J. Indian Inst. Sci. 62 (B), Feb. 1980, Pp. 25–33

A note on diffraction by a strip under mixed boundary conditions

A. CHAKRABARTI AND V. V. S. S. SASTRY
Department of Applied Mathematics, Indian Institute of Science, Bangalore 560 012.

Received on September 27, 1979.

Abstract

Higher order approximate solution is obtained for the problem of diffraction by a strip under mixed boundary conditions based on the solution obtained by Chakrabarti[1], by the method of successive approximation. Using this solution, the additional terms in the scattering coefficient are shown to be of order $l^{-3/2}$ for large l, where l is the width of the strip.

Key words : Diffraction, mixed boundary conditions, higher order approximation, integral equations.

1. Introduction

A direct technique of solving the problem of diffraction by a strip under mixed boundary conditions is given by Chakrabarti[1]. The solution is obtained by an application of Wiener–Hopf technique which reduces the problem to solving a system of coupled integral equations. An approximate solution of these integral equations is sought, by assuming the width of the strip, l, to be large. With this approximation, the resulting integral equations are reduced to 'a functional equation' of the type

$$\Phi_+(\gamma) = -i\gamma\sqrt{\gamma^2 - 2ik}\,\Psi_+(-\gamma),$$

first encountered by Rawlins[2]. The scattered field is obtained, from which the scattering coefficient is computed and is shown to agree with the first term obtained by Jones[3] (p. 607)*.

In this note, we propose to obtain the second order approximate solution by the method of successive approximations. Finally scattering coefficient is computed, using

* The expression for the scattering coefficient obtained by Jones on p. 607 is in error by a multiplication factor of 2 sin ϕ_0 on the right hand side.

the second order solution and the additional terms are shown to be of the order $l^{-3/2}$, for large l, l being the width of the strip.

In Section 2, the mathematical problem is stated and the first order solution presented, without derivation, for future reference. For the derivation of these results, the interested reader is referred to ref. 1. Section 3 deals with the second order solution and the computation of scattering coefficient.

2. Formulation of the problem : First order solution

The mathematical problem is that of solving the partial differential equation

$$(\nabla^2 + k^2)\, v = 0 \tag{1.1}$$

under the boundary conditions

$$v\,(x, 0+) = -\,u_0\,(x, 0)$$

$$\frac{\partial v}{\partial y}(x, 0-) = \frac{-\,\partial u_0}{\partial y}(x, 0)\,(-\,l < x < 0), \tag{1.2}$$

and the continuity conditions

$$\left.\begin{aligned} v\,(x, 0+) &= v\,(x, 0-) \\ \frac{\partial v}{\partial y}(x, 0+) &= \frac{\partial v}{\partial y}(x, 0-) \end{aligned}\right\}, \quad (-\,\infty < x < -\,l;\, 0 < x < \infty) \tag{1.3}$$

where $u_0\,(x, y)$ is the incident field and $v\,(x, y)$ the scattered field. We require that the field be outgoing at infinity and for uniqueness of the solution, we also require proper edge behaviour of the field. The appropriate edge conditions at $x = -\,l$ and $x = 0$ are[2] :

$$v\,(x, 0) \sim 0\,(x^{1/4})$$

$$\frac{\partial v}{\partial y}(x, 0) \sim 0\,(x^{-3/4})$$

$$\text{as } x \to 0+. \tag{1.4}$$

When the incident field is a plane wave, the solution of the above diffraction problem (1.1)–(1.4) is obtained in the form (for details see ref 1),

$$v\,(x, y) = u_0\,(x, y) + \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} V\,(s, y)\, e^{sx}\, ds, \tag{1.5}$$

$$k_i \cos\phi_0 < c < k_i$$

where

$$V(s, y) = A\, e^{-i\kappa y}, \quad y > 0$$
$$= B\, e^{i\kappa y}, \quad y < 0, \tag{1.6}$$

$\kappa = \sqrt{s^2 + k^2}$, the branch of the square root function is chosen in such a way that $\kappa = k$ when $s = 0$, and

$$2A = \mu_+(s) + \theta_+(s) + e^{sl}\,[\mu_+(-s) - \theta_+(-s)] \tag{1.7}$$

$$2i\kappa B = \lambda_+(s) + \nu_+(s) + e^{sl}\,[\lambda_+(-s) - \nu_+(-s)] \tag{1.8}$$

where $\mu_+, \theta_+, \lambda_+, \nu_+$ are the solutions of the following coupled integral equations :

$$(s + ik)^{-1/2}\, \lambda_+(s) = l_1(s) +$$

$$\frac{1}{2\pi} \int_{c-i\infty}^{c+i\infty} \frac{dw}{(w+s)} \left[(w - ik)^{-1/2}\, \lambda_+(w)\, e^{-wl} - i\,(w + ik)^{1/2}\, \mu_+(-w)\right]$$

$$(s + ik)^{1/2}\, \mu_+(s) = l_2(s)$$

$$-\frac{1}{2\pi} \int_{c-i\infty}^{c+i\infty} \frac{dw}{(w+s)} \left[(w - ik)^{1/2}\, \mu_+(w)\, e^{-wl} - i\,(w + ik)^{-1/2}\, \lambda_+(-w)\right]$$

$$(s + ik)^{-1/2}\, \nu_+(s) = m_1(s)$$

$$-\frac{1}{2\pi} \int_{c-i\infty}^{c+i\infty} \frac{dw}{(w+s)} \left[(w - ik)^{-1/2}\, \nu_+(w)\, e^{-wl} + i\,(w + ik)^{1/2}\, \theta_+(-w)\right]$$

$$(s + ik)^{1/2}\, \theta_+(s) = m_2(s)$$

$$+\frac{1}{2\pi} \int_{c-i\infty}^{c+i\infty} \frac{dw}{(w+s)} \left[(w - ik)^{1/2}\, \theta_+(w)\, e^{-wl} + i\,(w + ik)^{-1/2}\, \nu_+(-w)\right] \tag{1.9}$$

where

$$l_i(s) = \frac{a_i}{(s + ik\cos\phi_0)} + \frac{b_i}{(s - ik\cos\phi_0)}, \quad i = 1, 2$$

$$m_i(s) = \frac{a_i'}{(s + ik\cos\phi_0)} + \frac{b_i'}{(s - ik\cos\phi_0)}, \quad i = 1, 2 \tag{1.10}$$

and

$$
\begin{aligned}
a_1 &= a_1' = -\sqrt{2ik}\cos\phi_0/2 \\
b_1 &= -b_1' = -\sqrt{2ik}\sin\phi_0/2\exp(ikl\cos\phi_0) \\
a_2 &= a_2' = \sqrt{2ik}\sin\phi_0/2 \\
b_2 &= -b_2' = \sqrt{2ik}\cos\phi_0/2\exp(ikl\cos\phi_0)
\end{aligned} \tag{1.11}
$$

The integral equations (1.9) are best solved by approximated methods. It is assumed, that the width of the strip l is large and as a result the contribution from the integrals involving e^{-wl} is negligible (see ref. 1, Eqn. 3.7). To this order of approximation, the solution is obtained by solving the following coupled integral equations:

$$(s+ik)^{-1/2}\lambda_+^{(1)}(s) = l_1(s) + \frac{1}{2\pi}\int_{-c+i\infty}^{-c-i\infty} \frac{\mu_+^{(1)}(w)(w-ik)^{1/2}\,dw}{(w-s)}$$

$$(s+ik)^{1/2}\mu_+^{(1)}(s) = l_2(s) + \frac{1}{2\pi}\int_{-c+i\infty}^{-c-i\infty} \frac{\lambda_+^{(1)}(w)(w-ik)^{-1/2}\,dw}{(w-s)}$$

$$(s+ik)^{-1/2}\nu_+^{(1)}(s) = m_1(s) + \frac{1}{2\pi}\int_{-c+i\infty}^{-c-i\infty} \frac{\theta_+^{(1)}(w)(w-ik)^{1/2}\,dw}{(w-s)}$$

$$(s+ik)^{1/2}\theta_+^{(1)}(s) = m_2(s) + \frac{1}{2\pi}\int_{-c+i\infty}^{-c-i\infty} \frac{\nu_+^{(1)}(w)(w-ik)^{-1/2}\,dw}{(w-s)} \tag{1.12}$$

whose solutions are given by

$$\begin{Bmatrix}\lambda_+^{(1)}(s)\\ \nu_+^{(1)}(s)\end{Bmatrix} = \frac{[A_{-1}^{(0)} + A_0^{(0)}\gamma]\sqrt{\gamma+\sqrt{2ik}}}{(s+ik\cos\phi_0)} \pm \frac{[A_{-1}^{(1)} + A_0^{(1)}\gamma]\sqrt{\gamma+\sqrt{2ik}}}{(s-ik\cos\phi_0)} \tag{1.13}$$

$$\begin{Bmatrix}\mu_+^{(1)}(s)\\ \theta_+^{(1)}(s)\end{Bmatrix} = \frac{[A_{-1}^{(0)} - A_0^{(0)}\gamma]}{(s+ik\cos\phi_0)\,\gamma\sqrt{\gamma+\sqrt{2ik}}} \pm \frac{[A_{-1}^{(1)} - A_0^{(1)}\gamma]}{(s-ik\cos\phi_0)\,\gamma\sqrt{\gamma+\sqrt{2ik}}} \tag{1.14}$$

The upper sign corresponds to $\lambda_+^{(1)}$, $\mu_+^{(1)}$ and the lower sign corresponds to $\nu_+^{(1)}$ and $\theta_+^{(1)}$ functions respectively. The constants $A_{-1}^{(i)}$, $A_0^{(i)}$, $i = 1, 0$ are given by

$$A_{-1}^{(0)} = \frac{1}{2}\left[\frac{a_1\xi_0}{\sqrt{\xi_0+\sqrt{2ik}}} + a_2\sqrt{\xi_0+\sqrt{2ik}}\right] \tag{1.15}$$

$$A_0^{(0)} = \frac{1}{2\xi_0}\left[\frac{a_1\xi_0}{\sqrt{\xi_0+\sqrt{2ik}}} - a_2\sqrt{\xi_0+\sqrt{2ik}}\right] \tag{1.16}$$

$$A_{-1}^{(1)} = \frac{1}{2}\left[\frac{b_1 \eta_0}{\sqrt{\eta_0 + \sqrt{2ik}}} + b_2 \sqrt{\eta_0 + \sqrt{2ik}}\right] \quad (1.17)$$

$$A_0^{(1)} = \frac{1}{2\eta_0}\left[\frac{b_1 \eta_0}{\sqrt{\eta_0 + \sqrt{2ik}}} - b_2 \sqrt{\eta_0 + \sqrt{2ik}}\right] \quad (1.18)$$

a_i, b_i, $i = 1, 2$ are given by (1.11) and

$$\xi_0 = \sqrt{2ik} \sin \phi_0/2, \; \eta_0 = \sqrt{2ik} \cos \phi_0/2.$$

In the above, we have set $\gamma = \sqrt{s + ik}$ and we have denoted the first order solution by a superscript within the parenthesis (*e.g.*, $\lambda_+^{(1)}$, etc.). This notation should not be confused with the notation used in (3.8) of ref 1. Similarly, the second order solutions are denoted by $\lambda_+^{(2)}$, etc.

3. Second order solution

The coupled integral equations (1.12) are solved by reducing them to a 'functional equation' which is encountered by Rawlins, while solving the problem of acoustic diffraction by a semi-infinite plane under mixed boundary conditions of the present type[2].

Higher order approximate solutions of (1.12) can be obtained by successively substituting the lower order approximations in RHS of (1.12) and evaluating the known integrals approximately. In this note, due to mathematical complexity, we restrict our attention to second order approximation, which is obtained by making use of the first order solutions (1.13) and (1.14) in RHS of (1.12), *viz.*,

$$(s + ik)^{-1/2} \lambda_+^{(2)}(s) = l_1(s) + \frac{1}{2\pi} \int_{c-i\infty}^{c+i\infty} \frac{dw}{(w+s)} (w - ik)^{-1/2} \lambda_+^{(1)}(w)\, e^{-wl}$$

$$+ \frac{1}{2\pi} \int_{-c+i\infty}^{-c-i\infty} \frac{dw}{(w-s)} (w - ik)^{1/2} \mu_+^{(1)}(w)$$

$$(s + ik)^{1/2} \mu_+^{(2)}(s) = l_2(s) - \frac{1}{2\pi} \int_{c-i\infty}^{c+i\infty} \frac{dw}{(w+s)} (w - ik)^{1/2} \mu_+^{(1)}(w)\, e^{-wl}$$

$$+ \frac{1}{2\pi} \int_{-c+i\infty}^{-c-i\infty} \frac{dw}{(w-s)} \lambda_+^{(1)}(w) (w - ik)^{-1/2}$$

$$(s+ik)^{-1/2}\,\nu_+^{(2)}(s) = m_1(s) - \frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty}\frac{dw}{(w+s)}(w-ik)^{-1/2}\,\nu_+^{(1)}(w)\,e^{-wl}$$

$$+\frac{1}{2\pi}\int_{-c+i\infty}^{-c-i\infty}\frac{dw}{(w-s)}(w-ik)^{1/2}\,\theta_+^{(1)}(w)$$

$$(s+ik)^{1/2}\,\theta_+^{(2)}(s) = m_2(s) + \frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty}\frac{dw}{(w+s)}(w-ik)^{1/2}\,\theta_+^{(1)}(w)\,e^{-wl}$$

$$+\frac{1}{2\pi}\int_{-c+i\infty}^{-c-i\infty}\frac{dw}{(w-s)}(w-ik)^{-1/2}\,\nu_+^{(1)}(w) \tag{2.1}$$

Now, by making use of the fact that $\lambda_+^{(1)}$, $\mu_+^{(1)}$, $\nu_+^{(1)}$, $\theta_+^{(1)}$ satisfy integral equations (1.12), it follows from (2.1):

$$(s+ik)^{-1/2}\,\lambda_+^{(2)}(s) = (s+ik)^{-1/2}\,\lambda_+^{(1)}(s)$$

$$+\frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty}\frac{dw}{(w+s)}(w-ik)^{-1/2}\,\lambda_+^{(1)}(w)\,e^{-wl}$$

$$(s+ik)^{1/2}\,\mu_+^{(2)}(s) = (s+ik)^{1/2}\,\mu_+^{(1)}(s)$$

$$-\frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty}\frac{dw}{(w+s)}(w-ik)^{1/2}\,\mu_+^{(1)}(w)\,e^{-wl}$$

$$(s+ik)^{-1/2}\,\nu_+^{(2)}(s) = (s+ik)^{-1/2}\,\nu_+^{(1)}(s)$$

$$-\frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty}\frac{dw}{(w+s)}(w-ik)^{-1/2}\,\nu_+^{(1)}(w)\,e^{-wl}$$

$$(s+ik)^{1/2}\,\theta_+^{(2)}(s) = (s+ik)^{1/2}\,\theta_+^{(1)}(s)$$

$$+\frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty}\frac{dw}{(w+s)}(w-ik)^{1/2}\,\theta_+^{(1)}(w)\,e^{-wl}. \tag{2.2}$$

The integrals appearing on the RHS of (2.2) are approximately evaluated using Watson's lemma (see Jones[3], p. 438). Taking into consideration, the contribution from these integrals, one can write the second order solutions as

$$\lambda_+^{(2)}(s) = \lambda_+^{(1)}(s) + \frac{e^{-ikl}(s+ik)^{1/2}}{\sqrt{\pi}}\left[\frac{p_3+\bar{p}_3}{\sqrt{l}} - \frac{q_3+\bar{q}_3}{l^{3/2}}\right]$$

$$\mu_+^{(2)}(s) = \mu_+^{(1)}(s) + (s+ik)^{1/2}\,\frac{e^{-ikl}}{\sqrt{\pi}\,l^{3/2}}(a_3+\bar{a}_3)$$

$$v_+^{(2)}(s) = v_+^{(1)}(s) + \frac{(s+ik)^{1/2}}{\sqrt{\pi}} e^{-ikl} \left[\frac{-(c_3+\bar{c}_3)}{\sqrt{l}} + \frac{d_3+\bar{d}_3}{l^{3/2}} \right]$$

$$\theta_+^{(2)}(s) = \theta_+^{(1)}(s) - (s+ik)^{-1/2} \frac{e^{-ikl}}{\sqrt{\pi}} (e_3+\bar{e}_3) \tag{2.3}$$

where

$$p_3 = \frac{\sqrt{2}\,(2ik)^{1/4}}{(s+ik)\,\eta_0^2} [A_{-1}^{(0)} + A_0^{(0)'} \sqrt{2ik}]$$

$$\bar{p}_3 = \frac{\sqrt{2}\,(2ik)^{1/4}}{(s+ik)\,\xi_0^2} [A_{-1}^{(1)} + A_0^{(1)} \sqrt{2ik}]$$

$$q_3 = \frac{\sqrt{2}\,(2ik)^{1/4}}{(s+ik)\,\eta_0^2} \left[(A_{-1}^{(0)} + A_0^{(0)} \sqrt{2ik}) \left(\frac{1}{16ik} - \frac{1}{s+ik} - \frac{1}{\eta_0^2} \right) + \frac{A_0^{(0)}}{2\sqrt{2ik}} \right]$$

$$\bar{q}_3 = \frac{\sqrt{2}\,(2ik)^{1/4}}{(s+ik)\,\xi_0^2} \left[(A_{-1}^{(1)} + A_0^{(1)} \sqrt{2ik}) \left(\frac{1}{16ik} + \frac{1}{s+ik} - \frac{1}{\xi_0^2} \right) + \frac{A_0^{(1)}}{2\sqrt{2ik}} \right]$$

$$a_3 = \frac{(2ik)^{-3/4}}{\sqrt{2}\,(s+ik)\,\eta_0^2} [A_{-1}^{(0)} - A_0^{(0)} \sqrt{2ik}]$$

$$\bar{a}_3 = \frac{(2ik)^{-3/4}}{\sqrt{2}\,(s+ik)\,\xi_0^2} [A_{-1}^{(1)} - A_0^{(1)} - A_0^{(1)} \sqrt{2ik}]$$

$$c_3 = p_3,\ d_3 = q_3,\ \bar{c}_3 = -\bar{p}_3,\ \bar{d}_3 = -\bar{q}_3,\ e_3 = a_3,\ \bar{e}_3 = -\bar{a}_3. \tag{2.4}$$

By making use of (1.5), we obtain the final field, after using the formula (1.7) and (1.8) for A and B, the second order solutions obtained in (2.3).

The diffracted far field for large kr is obtained by writing $x = r\cos\phi$, $|y| = r\sin\phi$, $0 < \pi$ in (1.5) and can be written as

$$v_d(x, y) = A(-ik\cos\phi) \left(\frac{k\sin^2\phi}{2\pi r} \right)^{1/2} e^{-ikr+i\pi/4},\ y > 0$$

$$= M(-ik\cos\phi) \left(\frac{k\sin^2\phi}{2\pi r} \right)^{1/2} e^{-ikr+i\pi/4},\ y < 0 \tag{2.5}$$

where

$$A(-ik\cos\phi) = \tfrac{1}{2} \{ \mu_-^{(2)}(-ik\cos\phi) + \theta_+^{(2)}(-ik\cos\phi) + e^{-ikl\cos\phi} [\mu_+^{(2)}(ik\cos\phi) - \phi_+^{(2)}(ik\cos\phi)] \}$$

$$M(-ik\cos\phi) = (2ik\sin\phi)^{-1}\{\lambda_+^{(2)}(-ik\cos\phi) + \nu_+^{(2)}(-ik\cos\phi) + e^{-ikl\cos\phi} \times [\lambda_+^{(2)}(ik\cos\phi) - \theta_+^{(2)}(ik\cos\phi)]\}. \tag{2.6}$$

The scattering coefficient is calculated, using the formula

$$\sigma_s + \sigma_A = \frac{-2\sin\phi_0}{l}\,\mathrm{Re\ of}\ \{A(-ik\cos\phi_0) + M(-ik\cos\phi_0)\}. \tag{2.7}$$

(see (4.7) of ref. 1). Upon substitution for $A(s)$, and $B(s)$ and taking the limiting values as $\phi \to \phi_0$, we finally express the scattering coefficient as :

$$\begin{aligned}\sigma_s + \sigma_A = 4\sin\phi_0\,\frac{-2\sin\phi_0}{l}\Bigg[&\frac{\cos 5\pi/4\cos(2kl\sin^2\phi_0/2)\,S_6(\phi_0)}{(2k)^{5/2}\sqrt{\pi}\,l^{3/2}\sin^5\phi_0/2}\\ &+\frac{\cos 5\pi/4\cos(2kl\cos^2\phi_0/2)\,S_5(\phi_0)}{(2k)^{5/2}\,2\sqrt{2}\sqrt{\pi}\,l^{3/2}\cos^5\phi_0/2}\\ &-\frac{4\cos 3\pi/4\cos(2kl\sin^2\phi_0/2)\,S_1(\phi_0)}{(2k)^{3/2}\sqrt{\pi}\sin^3\phi_0/2\sin\phi_0\sqrt{l}}\\ &+\frac{4\cos 5\pi/4\cos(2kl\sin^2\phi_0/2)\,S_2(\phi_0)}{(2k)^{3/2}\sqrt{\pi}\sin^3\phi_0/2\sin\phi_0\,l^{3/2}}\\ &+\frac{\sqrt{2}\cos 3\pi/4\cos(2kl\cos^2\phi_0/2)\,S_3(\phi_0)}{(2k)^{3/2}\sqrt{\pi}\sin\phi_0\cos^3\phi_0/2\sqrt{1}}\\ &-\frac{\sqrt{2}\cos 5\pi/4\cos(2kl\cos^2\phi_0/2)\,S_4(\phi_0)}{(2k)^{5/2}\sqrt{\pi}\sin\phi_0\cos^3\phi_0/2\,l^{3/2}}\Bigg]\end{aligned} \tag{2.8}$$

where

$$\begin{aligned}S_1(\phi_0) &= \sin\phi_0/4\cos\phi_0/4(\sin\phi_0/4+\cos\phi_0/4)\\ S_2(\phi_0) &= \frac{(\sin^2\phi_0/2-16)}{8\sin^2\phi_0/2}S_1(\phi_0)+\frac{1}{4}(\sin\phi_0/4+\cos\phi_0/4)\\ S_3(\phi_0) &= (1+\sin\phi_0/2)^{1/2}(\sin\phi_0/2-\cos\phi_0/2-1)\\ S_4(\phi_0) &= \frac{(\cos^2\phi_0/2-16)}{8\cos^2\phi_0/2}-\frac{1}{2}\{(1+\sin\phi_0/2)^{1/2}+\cos\phi_0/2\,(1+\sin\phi_0/2)^{-1/2}\}\\ S_5(\phi_0) &= (\sin\phi_0/4+\cos\phi_0/4)^3+\cos\phi_0/2\,\frac{(\sin\phi_0/4-\cos\phi_0/4)^2}{(\sin\phi_0/4+\cos\phi_0/4)}\\ S_6(\phi_0) &= (\cos^3\phi_0/4+\sin^3\phi_0/4).\end{aligned} \tag{2.9}$$

We conclude by noting that the first term of the scattering coefficient is obtained by using first order solution (see ref. 1, eqn. 4.9) and is worth comparing with the expressions for the scattering coefficient obtained by Jones[3] (p. 607) for the perfectly conducting strip.

References

1. CHAKRABARTI, A. — Diffraction by a strip under mixed boundary conditions, *J. Indian Inst. Sci.*, 1979, **61 B**, 163.

2. RAWLINS, A. D. — The solution of a mixed boundary value problem in the theory of diffraction by a semi-infinite plane, *Proc. Roy. Soc. Lond.*, 1973, **A 346**, 469.

3. JONES, D. S. — *Theory of electromagnetism*, Pergamon Press, 1964.