

Short Communication

The spectrum of a matrix differential operator

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Abstract

In this paper we extend the problem contained in Chakravarty-Sen Gupta² in a more general form

$$\begin{pmatrix} -D^2 + p & q \\ q & -D^2 + r \end{pmatrix} f = \lambda \begin{pmatrix} s & h \\ h & t \end{pmatrix} f \quad \left(D \equiv \frac{d}{dx} \right)$$

We prove, under certain restrictions on the co-efficients, the spectrum to be discrete.

Key words : Spectrum, differential operator, weight matrix, number of eigenvalues, pseudo-monotonic.

1. Introduction

The differential expression under consideration is

$$M[f] = -D^2 f(x) + P(x)f(x) = \lambda S(x)f(x) \text{ on } I \quad (1.1)$$

where $P = \begin{pmatrix} p & q \\ q & r \end{pmatrix}$ and $S = \begin{pmatrix} s & h \\ h & t \end{pmatrix}$

and $f(x) = \{f_1(x), f_2(x)\}$, for all $x \in I$ ($I : \mathcal{R}^2; (0, \infty)$, the weighted integrable-square vector space generated by the weight matrix $S(x)$) and for $\lambda \in \mathbb{C}$ denotes a solution of (1.1) if it belongs to I , i.e., $\int_I f^* S f < \infty$. The interval I may be bounded or unbounded, and then open, half-open or closed. However, it is sufficient for the problem discussed here, to consider the case when I is half-open interval $[0, \infty)$.

To define the differential operator T , generated by the differential expression $S^{-1}M[\cdot]$ in $\mathcal{X}_2^2(0, \infty)$, it is convenient to introduce the linear manifold Δ of $\mathcal{X}_2^2(0, \infty)$ by $\Delta \stackrel{\text{def}}{=} \{(f \in \mathcal{X}_2^2(0, \infty); f' \in AC_{loc}, S^{-1}M[f] \in \mathcal{X}_2^2(0, \infty), f(0) = 0\}$. Absolute continuity is denoted by AC and the super-script 'loc' denotes a property to be satisfied on all compact sub-intervals of $[0, \infty)$. The differential operator T is defined as follows:

$$T : \text{the domain } \mathcal{D}(T) \text{ is } \Delta \text{ and } Tf \stackrel{\text{def}}{=} S^{-1}M[f] \quad (f \in \mathcal{D}(T)) \quad (1.2)$$

where the coefficient of the matrices are real-valued on $[0, \infty)$ and belong to $C^{(0)}[0, \infty)$, $\mathcal{L}(0, \infty)$ and AC_{loc} , also $p(x), s(x) > 0$ $\det P, \det S \geq 0$ for $x \in [0, \infty)$.

To ensure the problem in the limit-2 case the matrix $P(x)$ satisfies similar conditions like Chakravarty², Th. II or Lidskii⁶,

2. Some useful results

Result 2.1. If $p/s > c$ and $\det(P - cS) \geq 0$, then all the eigenvalues are greater than or equal to c ($c = \text{constant}$). All the other results given in § 3.4 in Chakravarty-Sen Gupta⁸ remain the same with an exception $C_m = \int_0^b \Psi_m^* S \Psi_m dx$ as the Fourier Coefficient of $f(x)$ corresponding to Dirichlet BVP .

We define $N(\lambda, P_j)$ as the number of eigenvalues not exceeding λ for the Dirichlet (or Neumann) problem with matrix P replaced by P_j for the n th eigenvalue λ_n and $N(\lambda, P_k)$ be the same in the case of the n th eigenvalue μ_n .

Result 2.2. $\lambda_n \geq \mu_n$ and $N(\lambda, P_k) \geq N(\lambda, P_j)$ for $j > k$.

Result 2.3. $\lambda_n \geq \mu_n$ and $N(\lambda, S_k) \geq N(\lambda, S_j)$ for $j > k$.

In the case of variation of the matrices P and S at a time, we define

$$P(x) = [0], \text{ a null matrix in } 0 < x < a$$

and

$$P(x) = (\gamma S(x)) = \begin{pmatrix} \gamma_1 s(x) & \gamma_2 h(x) \\ \gamma_2 h(x) & \gamma_3 t(x) \end{pmatrix} \text{ in } a \leq x \leq b.$$

where $\gamma_j, j = 1, 2, 3$ are constants with $\gamma_1, \gamma_3 > 0$ and $(0, a)$ is contained in $[0, b]$. The differential system then reduces to

$$M[f] \equiv \begin{cases} D^2 f + \lambda S f = 0 & \text{in } 0 < x < a \\ D^2 f + \{\lambda S - (\gamma S)\} f = 0 & \text{in } a \leq x \leq b. \end{cases} \quad (2.1)$$

Result 2.4. The n th eigenvalue decreases as the matrix $S(x)$ increases pseudo-monotonically, i.e., $\lambda_n \geq \mu_n$ and $N(\lambda, S_k) \geq N(\lambda, S_j)$ for $k > j$. (2.2)

Result 2.5. $N_b(\lambda) \leq N_B(\lambda)$ for $B > b$

[for definition $N_b(\lambda)$ and $N_B(\lambda)$ see Chakravarty-Sen Gupta³].

Result 2.6. If $p(x)/s(x)$ and $r(x)/t(x)$ are increasing functions of x , along with the conditions given in § 1 then $N_x(\lambda)$ is bounded independently of x .

Where $X > Y$ and Y is defined as $\det(P(Y) - \lambda S(Y)) = 0$.

Proof of this result follows in the same way as in Chakravarty-Sen Gupta³, Chaudhuri-Everitt⁴, but here we choose λ in such a way that we can have $p/s > \lambda$ and $\det(P - \lambda S) \geq 0$ following Hardy, Littlewood and Polya⁵ (result 2.7.1, p-21).

3. Proof of the main theorem

Theorem I. If all the conditions given in § 1 are satisfied and further $p(x)/s(x)$ and $r(x)/t(x)$ steadily increase and tend to infinity as x tend to infinity then the spectrum of the operator T as defined in (1.2) is discrete over the whole λ -range.

Proof: Let $p(x) > \gamma_1 s(x)$ and $\det(P(x) - (\gamma S(x))) \geq 0$ for $x \geq a$, ($a < b$).

If $\lambda_{n,b}$ denote the eigenvalues in the problem of the Dirichlet BVP in the interval $x \leq b$ and $\mu_{n,b}$ denote the same when the Dirichlet problem reduces to

$$D^2 f + \lambda S f = 0 \quad (0 < x < a)$$

$$D^2 f + \{\lambda S - (\gamma S)\} f = 0 \quad (a \leq x \leq b) \quad (3.1)$$

then $\lambda_{n,b} \geq \mu_{n,b}$ [See Result 2.2].

Next let $s(x) < S_j(x)$ and $\det(S_j - S) \geq 0$ and $\nu_{n,b}$ denote the eigenvalue in the Dirichlet BVP (3.1) when $S(x)$ is replaced by $S_j(x)$, it then follows from (2.2) $\mu_{n,b} > \nu_{n,b}$ provided $\nu_{n,b}$, greater than γ_j , for $j = 1, 3$, is restricted by

$$(\nu_{n,b} - \gamma_1) (\nu_{n,b} - \gamma_3) (s_j - s) (t_j - t) - (\nu_{n,b} - \gamma_2)^2 (h_j - h)^2 \geq 0$$

and hence

$$(\lambda_{n,b} \geq \mu_{n,b} \geq \nu_{n,b}) \quad (3.2)$$

Let N denote the number of numbers $\nu_{n,b}$, neither exceeding γ_1 nor exceeding γ_3 , then for large b , Result 2.6 leads to

$$\nu_{0,b} < \nu_{1,b} < \dots < \nu_{N,b} < \gamma_j < \nu_{N+1,b} \dots, \quad (j = 1, 3) \quad (3.3)$$

that is, the number of eigenvalues not exceeding γ_j , ($j = 1, 3$) is at most N .

It can be proved as in Bhagat¹ that each eigenvalue λ_n is at most a simple pole of the Green's matrix for the problem (1.1), where the uniqueness of the Green's matrix follows from the limit-2 restrictions given in § 1. Now as each eigenvalue $\lambda_{n,b}$ is non-decreasing as b steadily tends to infinity (see Result 2.5), then by arguments similar to those of Chakravarty-Sen Gupta³ and Titchmarsh⁸ (p. 147) that the spectrum of the problem to be discrete for $\lambda < \gamma_j$ ($j = 1, 3$) and since γ_1, γ_3 are arbitrary, the spectrum of the operator T is discrete over the whole λ -range.

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Short Communication

Spectrophotometric determination of micro amounts of vanadium (V) with resacetophenone—indirect method

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Abstract

A method has been developed for the indirect spectrophotometric determination of vanadium(V) using resacetophenone as reagent. It involves the oxidation of iron(II) to (III) by vanadium(V) and the resulting ferric iron gives wine-red coloured complex with resacetophenone. The complex was stable for 20 hr. The absorbance of the complex measured at 365 nm obeys Beer's law over the concentration range 2-8 $\mu\text{g/ml}$ of vanadium. The molar absorptivity and Sandell sensitivity are $(2.25 \pm 0.05) \times 10^3 \text{ lit. mol}^{-1} \text{ cm}^{-1}$ and $0.0226 \mu\text{g/cm}^2$ respectively. The effect of various ions was studied.

Key words : Spectrophotometry, vanadium (V), resacetophenone.

1. Introduction

Resacetophenone (2-4 dihydroxy-acetophenone) was first introduced by Cooper¹ as analytical reagent for the detection of iron in slightly acid medium. The reagent was applied for the fluorimetric detection of boron in concentrated sulphuric acid medium². The ketone was employed for the estimation of copper by many authors³⁻⁷. The reactions of the reagent with various metal ions was studied both in acid and alkaline media⁸. In acid medium the reagent gives no colour reactions with vanadium (IV) and iron (II). This fact was utilized for the spectrophotometric determination of microgram amounts of vanadium(V) with the reagent.

2. Experimental

Apparatus

ELICO Spectrophotometer Model GS 865A and ELICO Digital pH Meter Model L1-10 were used for these investigations.

Reagents

Resacetophenone was prepared from resorcinol⁹. The reagent solution was prepared in 50% methanol.

Stock solutions of ferrous ammonium sulphate and ammonium metavanadate were prepared by dissolving AnalaR grade samples in water. The concentrations were checked by standard methods.

The buffer solution of pH 3.0 was prepared by mixing sodium acetate (1 M)-hydrochloric acid (1 M).

3. Results and discussion

Oxidation of iron (II) to (III) by vanadium is well known¹⁰. The ferric iron formed gives wine-red coloured complex with resacetophenone⁸. The complex has absorption maximum at 365 nm and was stable for about 20 hr. Under the experimental conditions neither the vanadium(IV) formed nor iron(II) present in excess do not form any coloured complexes with the reagent. The system obeyed Beer's law over the concentration range 2–8 $\mu\text{g/ml}$ of vanadium. The molar absorptivity and Sandell sensitivity are $(2.25 \pm 0.05) \times 10^3 \text{ lit. mol}^{-1} \text{ cm}^{-1}$ and $0.0226 \mu\text{g/cm}^2$ respectively.

Procedure

A 15 ml aliquots of sodium acetate-hydrochloric acid buffer (pH 3.0) were taken into different 25 ml standard flasks. 2 ml of 0.01 M ferrous iron and 2 ml of resacetophenone (0.01 M) solutions were also added to the same standard flasks. A known volume of the standard vanadium(V) solution containing 50–200 μg of vanadium were then added and the solutions were made up to the mark with water. The contents in the flasks were shaken well and their absorbance was measured at 365 nm against the blank containing ferrous iron and the reagent. The data obtained showed suitability of the method for the determination of micro amounts (2–8 $\mu\text{g/ml}$) of vanadium.

Effect of various ions

K^+ , Na^+ , Cl^- , NO_3^- , SO_4^- did not interfere. Oxalate, citrate, tartrate, phosphate, ascorbate, Mo (VI), W(VI) and Al(III) interfered. Cr (III), Zn(II) and acetate interfered when

present in 100 fold excess. 80 fold excess of Co(II), Ni(II), Mn(II), Mg(II), Pb(II), 25 fold excess of Cd(II), Cu(II), 10 fold excess of V(IV), Br⁻, I⁻, SCN⁻ and 2 fold excess of U(VI) did not interfere.

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