J. Indian Inst. Sci. 62 (B), Feb. 1980, Pp. 17–23

On Lauricella's n-variable function $F_A^{(n)}$

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Received on September 10, 1979.

Abstract

In the present paper we have established four summation formulae including one s-tuple finite series and one s-tuple infinite series. In particular cases we have obtained various interesting formulae involving Jacobi polynomials.

Key words: Lauricella's n-variable function $F_A^{(n)}$, Pochhammer symbol, generalised Laguerre polynomial, Jacobi polynomial, Vandermonde's theorem.

1. Introduction

The formula

$$F_A^{(n)}[a,(b);\ (c);\ (z)]$$

$$= \sum_{m_1, m_2, \ldots, m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n} \prod_{j=1}^{n} (b_j)_{m_j} \prod_{j=1}^{n} (x_j)^{m_j}}{\prod_{j=1}^{n} (m_j)! \prod_{j=1}^{n} (c_j)_{m_j}} \tag{1}$$

where $\sum_{j=1}^{n} |x_j| < 1$, with usual meaning of Pochhammer symbol $(a)_n$ as

$$(a)_n = \Gamma(a+n)/\Gamma(a),\ n \in [0, 1, 2, \ldots] \tag{2}$$

is the definition of the n-variable generalised hypergeometric function $F_A^{(n)}$ given by Lauricella[6] [p. 135] whose integral representation

$$F_A^{(n)}(a,(b);\ (c);\ (z))$$

$$= \frac{1}{\Gamma(a)} \int_0^{\infty} e^{-t}\, t^{a-1} \prod_{j=1}^{n} {}_1F_1(b_j;\ c_j;\ z_j t)\, dt \tag{3}$$

valid for $\mathrm{Re}(a) > 0$ and each z sufficiently small, was obtained by Erdelyi[3] [p. 694] in the year 1936.

The formula

$$L_n^{(c)}(xy) = \sum_{k=0}^{n} \frac{(1+c)_n \, y^k}{(n-k)! \, (1+c)_k} \, {}_2F_1(-n+k, b+k+1; c+k+1; y) \, L_k^{(b)}(x) \tag{4}$$

involving the generalised Laguerre polynomial

$$L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!} \, {}_1F_1(-n; 1+\alpha; x) \tag{5}$$

obtained by Halim, A. and Al-Salam[5] [p. 58] along with the following results of Bailey[1] [p. 36] and Erdelyi[3] [pp. 17 and 19] are required in our investigations:

$$\begin{aligned} &{}_1F_1(a; b; xy) \\ &= \sum_{k=0}^{\infty} \frac{(a)_k (1+b)_k (-y)^k}{k!} \, {}_2F_1(a+k, c+k+1; b+k+1; y) \, {}_1F_1(-k; 1+c; x); \end{aligned} \tag{6}$$

$$(1-z)^{-\alpha-1} \exp\left(\frac{xz}{z-1}\right) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x) \, z^n; \tag{7}$$

$$e^{-xz}(1+z)^{\alpha} = \sum_{n=0}^{\infty} L_n^{(\alpha-n)}(x) \, z^n. \tag{8}$$

2. Results

The main results to be established are:

$$\begin{aligned} &\sum_{k_1=0}^{n_1} \cdots \sum_{k_s=0}^{n_s} \frac{\prod_{j=1}^{s} (-n_j)_{k_j} \prod_{j=1}^{s} (1+b_j)_{k_j}}{\prod_{j=1}^{s} k_j! \prod_{j=1}^{s} (1+a_j)_{k_j}} \prod_{j=1}^{s} (-x_j)^{k_j} \\ &F_A[a, -k_1, \ldots, -k_s; 1+b_1, \ldots, 1+b_s; 1, 1, \ldots, 1] \\ &\prod_{j=1}^{s} {}_2F_1[-n_j+k_j, b_j+k_j+1; a_j+k_j+1; x_j] \\ &= F_A[a, -n_1, \ldots, -n_s; 1+a_1, \ldots, 1+a_s; x_1, \ldots, x_s] \end{aligned} \tag{9}$$

provided $|x_1| + |x_2| + \ldots + |x_s| < 1$.

$$\sum_{k_1=0}^{\infty} \cdots \sum_{k_s=0}^{\infty} \frac{\prod_{j=1}^{s} (b_j)_{k_j} \prod_{j=1}^{s} (1+c_j)_{k_j}}{\prod_{j=1}^{s} k_j!} \prod_{j=1}^{s} (-x_j)^{k_j}$$

$$\cdot F_A[a; -k_1, \ldots, -k_s; 1+d_1, \ldots, 1+d_s; 1, 1, \ldots, 1]$$

$$\cdot \prod_{j=1}^{s} {}_2F_1[b_j + k_j; d_j + k_j + 1; c_j + k_j + 1; x_j]$$

$$= F_A[a; b_1, \ldots, b_s; c_1, \ldots, c_s; x_1, \ldots, x_s] \tag{10}$$

provided the series involved is uniformly convergent and

$$|x_1| + |x_2| + \ldots + |x_s| < 1$$

$$\sum_{n=0}^{\infty} \frac{(-a)_n}{n!} F_A^{(m)}[a, -n, b_2, \ldots, b_m; 1+a-n, c_2, \ldots, c_m; x_1, \ldots, x_m](-z)^n$$

$$= (1+z)^a (1+x_1 z)^{-a} F_A^{(m-1)}\Big[a; b_2, \ldots, b_m; c_2, \ldots, c_m;$$

$$\frac{x_2}{1+x_1 z}, \ldots, \frac{x_m}{1+x_1 z}\Big]. \tag{11}$$

provided the series involved is uniformly convergent and

$$\left|\frac{x_2}{1+x_1 z}\right| + \ldots + \left|\frac{x_m}{1+x_1 z}\right| < 1$$

$$\sum_{n=0}^{\infty} \frac{(1+a)_n}{n!} F_A^{(m)}[a; -n, b_2, \ldots, b_m; 1+a, c_1, \ldots, c_{m-1}; x, x_1, \ldots, x_{m-1}](z)^n$$

$$= (1-z)^{-a-1}\left(1 - \frac{xz}{(z-1)}\right)^{-a} \cdot F_A^{(m-1)}\Big[a; b_1, \ldots, b_{m-1}; c_1, \ldots, c_{m-1};$$

$$\frac{x_1(1-z)}{1-z+xz}, \ldots, \frac{x_{m-1}(1-z)}{1-z+xz}\Big] \tag{12}$$

provided the series involved is uniformly convergent and

$$\left|\frac{x_1(1-z)}{1-z+xz}\right| + \ldots + \left|\frac{x_{m-1}(1-z)}{1-z+xz}\right| < 1.$$

(A) *Proof of the result* (9) *and* (10)

In order to establish (9), let us first prove an elementary result,

$$\sum_{k=0}^{n} \frac{(-n)_k (1+b)_k}{k!\,(1+d)_k} (-x_1)^k\, {}_2F_1(-n+k, b+k+1;\ d+k+1;\ x_1)$$

$$\cdot\, F_A^{(m)}[a, -k, b_2, \ldots, b_m;\ 1+b,\ c_2, \ldots, c_m;\ 1, x_2, \ldots, x_m]$$

$$= F_A^{(m)}[a, -n, b_2, \ldots, b_m;\ 1+d,\ c_2, \ldots, c_m;\ x_1, \ldots, x_m]. \tag{13}$$

Proof of (13) : Making use of (3) and (5), we have,

$$F_A^{(m)}[a, -n, b_2, \ldots, b_m;\ 1+d, c_2, \ldots, c_m;\ x_1, \ldots, x_m]$$

$$= \frac{n!}{(1+d)_n} \frac{1}{\Gamma(a)} \int_0^\infty e^{-t}\, t^{a-1} L_n^{(d)}(x_1 t) \prod_{j=2}^{m} {}_1F_1(b_j;\ c_j;\ x_j t)\, dt \tag{14}$$

which on using (4) and then integrating term by term yields,

$$= \frac{1}{\Gamma(a)} \sum_{k=0}^{n} \frac{(-n)_k (1+b)_k}{k!\,(1+d)_k} (-x_1)^k\, {}_2F_1(-n+k, b+k+1;\ d+k+1;\ x_1)$$

$$\cdot \int_0^\infty e^{-t}\, t^{a-1}\, {}_1F_1(-k;\ 1+b;\ t) \prod_{j=2}^{m} {}_1F_1(b_j;\ c_j;\ x_j t)\, dt \tag{15}$$

interpreting it as $F_A^{(n)}$ with the help of (3), we get the left hand side of (13). It completes the proof of (13).

Similarly, to prove the result (10), we need the result,

$$\sum_{k=0}^{\infty} \frac{(b_1)_k (1+c_1)_k}{k!} (-x_1)^k\, {}_2F_1(b_1+k, c+k+1;\ c_1+k+1;\ x_1)$$

$$\cdot\, F_A^{(n)}[a, -k, b_2, \ldots, b_n;\ 1+c,\ c_2, \ldots, c_n;\ 1, x_2, \ldots, x_n]$$

$$= F_A^{(n)}[a;\ b_1, b_2, \ldots, b_n;\ c_1, c_2, \ldots, c_n;\ x_1, x_2, \ldots, x_n] \tag{16}$$

provided the series involved is uniformly convergent and

$$|x_1| + |x_2| + \ldots + |x_n| < 1$$

which can be shown to exist by proceeding very parallel to the lines of proof of (13), except that we use the formula (6), instead of (4), and since the series involved are uniformly convergent under the conditions stated with it, the term by term integration is justified. Now iteration of process (13) and (14) then yields the results (9), and (10) respectively.

(B) *Proof of (11) and (12)*

Proceeding parallel to the proof of (13) and then using (8) we easily have left hand side of (11) equal to,

$$\frac{(1+z)^a}{\Gamma(a)} \int_0^\infty e^{-(1+x_1 z)t}\, t^{a-1} \prod_{j=1}^{n-1} {}_1F_1(b_j;\, c_j;\, x_j t)\, dt \tag{17}$$

The substitution $(1 + x_1 z)\, t = u$ then leads it immediately to the right hand side of (11). The result (12) can similarly be proved on using the result (7) instead of (8).

3. Particular cases

(i) On setting $n = 2$ in (13), we get

$$\sum_{k=0}^{n} \frac{(-n)_k (1+b)_k}{k!(1+d)_k} (-x)^k\, {}_2F_1(-n+k,\, b+k+1;\, d+k+1;\, x)$$

$$F_2(a, -k, c\,;\, 1+b,\, e\,;\, 1, y)$$

$$= F_2(a, -n, c\,;\, 1+b,\, e\,;\, x, y). \tag{18}$$

(ii) On replacing x by $\frac{1}{2}(1-x)$, d by α and b by $\alpha+\beta+n$ in (18) we immediately get

$$\sum_{k=0}^{n} \frac{(1+\alpha+\beta)_{n+k}}{k!} F_2(a, -k, c\,;\, 1+\alpha+\beta+n,\, e\,;\, 1, y)$$

$$\times\, P_{n-k}^{(\alpha+k,\, \beta+k)}(x) \left(\frac{1-x}{2}\right)^k$$

$$= \frac{(1+\alpha)_n (1+\alpha+\beta)_n}{n!} F_2\left(a, -n, c\,;\, 1+\alpha+\beta+n,\, e\,;\, \frac{1-x}{2},\, y\right) \tag{19}$$

where the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ is defined by Szego[9] [p. 62],

$$P_n^{(\alpha,\beta)}(x) = \binom{n+\alpha}{n} {}_2F_1[-n,\ 1+\alpha+\beta+n;\ 1+\alpha;\ \tfrac{1}{2}(1-x)]. \tag{20}$$

(iii) If we let $y \to 0$ and take $a = 1 + \alpha + \beta + \delta + 2n$ in (19) and use the well-known Vandermonde's theorem [Bailey[1] (*p.* 63)]

$$F(-n, b;\ c;\ 1) = (c-b)_n/(c)_n \tag{21}$$

we get

$$\sum_{k=0}^{n} \frac{(-\delta-n)_k}{k!} P_{n-k}^{(\alpha+k,\,\beta+k)}(x)\left(\frac{1-x}{2}\right)^k$$

$$= \frac{(1+\alpha)_n(1+\alpha+\beta)_n}{(1+\alpha+\beta)_{2n}} P_n^{(\alpha+\beta+n,\,\delta)}(x). \tag{22}$$

On the other hand, if we let $y \to 0$, and take $a = \beta$ in (18) we get, in a similar way the result,

$$\sum_{k=0}^{n} \frac{(1+\alpha)_{n+k}}{k!} P_{n-k}^{(\alpha+k,\,\beta+k)}(x)\left(\frac{1-x}{2}\right)^k$$

$$= \frac{[(1+\alpha)_n]^2}{n!} {}_2F_1\left[-n, \beta;\ 1+\alpha+\beta+n;\ \frac{1-x}{2}\right] \tag{23}$$

(iv) On putting $a = 1 + \alpha + \beta + n$, $b = \alpha$, $d = \alpha + \gamma - n$, and replacing x by $\frac{1}{2}(1-x)$ in (18) and letting $y \to 0$, we get

$$\sum_{k=0}^{n} \frac{\left(\frac{x-1}{2}\right)^k}{k!\,(1+\beta)_{n-k}} P_{n-k}^{(\alpha-\gamma-n+k,\,\gamma+k)}(x)$$

$$= \frac{(-1)^n(\gamma-\alpha)_n}{(1+\alpha)_n(1+\beta)_n} P_n^{(\alpha,\beta)}(x). \tag{24}$$

(v) If we take $n = 2$, in (16), we get

$$\sum_{k=0}^{\infty} \frac{(b_1)_k(1+c_1)_k}{k!}(-x_1)^k\, {}_2F_1(b_1+k,\ c+k+1;\ c_1+k+1;\ x_1)$$

$$F_2(a;\ -k,\ b_2;\ 1+c,\ c_2;\ 1,\ x_2)$$

$$= F_2(a;\ b_1, b_2;\ c_1,\ c_2;\ x_1, x_2) \tag{25}$$

(vi) Finally, if we take $m = 1$, $a = 1 + \alpha + \beta$ and replacing x_1 by $(1 - x)/2$ in (11) and then make use of the result Rainville[7] [p. 256],

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x) \tag{26}$$

we get the well-known result due to Feldheim[4] [p. 120], *viz.*,

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta-n)}(x)\, t^n = (1-t)^{\beta}\,[1 - \tfrac{1}{2}(1+x)\,t]^{-1-\alpha-\beta}. \tag{27}$$

4. Acknowledgements

The authors are thankful to Prof. R. S. Kushwaha for his constant encouragement during the preparation of the paper.

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