is easy to see that the set $G(f)$ is open in $[a,b]$ and hence $N(f)$ is a closed subset of $[a,b]$.

In view of Corollary 2.4 of De Sarkar and Das[3], the preceding Theorem 2.1 and the fact that the existence of $D^k f$ implies that of $AD^k f$ we have the following theorem, the proof of which is omitted.

Theorem 2.2: The point $x \in [a,b]$ is an AC_k point of f if and only if x is an AC_r point of $AD^{k-r}f$, $r=1,2 \ldots, k-1$.

Theorem 2.3: If the kth divided difference of f in $[a,b]$ are bounded then each $x \in [a,b]$ is an AC_r point of $AD^{k-r}f$, $r=1,2 \ldots, k$.

The proof is omitted.

Theorem 2.4: If $D^{k-1}f$ is continuous and f is BV_k on $[a,b]$, then $N(f)$ is either void or a perfect set.

Proof: The set $N(f)$ is void if f is AC_k on $[a,b]$. Let f be not AC_k on $[a,b]$. Then, by Theorem 3.1 of Das and Lahiri[8], f can be uniquely expressed in the form $f = \phi + S$ where ϕ is AC_k on $[a, b]$, $\phi^{(r)}(a) = f^{(r)}(a)$, $r = 0, 1, \ldots, k-1$ and S is a (k) singular function or an identically zero function. Then clearly, an AC_k point of f is an AC_k point of S and conversely. Let $G(f) = \cup\{(\alpha_i, \beta_i)\}$; $i = 1, 2, \ldots$. If $[\alpha, \beta]$ is any closed subinterval of (α_i, β_i), then S is AC_k on $[\alpha, \beta]$. Since $S^{(k)}$ vanishes almost everywhere on $[a,b]$, by Theorem 2 of Das and Lahiri[2], S is a polynomial of degree $(k-1)$ at most. Thus it follows that S is a polynomial of degree $(k-1)$ at most on $[\alpha_i, \beta_i]$. Hence, by Lemma 1 of Russell[1], $S^{(k)}(x) = 0$ for all $x \in G(f)$. The set $N(f)$ is not enumerable, for in that case S becomes a polynomial of degree $(k-1)$ at most in $[a,b]$ which contradicts the fact that S is a (k) singular function. Let $E(f)$ be the set of condensation points of $N(f)$. Then the set $N(f) - E(f) = D(f)$ is at most enumerable. We have $[a,b] = A(f) \cup E(f)$ where $A(f) = D(f) \cup G(f)$ and $E(f) \cap A(f) = 0$ (null set). Since $E(f)$ is closed, the set $A(f)$ is open and contains at most an enumerable set of points of $N(f)$. We take $A(f) = \cup \{(a_i, b_i)\} : i = 1, 2, \ldots$, where the intervals (a_i, b_i) are non-overlapping. By Theorem 2.3 of De Sarkar and Das[3], $D^{k-1}\phi$ is continuous on (a,b) and hence $D^{k-1}S$ is continuous on (a,b). Also $D^k S$ vanishes on $[a_i, b_i]$ except on enumerable set of points. Therefore, by Hobson[11] (p. 365), $D^{k-1}S$ is constant on $[a_i, b_i]$. So each point of $[a_i, b_i]$ is an AC point of $D^{k-1}S$ and hence by Theorem 2.1, is an AC_k point of S. Therefore, each point of $[a_i, b_i]$ is an AC_k point of f.

Thus $A(f) \subset G(f)$ and so $N(f) = E(f)$. Hence by Theorem 3 of Natanson[12] (p. 53), $N(f)$ is a perfect set. This proves the theorem.

Theorem 2.5: If $D^k f$ exists on (a,b), then $N(f)$ is a nowhere dense set in $[a,b]$.

Proof: Let C and D denote, respectively, the set of all continuity points and discontinuity points of $D^k f$. Denjoy[13] proved that if a Peano derivative of some order exists finitely so does the Riemann* derivative of the same order and *vice versa*. Later Oliver[14] showed that the Peano derivative belongs to the first Baire class. Hence $D^k f$ is a function in the first Baire class. Thus the set D is a set of first Baire category in $[a,b]$ (*cf.* Mukhopadhyay[15], p. 182).

Let x be any point in C. Then $D^k f$ is bounded in a certain neighbourhood of x. Hence by Theorem 3 of Oliver[14], $f^{(k)}$ is bounded in some neighbourhood of x. From this we easily see that $f^{(k-1)}$ satisfies Lipschitzian condition in an interval containing x. Therefore $f^{(k-1)}$ is absolutely continuous and hence (by Theorem 2.1) f is AC_k on a neighbourhood of x and so $x \in G(f)$. Thus $C \subset G(f)$ and so $N(f) \subset D \cup \{a\} \cup \{b\}$.

Since D is a set of first category it thus follows that $N(f)$ is also a set of first category. Since $N(f)$ is closed in $[a,b]$ it must be nowhere dense. This completes the proof.

Since the existence of the kth Riemann* derivative implies that of approximate kth Riemann* derivative we produce a generalisation of the above theorem as follows:

Theorem 2.6: If $AD^k f$ exists on (a,b), then $N(f)$ is a nowhere dense set in $[a,b]$.

Proof: Let C and D denote, respectively, the set of all continuity points and discontinuity points of $AD^k f$. Since, by Theorem 2.6 of De Sarkar *et al*[4], $AD^k f$ is a function in the first

Baire class, the set D is a set of first Baire category in $[a,b]$. Let $x \in C$. Then there exists an interval $I=[x-\delta, x+\delta] \subset [a,b]$ such that $AD^k f$ is bounded on I. Therefore there exists a $K>0$ such that $|AD^k f(\eta)| \leq K$ for all $\eta \in I$.

Let α, β be any two points in I. We consider the set of $2k$ points $\alpha < \alpha_1 < \alpha_2 < \ldots < \alpha_{k-1} < \beta_{k-1} < \ldots < \beta_1 < \beta$ in I. We relabel the set of points as

$$z_0 < z_1 < z_2 < \ldots < z_{k-1} < z_k < z_{k+1} < \ldots < z_{2k-2} < z_{2k-1}$$

where $z_0 = \alpha$, $z_{2k-1} = \beta$, $z_i = \alpha_i$; $i = 1,2,\ldots, k-1$ and $z_i = \beta_{2k-1-i}$, $i = k,k+1, \ldots, 2k-2$. Then we have, using Lemma 4 of Russell[1],

$$\begin{aligned} &|Q_{k-1}(f;z_0,z_1, \ldots, z_{k-1}) - Q_{k-1}(f;z_k, \ldots, z_{2k-1})| \\ &\leq \sum_{i=0}^{k-1} |Q_{k-1}(f;z_i,\ldots, z_{i+k-1}) - Q_{k-1}(f;z_{i+1}, \ldots, z_{i+k})| \\ &= \sum_{i=0}^{k-1} |(z_i - z_{i+k})| \, |Q_k(f;z_i, \ldots, z_{i+k})|. \end{aligned}$$

Hence, by Theorem 8 (Mean value theorem) of De Sarkar *et al*[4], we get

$$\begin{aligned} &|Q_{k-1}(f;\alpha,\alpha_1, \ldots, \alpha_{k-1}) - Q_{k-1}(f;\beta_{k-1},\ldots, \beta_1,\beta)| \\ &\leq \sum_{i=0}^{k-1} |(z_i - z_{i+k})| \, |AD^k f(\eta_i)|/k!, \quad \eta_i \in (z_i, z_{i+k}), \\ &\leq K|\alpha-\beta|/(k-1)! \end{aligned}$$

Since the existence of $AD^k f$ implies that of $AD^r f$, $1 \leq r < k$, it is now easy to see that

$$|AD^{k-1}f(\alpha) - AD^{k-1}f(\beta)| \leq K\,|\alpha-\beta|/(k-1)!$$

Hence $AD^{k-1}f$ is Lipschitzian on I, and so, by Theorem 2.1, $C \subset G(f)$. For the rest of the proof we proceed similarly as in the proof of Theorem 2.5.

From Theorem 2.6 we can easily deduce the following result.

Theorem 2.7: If $AD^k f$ exists on (a,b), then the set $[a,b] - L^*(AD^{k-1}f)$ is a nowhere dense set in $[a,b]$.

Since $P^*(f) \subset P(f)$, $L(f) \subset P(f)$ and $L^*(f) \subset P^*(f)$ we have

Corollary 2.1: If $AD^k f$ exists on (a,b), then each of the sets $[a,b] - L(AD^{k-1}f)$, $[a,b] - P^*(AD^{k-1}f)$, $[a,b] - P(AD^{k-1}f)$ is a nowhere dense set in $[a,b]$.

We set

$$A(a,b) = \{ f \in C(a,b) / G(f) \text{ is nonvoid}\}$$

and

$$B(a,b) = \{ f \in C(a,b) / N(f) = [a,b]\}$$

Then we have $B(a,b) = C(a,b) - A(a,b)$.
We now prove the following theorem.

Theorem 2.8: The class $A(a,b) \subset C(a,b)$ is an $F_{\sigma\delta\sigma}$-set of the first Baire category in $C(a,b)$.

Proof: Let R^0 be the set of all rational numbers of the interval (a,b). Let $q \in R^0$, $\delta > 0$, $\varepsilon > 0$, $\eta > 0$. We denote by $A(q,\delta,\varepsilon,\eta)$ the class of all such functions $f \in C(a,b)$ for which the following holds:

If $I(x_{i,1}, \ldots, x_{i,k-1}) : (x_{i,0}, x_{i,k})$; $i = 1,2 \ldots, n$ is an elementary system in $[q-\delta, q+\delta]$, $mI = \sum_{i=1}^{n} (x_{i,k} - x_{i,0}) \leqslant \eta$, then $\sigma|I| \leqslant \varepsilon$.

We shall show that $A(q,\delta,\varepsilon,\eta)$ is a closed subset of $C(a,b)$.

Let $f_m \in A(q, \delta, \varepsilon, \eta)$; $m = 1, 2, \ldots$, let the sequence $\{f_m\}$ converge to a function f in $C(a,b)$. We shall show that $f \in A(q,\delta,\varepsilon,\eta)$.

Let $I(x_{i,1}, \ldots, x_{i,k-1}) : (x_{i,0}, x_{i,k})$; $i = 1,2, \ldots, n$ be an elementary system in $[q-\delta, q+\delta]$ with $mI \leqslant \eta$. Since f_m and f are uniformly continuous in $[a,b]$, by a simple manipulation, it can be shown that for each set of k points $x_0, x_1, \ldots, x_{k-1}$ in $[a,b]$ and an arbitrarily chosen $v > 0$, we have for m sufficiently large

$$| Q_{k-1}(f_m; x_0, x_1, \ldots, x_{k-1}) - Q_{k-1}(f; x_0, x_1, \ldots, x_{k-1}) | < \varepsilon/2vn. \tag{1}$$

Now, using Lemma 4 of Russell[1],

$$\begin{aligned}
\sigma|I| &= \sum_{i=1}^{n} | Q_{k-1}(f; x_{i,1}, \ldots, x_{i,k}) - Q_{k-1}(f; x_{i,0}, \ldots, x_{i,k-1}) | \\
&\leqslant \sum_{i=1}^{n} | Q_{k-1}(f; x_{i,1}, \ldots, x_{i,k}) - Q_{k-1}(f_m; x_{i,1}, \ldots, x_{i,k}) | \\
&+ \sum_{i=1}^{n} | Q_{k-1}(f_m; x_{i,1}, \ldots, x_{i,k}) - Q_{k-1}(f_m; x_{i,0}, \ldots, x_{i,k-1}) | \\
&+ \sum_{i=1}^{n} | Q_{k-1}(f_m; x_{i,0}, \ldots, x_{i,k-1}) - Q_{k-1}(f; x_{i,0}, \ldots, x_{i,k-1}) |
\end{aligned}$$

Hence from (1) and (2) and the fact that $f_m \in A(q,\delta,\varepsilon,\eta)$

$$\sigma\,|I| \leqslant \varepsilon\,(1 + 1/v).$$

Since $v > 0$ is arbitrary, making $v \to \infty$ we get $\sigma\,|I| \leqslant \varepsilon$. Hence $f \in A(q,\delta,\varepsilon,\eta)$ and so the set $A(q,\delta,\varepsilon,\eta)$ is closed. Also we see that

$$A(q,1/\alpha) = \bigcap_{\beta=1}^{\infty} \bigcup_{\gamma=1}^{\infty} A(q,\ 1/\alpha, 1/\beta, 1/\gamma)$$

and so $A(q,\ 1/\alpha)$ is an $F_{\sigma\delta}$ - set in $C(a,b)$. Again

$$A(q) = \bigcup_{\alpha=1}^{\infty} A(q, 1/\alpha)$$

and is therefore an $F_{\sigma\delta\sigma}$ – set in $C(a,b)$. Since we have

$$A(a,b) = \bigcup_{q \in R^\circ} A(q)$$

it follows, in view of the countability of R°, that the set $A(a,b)$ is an $F_{\sigma\delta\sigma}$ – set in $C(a,b)$.

Further, if $f \in A(a,b)$, then by Corollary 2.4 of De Sarkar and Das[3] and Theorem 2.2, $AD^{k-1}f$ is absolutely continuous on a certain interval $I \subset (a,b)$ and $AD^k f$ exists almost everywhere on I. Hence in view of Theorem 2.1, it follows that $A(a,b)$ is a subset H of all such functions from $C(a,b)$ which have at least at one point of (a,b) a finite derivative. Since H is a set of first Baire category (by Hewitt and Stromberg[16], p. 260) the theorem follows.

Corollary 2.2: The set $B(a,b) \subset C(a,b)$ is a $G_{\delta\sigma\delta}$ – set residual in $C(a,b)$.

Acknowledgements

The authors wish to express their heartiest gratitude to Dr. A. G. Das for his kind help in the preparation of the paper. Also the authors are grateful to the referee for valuable criticism and helpful suggestions.

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