On the removal of the infinite self-energies of point-particles

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A general method is set up for modifying the energy-momentum tensor so as to remove the singularities in the flow of energy and momentum into the world-line of a particle without affecting the equations of motion of the particle. It is shown how the singularities of different order may be removed one by one.

In the case of the electromagnetic and meson fields it is shown that the modified tensor leads to a finite integral of energy and momentum over any space-like surface. In other cases the corresponding result may be secured by making a further modification in the tensor.

1. It has been shown in the preceding paper* that all singular terms in the expression for the rate of inflow† of energy and momentum are perfect differentials, and therefore the equations of motion of a point-particle calculated from considerations of the conservation of energy and momentum are always finite. One would now expect to be able to alter, without disturbing the equations of motion, the energy-momentum tensor $T^{\mu\nu}$ of the field so as to make the inflow finite, thus avoiding completely the appearance of infinities which are entirely spurious, at any rate, from the point of view of the equations of motion. It would be still better if it could be arranged that the total energy and momentum calculated from the modified tensor $\tilde{T}^{\mu\nu}$ were finite. This of course would automatically secure a finite inflow.

It is proved in A that the replacement of the energy-momentum tensor $T^{\mu\nu}$ by

$$\tilde{T}^{\mu\nu} = T^{\mu\nu} + \frac{\partial K^{\mu\nu\sigma}}{\partial x^{\sigma}}, \tag{1}$$

where $K^{\mu\nu\sigma}$ is antisymmetric in (ν, σ), does not alter the equations of motion. Therefore an attempt should be made to alter $T^{\mu\nu}$ in accordance with (1) in trying to get rid of the infinities. This method has already been used by Pryce (1938) to make the energy and momentum of the field finite in the case of the point electron. It will be shown quite generally in this paper that given an energy-momentum tensor $T^{\mu\nu}$ satisfying the conservation equation

$$\frac{\partial T^{\mu\nu}}{\partial x^{\nu}} = 0, \tag{2}$$

a tensor $K^{\mu\nu\sigma}$ can always be found so as to make the energy-momentum integrals calculated from $\tilde{T}^{\mu\nu}$ finite provided $T^{\mu\nu}$ fulfils certain very general restrictions. In general, $K^{\mu\nu\sigma}$ is not symmetric in μ, ν even when $T^{\mu\nu}$ is so, and therefore the modified

* Bhabha & Harish-Chandra (1944), referred to in this paper as A.
† This term is used in the same sense as in A.

tensor $\tilde{T}^{\mu\nu}$ is also not symmetric. But it will be shown that in the case of an electromagnetic charge or a dipole this symmetry can be achieved. However, the symmetry of the energy-momentum tensor, which is needed only for the purpose of constructing an angular-momentum tensor, is not necessary here, because it will be shown that a similar process can be applied to the angular-momentum tensor also, so that the total angular momentum calculated from this modified tensor is finite.

2. Keeping to the notation of A it is now necessary to determine $K^{\mu\nu\sigma}$. Equations (18) and (19) of A are

$$\frac{d}{d\tau}\left\{\int_{\epsilon}^{\eta} d\kappa \int_{\kappa} T^{\mu\nu}\frac{s_\nu}{\kappa}\,d\Omega\right\} = \int_{\kappa=\epsilon} T^{\mu\nu}\kappa_\nu\,d\Omega - \int_{\kappa=\eta} T^{\mu\nu}\kappa_\nu\,d\Omega, \tag{3}$$

$$\frac{d}{d\tau}\int_{\kappa} T^{\mu\nu}\frac{s_\nu}{\kappa}\,d\Omega = -\frac{d}{d\kappa}\int_{\kappa} T^{\mu\nu}\kappa_\nu\,d\Omega. \tag{4}$$

Here $\int_{\kappa}$ denotes integration over the sphere of intersection of the light cone at τ with the tube $\kappa =$ constant. This sphere will be called the 'retarded sphere' of radius κ at τ. In the appendix (equation (98)) it is shown that

$$\kappa\frac{d}{d\kappa}\int_{\kappa} T^{\mu\nu}\kappa_\nu\frac{s^\alpha}{\kappa}\frac{s^\beta}{\kappa}\cdots\frac{s^\lambda}{\kappa}\,d\Omega = \int_{\kappa}\left\{\frac{\partial}{\partial x^\sigma}\left(T^{\mu\nu}s^\sigma - T^{\mu\sigma}s^\nu\right)\right\}\kappa_\nu\frac{s^\alpha}{\kappa}\frac{s^\beta}{\kappa}\cdots\frac{s^\lambda}{\kappa}\,d\Omega, \tag{5}$$

where α, β, ..., λ are any number of free indices. Now introduce two operators D and θ defined in the following way:

$$D = \kappa\frac{d}{d\kappa}, \tag{6a}$$

$$\begin{aligned}\theta U^{\alpha\beta\ldots\lambda\nu} &= \frac{\partial}{\partial x^\sigma}\left(U^{\alpha\beta\ldots\lambda\nu}s^\sigma - U^{\alpha\beta\ldots\lambda\sigma}s^\nu\right)\\ &= 2U^{\alpha\beta\ldots\nu} + U^{\alpha\beta\ldots\sigma}\frac{s_\sigma}{\kappa}v^\nu + s^\sigma\frac{\partial U^{\alpha\beta\ldots\nu}}{\partial x^\sigma} - s^\nu\frac{\partial U^{\alpha\beta\ldots\sigma}}{\partial x^\sigma},\end{aligned} \tag{6b}$$

where $U^{\alpha\beta\ldots\lambda\nu}$ is any arbitrary tensor with any number of indices. It should be noted that

$$\frac{\partial}{\partial x^\nu}\left(\theta U^{\alpha\beta\ldots\lambda\nu}\right) = 0 \tag{7}$$

owing to the antisymmetry in ν, σ of the expression inside the brackets in (6b). Obviously both D and θ are linear operators. Also $D\kappa^n = n\kappa^n$ and therefore by repetition $D^r\kappa^n = n^r\kappa^n$, so that if $f(D)$ is a polynomial expressible in powers of D, $f(D)\,\kappa^n = f(n)\,\kappa^n$. This holds whether n is positive or negative.

Using D and θ, (5) can be written in the form

$$D\int_{\kappa} T^{\mu\nu}\kappa_\nu\frac{s^\alpha}{\kappa}\cdots\frac{s^\lambda}{\kappa}\,d\Omega = \int_{\kappa}\theta T^{\mu\nu}\,.\,\kappa_\nu\frac{s^\alpha}{\kappa}\cdots\frac{s^\lambda}{\kappa}\,d\Omega.$$

Since $\theta T^{\mu\nu}$ is conserved with respect to ν as $T^{\mu\nu}$ was, the same considerations as above can be applied to $\theta T^{\mu\nu}$ instead of $T^{\mu\nu}$ and hence

$$D^2 \int_\kappa T^{\mu\nu}\kappa_\nu \frac{s^\alpha}{\kappa}\cdots\frac{s^\lambda}{\kappa}d\Omega = D\int_\kappa (\theta T^{\mu\nu})\kappa_\nu \frac{s^\alpha}{\kappa}\cdots\frac{s^\lambda}{\kappa}d\Omega = \int_\kappa \theta^2 T^{\mu\nu}.\kappa_\nu \frac{s^\alpha}{\kappa}\cdots\frac{s^\lambda}{\kappa}d\Omega.$$

By repetition it can be proved that

$$D^r \int_\kappa T^{\mu\nu}\kappa_\nu \frac{s^\alpha}{\kappa}\cdots\frac{s^\lambda}{\kappa}d\Omega = \int_\kappa \theta^r T^{\mu\nu}.\kappa_\nu \frac{s^\alpha}{\kappa}\cdots\frac{s^\lambda}{\kappa}d\Omega,$$

and therefore
$$f(D)\int_\kappa T^{\mu\nu}\kappa_\nu \frac{s^\alpha}{\kappa}\cdots\frac{s^\lambda}{\kappa}d\Omega = \int_\kappa f(\theta)\, T^{\mu\nu}.\kappa_\nu \frac{s^\alpha}{\kappa}\cdots\frac{s^\lambda}{\kappa}d\Omega, \tag{8}$$

where $f(D)$ is a polynomial expressible in positive integral powers of D.

It will now be assumed that the part of $T^{\mu\nu}$ which becomes infinite as $\kappa\to 0$ can be expressed for sufficiently small values of κ in the form of a series involving a finite number of negative powers of κ, so that for sufficiently small κ one can write*

$$T^{\mu\nu} = T_0^{\mu\nu} + \sum_{r=1}^{r=m+2}\frac{a_{(-r)}^{\mu\nu}}{\kappa^r}, \tag{9}$$

where $T_0^{\mu\nu}$ remains finite as $\kappa\to 0$ and $a_{(-r)}^{\mu\nu}$ are functions of s^α/κ and τ only. It is assumed that these functions are finite and continuous. That the energy-momentum tensors of the electromagnetic and the meson fields actually satisfy these requirements is shown in the appendix. Therefore

$$\int_\kappa T^{\mu\nu}\kappa_\nu \frac{s^\alpha}{\kappa}\cdots\frac{s^\lambda}{\kappa}d\Omega = c_0^{\mu\alpha\ldots\lambda} + \sum_{r=1}^{r=m}\frac{c_{(-r)}^{\mu\alpha\ldots\lambda}}{\kappa^r} \tag{10}$$

can be written, where $c_0^{\mu\alpha\ldots\lambda}$ is a function of κ and τ which remains finite as $\kappa\to 0$ and $c_{(-r)}^{\mu\alpha\ldots\lambda}$ are functions of τ only. It is assumed that for sufficiently small κ, $c_0^{\mu\alpha\ldots\lambda}$ is expressible as a series in positive powers of κ. This also is true for the electromagnetic and meson fields.

Applying the operator $f(D)$ to both the sides of (10),

$$f(D)\int_\kappa T^{\mu\nu}\kappa_\nu \frac{s^\alpha}{\kappa}\cdots\frac{s^\lambda}{\kappa}d\Omega = f(D)\,c_0^{\mu\alpha\ldots\lambda} + \sum_{r=1}^{r=m} f(-r)\frac{c_{(-r)}^{\mu\alpha\ldots\lambda}}{\kappa^r}. \tag{11}$$

Taking
$$f(D) = \prod_{k=1}^{k=m}\left(1+\frac{D}{k}\right), \tag{12}$$

we get $f(-r) = 0$ for $1\leqslant r\leqslant m$.

From (11) and (8) it follows that

$$\prod_{k=1}^{k=m}\left(1+\frac{D}{k}\right)\int_\kappa T^{\mu\nu}\kappa_\nu \frac{s^\alpha}{k}\cdots\frac{s^\lambda}{\kappa}d\Omega = \int_\kappa \left\{\prod_{k=1}^{k=m}\left(1+\frac{\theta}{k}\right)T^{\mu\nu}\right\}\kappa_\nu \frac{s^\alpha}{\kappa}\cdots\frac{s^\lambda}{\kappa}d\Omega$$
$$= \prod_{k=1}^{k=m}\left(1+\frac{D}{k}\right)c_0^{\mu\alpha\ldots\lambda}. \tag{13}$$

* The highest singularity is written as of the order $m+2$ for convenience. The removal of only those singularities from $T^{\mu\nu}$ is necessary which are of an order higher than the 2nd, since singularities of the 2nd or lesser order give a finite energy integral (Pryce 1938).

Since the right side of this equation is non-singular, the left must be so, i.e. $\prod_{k=1}^{k=m}\left(1+\frac{\theta}{k}\right)T^{\mu\nu}$ gives a finite integral.

From this it follows that
$$\left\{\prod_{k=1}^{k=m}\left(1+\frac{\theta}{k}\right)T^{\mu\nu}\right\}\kappa_\nu \tag{14}$$

cannot contain any singularity of an order higher than the 2nd, because if it contained a term of the type b^μ/κ^{r+2} (where $r>0$), this would give after integration a singular term of the type $c^{\mu\alpha\ldots\lambda}/\kappa^r$, where

$$c^{\mu\alpha\ldots\lambda} = \int b^\mu \frac{s^\alpha}{\kappa}\cdots\frac{s^\lambda}{\kappa}d\omega \quad \left[d\omega = \frac{d\Omega}{\kappa^2} = \text{the element of solid angle}\right].$$

Since such a term is absent from the right side of (13) it follows that

$$c^{\mu\alpha\ldots\lambda} = \int b^\mu \frac{s^\alpha}{\kappa}\cdots\frac{s^\lambda}{\kappa}d\omega = 0.$$

As in the rest system s^α/κ are nothing else than the direction cosines (except s^0/κ which is 1), it follows from the expansibility of every continuous function defined on the surface of a sphere, in terms of surface harmonics, that $b^\mu = 0$. Thus (14) does not contain any singularities higher than of the 2nd order.

Because
$$\prod_{k=1}^{k=m}\left(1+\frac{\theta}{k}\right) = 1+a\theta+b\theta^2+\ldots = 1+\theta(a+b\theta+\ldots), \tag{15}$$

where a, b, ... are numerical coefficients, it is clear that

$$\prod_{k=1}^{k=m}\left(1+\frac{\theta}{k}\right)T^{\mu\nu} = T^{\mu\nu}+\theta(a+b\theta+\ldots)\,T^{\mu\nu} = T^{\mu\nu}+\theta K^{\mu\nu},$$

where
$$K^{\mu\nu} \equiv (a+b\theta+\ldots)\,T^{\mu\nu}.$$

Further, as $\theta K^{\mu\nu} = \partial K^{\mu\nu\sigma}/\partial x^\sigma$, where $K^{\mu\nu\sigma} = K^{\mu\nu}s^\sigma - K^{\mu\sigma}s^\nu$, the modified tensor

$$\prod_{k=1}^{k=m}\left(1+\frac{\theta}{k}\right)T^{\mu\nu} = T^{\mu\nu}+\frac{\partial K^{\mu\nu\sigma}}{\partial x^\sigma},$$

which is just of the required form. Put

$$\prod_{k=1}^{k=m}\left(1+\frac{\theta}{k}\right)T^{\mu\nu} = T'^{\mu\nu}. \tag{16}$$

It should be noted in passing that it can be shown directly that $T'^{\mu\nu}$ does not contain singularities of an order higher than the 3rd, because, as shown in the appendix (equation (102)),

$$D\int_\kappa T^{\mu\sigma}(\delta^\nu_\sigma - v_\sigma v^\nu)\frac{s^\alpha}{\kappa}\cdots\frac{s^\lambda}{\kappa}d\Omega = \int_\kappa \theta T^{\mu\sigma}.\,(\delta^\nu_\sigma - v_\sigma v^\nu)\frac{s^\alpha}{\kappa}\cdots\frac{s^\lambda}{\kappa}d\Omega.$$

Hence
$$\int_\kappa T'^{\mu\sigma}(\delta^\nu_\sigma - v_\sigma v^\nu)\frac{s^\alpha}{\kappa}\cdots\frac{s^\lambda}{\kappa}d\Omega = \prod_{k=1}^{k=m}\left(1+\frac{D}{k}\right)\int_\kappa T^{\mu\sigma}(\delta^\nu_\sigma - v_\sigma v^\nu)\frac{s^\alpha}{\kappa}\cdots\frac{s^\lambda}{\kappa}d\Omega. \tag{17}$$

Through the same reasoning as used before it can be shown that, since the right side of (17) is non-singular, so also must be the left. From this it can be inferred exactly as in the case of $T'^{\mu\nu}\kappa_\nu$ that $T'^{\mu\sigma}(\delta^\nu_\sigma - v_\sigma v^\nu)$ does not contain singularities of an order higher than the 2nd.

It is further found (appendix equation (100)) that

$$D\int_\kappa T^{\mu\nu}s_\nu \frac{s^\alpha}{\kappa}\cdots\frac{s^\lambda}{\kappa}\,d\Omega = \int_\kappa \theta T^{\mu\nu}.\,s_\nu\frac{s^\alpha}{\kappa}\cdots\frac{s^\lambda}{\kappa}\,d\Omega.$$

Hence
$$\int_\kappa T'^{\mu\nu}s_\nu \frac{s^\alpha}{\kappa}\cdots\frac{s^\lambda}{\kappa}\,d\Omega = \prod_{k=1}^{k=m}\left(1+\frac{D}{k}\right)\int_\kappa T^{\mu\nu}s_\nu\frac{s^\alpha}{\kappa}\cdots\frac{s^\lambda}{\kappa}\,d\Omega. \tag{18}$$

From this one infers precisely as before that $T'^{\mu\nu}s_\nu$ does not contain singularities higher than of the 2nd order, so that $T'^{\mu\nu}\dfrac{s_\nu}{\kappa}$ or $T'^{\mu\nu}\dfrac{s_\nu}{\kappa}(1-\kappa')$ cannot contain singularities higher than $1/\kappa^3$. Since it has also been proved that

$$T'^{\mu\nu}\kappa_\nu = T'^{\mu\nu}\left\{v_\nu - \frac{s_\nu}{\kappa}(1-\kappa')\right\}$$

does not have singularities higher than of the 2nd order, it follows that $T'^{\mu\nu}v_\nu$ does not possess terms of order higher than $1/\kappa^3$. Now write $T'^{\mu\nu}$ in the form

$$T'^{\mu\nu} = T'^{\mu\sigma}(\delta^\nu_\sigma - v_\sigma v^\nu) + (T'^{\mu\sigma}v_\sigma)\,v^\nu.$$

The first term does not contain terms of order higher than $1/\kappa^2$ and the second of order higher than $1/\kappa^3$. So the highest singularity in $T'^{\mu\nu}$ can at most be of the order $1/\kappa^3$.

3. We now proceed to examine in detail the process of removal of the singular terms of $T^{\mu\nu}$ by the operator $\prod_{k=1}^{k=m}\left(1+\dfrac{\theta}{k}\right)$. In addition to θ we define another operator ϕ by the equation

$$\phi U^{\alpha\ldots\nu} = 2U^{\alpha\ldots\nu} + s^\sigma\frac{\partial U^{\alpha\ldots\sigma}}{\partial x^\sigma} + U^{\alpha\ldots\sigma}\frac{s_\sigma}{\kappa}v^\nu, \tag{19}$$

so that from (6*b*) and (19) is obtained

$$\theta U^{\alpha\ldots\nu} = \phi U^{\alpha\ldots\nu} - s^\nu\frac{\partial U^{\alpha\ldots\sigma}}{\partial x^\sigma}. \tag{20}$$

When $U^{\alpha\ldots\sigma}$ is conserved with respect to σ,

$$\theta U^{\alpha\ldots\nu} = \phi U^{\alpha\ldots\nu}. \tag{21}$$

While $\theta U^{\alpha\ldots\nu}$ is always conserved with respect to ν, $\phi U^{\alpha\ldots\nu}$ is conserved in general only when $U^{\alpha\ldots\nu}$ itself is conserved.

Let $T^{\mu\nu}_{(-n)}$ denote the nth order term in $T^{\mu\nu}$ so that in accordance with (9) $T^{\mu\nu}_{(-n)} = \dfrac{a^{\mu\nu}_{(-n)}}{\kappa^n}$. Then

$$\phi T^{\mu\nu}_{-n)} = 2T^{\mu\nu}_{(-n)} + s^\sigma\frac{\partial T^{\mu\nu}_{(-n)}}{\partial x^\sigma} + T^{\mu\nu}_{(-n)}\frac{s_\sigma}{\kappa}v^\nu.$$

As before assume $a^{\mu\nu}_{(-n)}$ to be a function of the arguments s^α/κ and τ only. Now both τ and s^α/κ are to be treated as constants with respect to the operator $s^\sigma \frac{\partial}{\partial x^\sigma}$, because $s^\sigma \frac{\partial \tau}{\partial x^\sigma} = 0$ and $s^\sigma \frac{\partial}{\partial x^\sigma}\left(\frac{s^\alpha}{\kappa}\right) = 0$, so that $s^\sigma \frac{\partial a^{\mu\nu}_{(-n)}}{\partial x^\sigma} = 0$. Therefore

$$s^\sigma \frac{\partial T^{\mu\nu}_{(-n)}}{\partial x^\sigma} = -n\frac{a^{\mu\nu}_{(-n)}}{\kappa^{n+1}}\kappa_\sigma s^\sigma = -n\frac{a^{\mu\nu}_{(-n)}}{\kappa^n} = -nT^{\mu\nu}_{(-n)}. \tag{22}$$

Thus $$\phi T^{\mu\nu}_{(-n)} = (2-n)\,T^{\mu\nu}_{(-n)} + T^{\mu\sigma}_{(-n)}\frac{s_\sigma}{\kappa}v^\nu = \left[(2-n)\,\delta^\nu_\sigma + v^\nu\frac{s_\sigma}{\kappa}\right]T^{\mu\sigma}_{(-n)}. \tag{23}$$

Now since $s^\rho \frac{\partial}{\partial x^\rho}\left(\frac{s_\sigma}{\kappa}v^\nu\right) = 0$, it follows that

$$\phi^2 T^{\mu\nu}_{(-n)} = \left[(2-n)\,\delta^\nu_\rho + v^\nu\frac{s_\rho}{\kappa}\right]\left[(2-n)\,\delta^\rho_\sigma + v^\rho\frac{s_\sigma}{\kappa}\right]T^{\mu\sigma}_{(-n)}.$$

By repetition $$\phi^l T^{\mu\nu}_{(-n)} = \left(\left[(2-n)\,\delta + v\frac{s}{\kappa}\right]^l\right)^\nu_\sigma T^{\mu\sigma}_{(-n)} \tag{24}$$

for any positive integer l, where

$$\left(\left[(2-n)\,\delta + v\frac{s}{\kappa}\right]^l\right)^\nu_\sigma$$
$$\equiv\left[(2-n)\,\delta^\nu_{\rho_1} + v^\nu\frac{s_{\rho_1}}{\kappa}\right]\left[(2-n)\,\delta^{\rho_1}_{\rho_2} + v^{\rho_1}\frac{s_{\rho_2}}{\kappa}\right]\dots\left[(2-n)\,\delta^{\rho_{l-1}}_\sigma + v^{\rho_{l-1}}\frac{s_\sigma}{\kappa}\right].$$

Put $$a\delta^\nu_\sigma + bv^\nu\frac{s_\sigma}{\kappa} \equiv \left[A(a,b)\right]^\nu_\sigma \tag{25a}$$

and $$\left[\left(a\delta + bv\frac{s}{\kappa}\right)^l\right]^\nu_\sigma \equiv \left[A^l(a,b)\right]^\nu_\sigma \tag{25b}$$

The indices will sometimes be omitted. For example, we shall write

$$\phi^l T_{(-n)} = A^l(2-n,1)\,T_{(-n)}, \tag{26a}$$

meaning thereby $$\phi^l T^{\mu\nu}_{(-n)} = \left[A^l(2-n,1)\right]^\nu_\sigma T^{\mu\sigma}_{(-n)}. \tag{26b}$$

It is also seen that $$cA(a,b) = A(ca,cb), \tag{27a}$$

$$A(a_1,b_1) + A(a_2,b_2) = A(a_1+a_2, b_1+b_2), \tag{27b}$$

and $$c = A(c,0), \tag{27c}$$

where c is an ordinary number. On using (26) and (27)

$$\prod_{k=1}^{k=m}\left(1+\frac{\phi}{k}\right)T_{(-n)} = \prod_{k=1}^{k=m}\left[1+\frac{A(2-n,1)}{k}\right]T_{(-n)} = \prod_{k=1}^{k=m}A\left(1+\frac{2-n}{k},\frac{1}{k}\right)T_{(-n)}. \tag{28}$$

Consider now $A(a_1, b_1)\, A(a_2, b_2)$. It is found that

$$\left[A(a_1,b_1)\,A(a_2,b_2)\right]^{\nu}_{\sigma} = \left(a_1\delta^{\nu}_{\rho} + b_1 v^{\nu}\frac{s_\rho}{\kappa}\right)\left(a_2\delta^{\rho}_{\sigma} + b_2 v^{\rho}\frac{s_\sigma}{\kappa}\right)$$

$$= a_1 a_2 \delta^{\nu}_{\sigma} + [(a_1+b_1)(a_2+b_2) - a_1 a_2]\, v^{\nu}\frac{s_\sigma}{\kappa}$$

$$= \left[A(a_1 a_2, [a_1+b_1][a_2+b_2] - a_1 a_2)\right]^{\nu}_{\sigma}.$$

It can easily be proved by induction that

$$A(a_1,b_1)\,A(a_2,b_2)\ldots A(a_n,b_n)$$
$$= A(a_1 a_2 \ldots a_n, [a_1+b_1][a_2+b_2]\ldots[a_n+b_n] - a_1 a_2 \ldots a_n). \quad (29)$$

Thus we get immediately

$$\prod_{k=1}^{k=m} A\left(1+\frac{2-n}{k}, \frac{1}{k}\right) = A\left(\prod_{k=1}^{k=m}\left(1+\frac{2-n}{k}\right),\ \prod_{k=1}^{k=m}\left(1+\frac{3-n}{k}\right) - \prod_{k=1}^{k=m}\left(1+\frac{2-n}{k}\right)\right). \quad (30)$$

Hence if $m \geqslant n-2$ and $n \geqslant 4$,

$$\prod_{k=1}^{k=m} A\left(1+\frac{2-n}{k}, \frac{1}{k}\right) = A(0,0) = 0.$$

Thus
$$\prod_{k=1}^{k=m}\left(1+\frac{\phi}{k}\right) T_{(-n)} = 0 \quad \text{for} \quad m+2 \geqslant n \geqslant 4.$$

Since $m+2$ has already been chosen equal to the order of the highest singularity in $T^{\mu\nu}$, it follows that the operator $\prod_{k=1}^{k=m}\left(1+\frac{\phi}{k}\right)$ when applied to $T^{\mu\nu}$ removes all singular terms higher than the 3rd. For the term of the 3rd order, from (28) and (30),

$$\prod_{k=1}^{k=m}\left(1+\frac{\phi}{k}\right) T_{(-3)} = A(0,1)\, T_{(-3)} = v^{\nu}\frac{s_\sigma}{\kappa} T^{\mu\sigma}_{(-3)}.$$

Thus
$$\prod_{k=1}^{k=m}\left(1+\frac{\phi}{k}\right) T^{\mu\nu} = \prod_{k=1}^{k=m}\left(1+\frac{\phi}{k}\right) T^{\mu\nu}_{-2)} + T^{\mu\sigma}_{(-3)}\frac{s_\sigma}{\kappa} v^{\nu},$$

where $T^{\mu\nu}_{-2)}$ denotes the part of $T^{\mu\nu}$ remaining after omitting terms of 3rd order or higher.*

* $T^{\mu\nu}_{-n)}$ denotes the part of $T^{\mu\nu}$ left over after omitting terms of order higher than the nth, and $T^{\mu\nu}_{(-n}$ the part obtained by retaining only terms of the nth or higher order, so that

$$T^{\mu\nu}_{-n+1)} + T^{\mu\nu}_{(-n} = T^{\mu\nu}.$$

The suffixes $-n)$ and $(-n$ attached to any other quantity have analogous significance. Similarly the suffix $(-n)$ always denotes the term of the nth order.

Since $T^{\mu\nu}$ is conserved with respect to ν, $\phi T^{\mu\nu} = \theta T^{\mu\nu}$ and $\phi^l T^{\mu\nu} = \theta^l T^{\mu\nu}$, so that

$$T'^{\mu\nu} = \prod_{k=1}^{k=m}\left(1+\frac{\theta}{k}\right) T^{\mu\nu} = \prod_{k=1}^{k=m}\left(1+\frac{\phi}{k}\right) T^{\mu\nu} = \prod_{k=1}^{k=m}\left(1+\frac{\phi}{k}\right) T^{\mu\nu}_{-2)} + T^{\mu\sigma}_{(-3)}\frac{s_\sigma}{\kappa}v^\nu, \tag{31}$$

which is in conformity with the proved fact that $T'^{\mu\nu}$ does not contain singularities higher than of the 3rd order.

In accordance with (15)

$$\prod_{k=1}^{k=m}\left(1+\frac{\phi}{k}\right) = 1+\phi(a+b\phi+\ldots),$$

so that

$$\begin{aligned} T'^{\mu\nu} &= T^{\mu\nu}_{-2)} + \phi(a+b\phi+\ldots)\, T^{\mu\nu}_{-2)} + T^{\mu\sigma}_{(-3)}\frac{s_\sigma}{\kappa}v^\nu \\ &= T^{\mu\nu}_{-2)} + \theta(a+b\phi+\ldots)\, T^{\mu\nu}_{-2)} + T^{\mu\sigma}_{(-3)}\frac{s_\sigma}{\kappa}v^\nu + s^\nu\frac{\partial}{\partial x^\sigma}[(a+b\phi+\ldots)\, T^{\mu\sigma}_{-2)}]. \end{aligned} \tag{32}$$

A new tensor $T''^{\mu\nu}$ is now introduced, defined by

$$T''^{\mu\nu} \equiv T'^{\mu\nu} - \theta(a+b\phi+\ldots)\, T^{\mu\nu}_{-2)} = T^{\mu\nu}_{-2)} + T^{\mu\sigma}_{(-3)}\frac{s_\sigma}{\kappa}v^\nu + s^\nu\frac{\partial}{\partial x^\sigma}[(a+b\phi+\ldots)\, T^{\mu\sigma}_{-2)}]. \tag{33}$$

$T''^{\mu\nu}$ is obviously conserved and is of the form $T^{\mu\nu} + \dfrac{\partial K^{\mu\nu\sigma}}{\partial x^\sigma}$, where $K^{\mu\nu\sigma}$ is antisymmetrical in ν, σ and therefore $T''^{\mu\nu}$ instead of $T'^{\mu\nu}$ may be taken as the modified tensor.

It will now be shown that

$$\frac{\partial}{\partial x^\sigma}[(a+b\phi+\ldots)\, T^{\mu\sigma}_{-2)}] = a\left[\frac{\partial T^{\mu\sigma}_{(-2)}}{\partial x^\sigma}\right]_{(-3)} = a\frac{\partial T^{\mu\sigma}_{-2)}}{\partial x^\sigma} \tag{34}$$

It has been seen above that the order of a term is not changed by application of the operator ϕ. Hence $(a+b\phi+\ldots)\, T^{\mu\nu}_{-2)}$ contains no singularities higher than of the 2nd order. Since

$$T^{\mu\nu}_{(-3} = T^{\mu\nu} - T^{\mu\nu}_{-2)},$$

it follows from the conservation of $T^{\mu\sigma}$ and therefore of $(a+b\phi+\ldots)\, T^{\mu\sigma}$ that

$$\frac{\partial}{\partial x^\sigma}[(a+b\phi+\ldots)\, T^{\mu\sigma}_{-2)}] = -\frac{\partial}{\partial x^\sigma}[(a+b\phi+\ldots)\, T^{\mu\sigma}_{(-3}]. \tag{35}$$

Now

$$\begin{aligned} \frac{\partial T^{\mu\nu}_{(-n)}}{\partial x^\sigma} &= \frac{1}{\kappa^n}\left[\dot{a}^{\mu\nu}_{(-n)}\frac{s_\sigma}{\kappa} + a^{\mu\nu,\rho}_{(-n)}\frac{\partial}{\partial x^\sigma}\left(\frac{s_\rho}{\kappa}\right) - n\frac{a^{\mu\nu}_{(-n)}}{\kappa}\kappa_\sigma\right] \\ &= \frac{\dot{a}^{\mu\nu}_{(-n)}}{\kappa^n}\frac{s_\sigma}{\kappa} + \frac{a^{\mu\nu,\rho}_{(-n)}}{\kappa^{n+1}}\left\{g_{\rho\sigma} - \frac{s_\sigma}{\kappa}v_\rho - \frac{s_\rho}{\kappa}v_\sigma + \frac{s_\rho}{\kappa}\frac{s_\sigma}{\kappa}(1-\kappa')\right\} - n\frac{a^{\mu\nu}_{(-n)}}{\kappa^{n+1}}\left[v_\sigma - (1-\kappa')\frac{s_\sigma}{\kappa}\right], \end{aligned} \tag{36}$$

where

$$\dot{a}^{\mu\nu} = \frac{\partial a^{\mu\nu}_{(-n)}}{\partial\tau} \quad\text{and}\quad a^{\mu\nu,\rho} = \frac{\partial a^{\mu\nu}_{(-n)}}{\partial\left(\frac{s_\rho}{\kappa}\right)}, \tag{37}$$

treating $\tau, s_\rho/\kappa$ as independent variables. It follows therefore that on differentiation $T^{\mu\nu}_{(-n)}$ can give terms only of the nth and $n+1$th order. The same result is obviously applicable to terms of different orders in $(a+b\phi+\ldots)\,T^{\mu\nu}$. Thus the highest singularity in $\frac{\partial}{\partial x^\sigma}[(a+b\phi+\ldots)\,T^{\mu\nu}]$ is of the 3rd order, which is also the order of the lowest singularity in $\frac{\partial}{\partial x^\sigma}[(a+b\phi+\ldots)\,T^{\mu\sigma}_{(-3)}]$. From equation (35) it follows therefore that $\frac{\partial}{\partial x^\sigma}[(a+b\phi+\ldots)\,T^{\mu\sigma}_{(-2)}]$ contains only one term of the 3rd order, because the terms of lower order which it could contain do not exist on the right side of (35). Since only the 2nd order term in $(a+b\phi+\ldots)\,T^{\mu\sigma}_{(-2)}$ can contribute to a 3rd order term in the divergence, it is sufficient to calculate the 2nd order term of $(a+b\phi+\ldots)\,T^{\mu\sigma}_{(-2)}$ and find out the 3rd order term that results from it on differentiation.

Now proceed to determine $(a+b\phi+\ldots)\,T^{\mu\nu}_{(-n)}$. From (26), (27) and (29) it follows immediately that

$$(a+b\phi+\ldots)\,T_{(-n)} = A(p,q)\,T_{(-n)}, \tag{38}$$

where p, q are some numerical numbers. However, on using (27)–(30) one gets

$$\begin{aligned}\prod_{k=1}^{k=m}\left(1+\frac{\phi}{k}\right)T_{(-n)} &= [1+\phi(a+b\phi+\ldots)]\,T_{(-n)} = T_{(-n)}+\phi A(p,q)\,T_{(-n)}\\ &= T_{(-n)}+A(2-n,1)\,A(p,q)\,T_{(-n)}\\ &= A(1+p(2-n),\ q(3-n)+p)\,T_{(-n)}\\ &= A\left[\prod_{k=1}^{k=m}\left(1+\frac{2-n}{k}\right),\ \prod_{k=1}^{k=m}\left(1+\frac{3-n}{k}\right)-\prod_{k=1}^{k=m}\left(1+\frac{2-n}{k}\right)\right]T_{(-n)}.\end{aligned}$$

Thus

$$1+p(2-n) = \prod_{k=1}^{k=m}\left(1+\frac{2-n}{k}\right),\quad q(3-n)+p = \prod_{k=1}^{k=m}\left(1+\frac{3-n}{k}\right)-\prod_{k=1}^{k=m}\left(1+\frac{2-n}{k}\right),$$

i.e.

$$p = \frac{1}{2-n}\left[\prod_{k=1}^{k=m}\left(1+\frac{2-n}{k}\right)-1\right], \tag{39a}$$

$$q = \frac{1}{3-n}\left[\prod_{k=1}^{k=m}\left(1+\frac{3-n}{k}\right)-\left(1+\frac{1}{2-n}\right)\prod_{k=1}^{k=m}\left(1+\frac{2-n}{k}\right)+\frac{1}{2-n}\right]. \tag{39b}$$

By passing to the limit $n\to 2$ it is easy to show that in this case $p = a = 1+\frac{1}{2}+\ldots+\frac{1}{m}$ and $q = m-a$. Therefore

$$\frac{\partial}{\partial x^\sigma}[(a+b\phi+\ldots)\,T^{\mu\sigma}_{(-2)}] = \left\{\frac{\partial}{\partial x^\sigma}\left[aT^{\mu\sigma}_{(-2)}+(m-a)\,T^{\mu\rho}_{(-2)}\frac{s_\rho}{\kappa}v^\sigma\right]\right\}_{(-3)}, \tag{40}$$

the suffix denoting that only the 3rd order term resulting after differentiation is to be retained. Now, since

$$\frac{\partial}{\partial x^\sigma}\left(\frac{s_\rho}{\kappa}v^\sigma\right) = 0, \tag{41}$$

it is found that

$$\frac{\partial}{\partial x^\sigma}\left[T^{\mu\rho}_{(-n)}\frac{s_\rho}{\kappa}v^\sigma\right]=\frac{s_\rho}{\kappa}v^\sigma\frac{\partial T^{\mu\nu}_{(-n)}}{\partial x^\sigma}=\frac{s_\rho}{\kappa}\left[\frac{\dot{a}^{\mu\rho}_{(-n)}}{\kappa^n}-\frac{a^{\mu\rho,\lambda}_{(-n)}}{\kappa^n}\frac{s_\lambda}{\kappa}\frac{\kappa'}{\kappa}-n\frac{a^{\mu\rho}_{(-n)}}{\kappa^n}\frac{\kappa'}{\kappa}\right], \tag{42}$$

where the various symbols have the same meaning as in (37). Since $\dot{a}^{\mu\rho}_{(-n)}$, $a^{\mu\rho,\lambda}_{(-n)}$, $a^{\mu\rho}_{(-n)}$ are finite functions of s^α/κ and τ only, it is obvious that $\frac{\partial}{\partial x^\sigma}\left[T^{\mu\rho}_{(-n)}\frac{s_\rho}{\kappa}v^\sigma\right]$ is of the nth order. So that

$$\left\{\frac{\partial}{\partial x^\sigma}\left[T^{\mu\rho}_{(-2)}\frac{s_\rho}{\kappa}v^\sigma\right]\right\}_{(-3)}=0. \tag{43}$$

By comparing (36) and (42) it is also seen that

$$\frac{s_\rho}{\kappa}v^\sigma\frac{\partial T^{\mu\nu}_{(-n)}}{\partial x^\sigma}=\left[\frac{\partial T^{\mu\nu}_{(-n)}}{\partial x^\rho}\right]_{(-n)}, \tag{44}$$

a result which will be used later. So from (33), (40) and (43)

$$T''^{\mu\nu}=T^{\mu\nu}_{-2)}+T^{\mu\sigma}_{(-3)}\frac{s_\sigma}{\kappa}v^\nu+s^\nu\left\{\frac{\partial T^{\mu\nu}_{(-2)}}{\partial x^\sigma}\right\}_{(-3)}a. \tag{45}$$

It should be observed that

$$\frac{\partial}{\partial x^\nu}\left[s^\nu\left\{\frac{\partial T^{\mu\sigma}_{(-2)}}{\partial x^\sigma}\right\}_{(-3)}\right]=3\left\{\frac{\partial T^{\mu\sigma}_{(-2)}}{\partial x^\sigma}\right\}_{(-3)}+s^\nu\frac{\partial}{\partial x^\nu}\left\{\frac{\partial T^{\mu\sigma}_{(-2)}}{\partial x^\sigma}\right\}_{(-3)}=0, \tag{46}$$

since $\left\{\frac{\partial T^{\mu\sigma}_{(-2)}}{\partial x^\sigma}\right\}_{(-3)}$ is of the 3rd order, and therefore by a relation analogous to (22)

$$s^\nu\frac{\partial}{\partial x^\nu}\left\{\frac{\partial T^{\mu\sigma}_{(-2)}}{\partial x^\sigma}\right\}_{(-3)}=-3\left\{\frac{\partial T^{\mu\sigma}_{(-2)}}{\partial x^\sigma}\right\}_{(-3)}.$$

Now since $T''^{\mu\nu}$ is conserved with respect to ν, as also $s^\nu\left\{\frac{\partial T^{\mu\sigma}_{(-2)}}{\partial x^\sigma}\right\}_{(-3)}$, it follows from (45) that

$$\frac{\partial T^{\mu\nu}_{-2)}}{\partial x^\nu}+\frac{\partial}{\partial x^\nu}\left[T^{\mu\sigma}_{(-3)}\frac{s_\sigma}{\kappa}v^\nu\right]=0. \tag{47}$$

Further, using again the relation expressed by (22)

$$\begin{aligned}\theta\left[T^{\mu\sigma}_{(-3)}\frac{s_\sigma}{\kappa}v^\nu\right]&=-s^\nu\frac{\partial}{\partial x^\rho}\left[T^{\mu\sigma}_{(-3)}\frac{s_\sigma}{\kappa}v^\rho\right]\\&=s^\nu\frac{\partial T^{\mu\rho}_{-2)}}{\partial x^\rho}\quad\text{from (47).}\end{aligned} \tag{48}$$

Employing an argument similar to that used in deriving (35) and (40) it can be shown that

$$\frac{\partial T^{\mu\rho}_{-2)}}{\partial x^\rho}=-\frac{\partial T^{\mu\rho}_{(-3}}{\partial x^\rho}=\left[\frac{\partial T^{\mu\rho}_{(-2)}}{\partial x^\rho}\right]_{(-3)}=-\left[\frac{\partial T^{\mu\rho}_{(-3)}}{\partial x^\rho}\right]_{(-3)} \tag{49}$$

Therefore from (45), (48) and (49)

$$T'''^{\mu\nu} \equiv T''^{\mu\nu} - \theta\left[aT^{\mu\sigma}_{(-3)}\frac{s_\sigma}{\kappa}v^\nu\right] = T^{\mu\nu}_{-2)} + T^{\mu\sigma}_{(-3)}\frac{s_\sigma}{\kappa}v^\nu. \tag{50}$$

It is easy to see from (15), (16), (33) and (50) that

$$T'''^{\mu\nu} = T^{\mu\nu} + \theta K^{\mu\nu};$$

where $$\theta K^{\mu\nu} = \theta(a+b\phi+\ldots)\,T^{\mu\nu} - \theta(a+b\phi+\ldots)\,T^{\mu\nu}_{-2)} - \theta\left[aT^{\mu\sigma}_{(-3)}\frac{s_\sigma}{\kappa}v^\nu\right]$$

$$= \theta(a+b\phi+\ldots)\,T^{\mu\nu}_{(-3} - \theta\left[aT^{\mu\sigma}_{(-3)}\frac{s_\sigma}{\kappa}v^\nu\right].$$

We can therefore write

$$K^{\mu\nu} = (a+b\phi+\ldots)\,T^{\mu\nu}_{(-3} - aT^{\mu\sigma}_{(-3)}\frac{s_\sigma}{\kappa}v^\nu.$$

Using (39) one gets explicitly

$$K^{\mu\nu} = \left[T^{\mu\nu}_{(-3)} - T^{\mu\sigma}_{(-3)}\frac{s_\sigma}{\kappa}v^\nu\right] + \sum_{n=4}^{n=m}\left[\frac{T^{\mu\nu}_{(-n)}}{n-2} + \frac{T^{\mu\sigma}_{(-n)}\,s_\sigma/\kappa\,.\,v^\nu}{(n-2)\,(n-3)}\right]. \tag{51}$$

Thus $$T'''^{\mu\nu} = T^{\mu\nu}_{-2)} + T^{\mu\sigma}_{(-3)}\frac{s_\sigma}{\kappa}v^\nu = T^{\mu\nu} + \frac{\partial}{\partial x^\sigma}\left[K^{\mu\nu}s^\sigma - K^{\mu\sigma}s^\nu\right], \tag{52}$$

and is therefore of the required form.

Now $T'''^{\mu\nu}\kappa_\nu = T^{\mu\nu}_{-2)}\kappa_\nu + T^{\mu\sigma}_{(-3)}\frac{s_\sigma}{\kappa}\kappa'$ and therefore it possesses terms only up to the 2nd order, thus giving a finite integral over the world tube.

Since $$\frac{\partial}{\partial x^\rho} = \left(\delta^\sigma_\rho - \frac{s_\rho}{\kappa}v^\sigma\right)\frac{\partial}{\partial x^\sigma} + \frac{s_\rho}{\kappa}v^\sigma\frac{\partial}{\partial x^\sigma}, \tag{53}$$

it follows from (36) and (44) that

$$\left(\delta^\sigma_\rho - \frac{s_\rho}{\kappa}v^\sigma\right)\frac{\partial T^{\mu\nu}_{(-n)}}{\partial x^\sigma} = \left[\frac{\partial T^{\mu\nu}_{(-n)}}{\partial x^\rho}\right]_{(-n-1)}. \tag{54}$$

(44) and (54) hold not only for $T^{\mu\nu}_{(-n)}$ but also for any term of the nth order which is expressed as a/κ^n, where a is dependent on s^α/κ and τ only.

In view of (44) the conservation of $T'''^{\mu\nu}$ is easy to understand. Because, using (49),

$$\frac{\partial T'''^{\mu\nu}}{\partial x^\nu} = \left[\frac{\partial T^{\mu\nu}_{(-2)}}{\partial x^\nu}\right]_{(-3)} + \frac{\partial}{\partial x^\nu}\left[T^{\mu\sigma}_{(-3)}\frac{s_\sigma}{\kappa}v^\nu\right]$$

$$= \left[\frac{\partial T^{\mu\nu}_{(-2)}}{\partial x^\nu}\right]_{(-3)} + \frac{s_\sigma}{\kappa}v^\nu\frac{\partial T^{\mu\sigma}_{(-3)}}{\partial x^\nu} \quad \text{from (41)}$$

$$= \left[\frac{\partial T^{\mu\nu}_{(-2)}}{\partial x^\nu}\right]_{(-3)} + \left[\frac{\partial T^{\mu\nu}_{(-3)}}{\partial x^\nu}\right]_{(-3)} = 0 \quad \text{from (44) and (49).}$$

Precisely in the same way it can be proved that

$$T^{\mu\nu}_{-n)} + T^{\mu\sigma}_{(-n-1)} \frac{s_\sigma}{\kappa} v^\nu \tag{55}$$

is conserved with respect to ν. Further it can be shown that if $n \geqslant 2$

$$T^{\mu\nu}_{-n)} + T^{\mu\sigma}_{(-n-1)} \frac{s_\sigma}{\kappa} v^\nu = T^{\mu\nu} + \frac{\partial K^{\mu\nu\sigma}}{\partial x^\sigma},$$

where $K^{\mu\nu\sigma}$ is a tensor antisymmetrical in ν, σ. This follows from the fact that

$$S^{\mu\nu} \equiv -T'''^{\mu\nu} + T^{\mu\nu}_{-n)} + T^{\mu\sigma}_{(-n-1)} \frac{s_\sigma}{\kappa} v^\nu$$

$$= \left(\delta^\nu_\sigma - \frac{s_\sigma}{\kappa} v^\nu\right) T^{\mu\sigma}_{(-3)} + T^{\mu\nu}_{(-4)} + \ldots + T^{\mu\nu}_{(-n)} + T^{\mu\sigma}_{(-n-1)} \frac{s_\sigma}{\kappa} v^\nu$$

is conserved. It is then obvious that exactly as in the case of $T^{\mu\nu}$ a tensor

$$S'''^{\mu\nu} = S^{\mu\nu}_{-2)} + S^{\mu\sigma}_{(-3)} \frac{s_\sigma}{\kappa} v^\nu = S^{\mu\nu} + \frac{\partial K'^{\mu\nu\sigma}}{\partial x^\sigma}$$

can be formed, where $K'^{\mu\nu\sigma}$ is a suitable tensor antisymmetrical in ν, σ. But since $S^{\mu\nu}$ does not contain terms of the 2nd order or lower, and

$$S^{\mu\rho}_{(-3)} \frac{s_\rho}{\kappa} v^\nu = v^\nu \frac{s_\rho}{\kappa} \left(\delta^\rho_\sigma - \frac{s_\sigma}{\kappa} v^\rho\right) T^{\mu\sigma}_{(-3)} = 0,$$

then $S'''^{\mu\nu} = 0$, i.e. $S^{\mu\nu} = -\dfrac{\partial K'^{\mu\nu\sigma}}{\partial x^\sigma}$, so that

$$T^{\mu\nu}_{(-n)} + T^{\mu\sigma}_{(-n-1)} \frac{s_\sigma}{\kappa} v^\nu = T'''^{\mu\nu} - \frac{\partial K'^{\mu\nu\sigma}}{\partial x^\sigma} = T^{\mu\nu} + \frac{\partial K^{\mu\nu\sigma}}{\partial x^\sigma}.$$

Thus it is found that there exist a large number of tensors formed according to the simple rule (55) and capable of being put in the form (1) in which the order k of the highest singularity can be any number such that $3 \leqslant k \leqslant m$.

4. It will now be proved that the 3rd order term in $T'''^{\mu\nu}$ does not give rise to a singularity in the energy-momentum integral, in the case of the electromagnetic and meson fields.

It has already been shown that $T'''^{\mu\nu}\kappa_\nu$ has terms only up to the order $1/\kappa^2$, even though $T'''^{\mu\nu}$ contains terms up to the order $1/\kappa^3$. If equation (4) is rewritten for $T'''^{\mu\nu}$, then

$$\frac{d}{d\tau}\left(\int_\kappa T'''^{\mu\nu} \frac{s_\nu}{\kappa} d\Omega\right) = -\frac{d}{d\kappa}\int_\kappa T'''^{\mu\nu}\kappa_\nu d\Omega.$$

Since the right side of this equation is finite the left side must also be so. It is, however, evident from (50) that the left side contains a singular term of the order $1/\kappa$ which is

$$\frac{d}{d\tau}\left(\int_{\kappa} T^{\mu\sigma}_{(-3)}\frac{s_{\sigma}}{\kappa}d\Omega\right). \tag{56}$$

Therefore this term must be put equal to zero. Therefore

$$\int_{\kappa} T^{\mu\sigma}_{(-3)}\frac{s_{\sigma}}{\kappa}d\Omega = \frac{c^{\mu}}{\kappa}, \tag{57}$$

where c^{μ} are constants. The physical meaning of this is obvious. Because the rate of outflow of energy and momentum from the particle to the outer space, given by $\int_{\kappa\to 0} T'''^{\mu\nu}\kappa_{\nu}d\Omega$ is finite, any singularity in total field energy or field momentum must remain independent of τ.*

It is now assumed that for values of τ very far in the past the world-line is straight so that the particle (whose motion the world-line represents) was moving a long time back in a straight line. It is further assumed that at that time it had no rotational motion. Consider now the following motion of the particle. In the remote past it was moving uniformly in the absence of an external field. It is then acted upon by an external field which causes acceleration. Finally, the external field dies down and the particle has again in the distant future uniform unaccelerated motion but with a velocity entirely different from the one it had far back in the past. If such a motion is permissible, then since c^{μ} is independent of τ, it must remain the same in the initial and final stages of motion, i.e. it should be the same whatever the velocity of the particle in the course of its uniform motion. This means that c^{μ} is independent of the choice of the Lorentz-frame of reference. The only tensors which are so independent are 0 and $g^{\mu\nu}$. The latter alternative is obviously inadmissible. Therefore $c^{\mu} = 0$.

* $\int_{\kappa=\epsilon}^{\kappa=\eta} d\kappa \int_{\kappa} T^{\mu\sigma}\frac{s_{\sigma}}{\kappa}\,d\Omega$ is not really the energy and momentum contained inside the volume enclosed between the two spheres $\kappa=\epsilon$ and $\kappa=\eta$. Energy-momentum is in fact defined as the integral of $T^{\mu\nu}$ over a plane three-dimensional space-like surface. However, if in equation (10) of A we take S as the plane defined by $[x^{\mu}-z^{\mu}(\tau)]\,v_{\mu}(\tau)=\epsilon$ (which is nothing but the space of the rest system taken a time ϵ later than τ), it is easy to see that P intersects S in the retarded sphere of radius ϵ. Remembering that for the plane $dS_{\nu}=v_{\nu}(\tau)\,dV$, where dV is the element of three-dimensional volume, relation (10) of A for $T'''^{\mu\nu}$ can be written as

$$\frac{d}{d\tau}\int_{S} T'''^{\mu\nu}v_{\nu}dV = \int_{\kappa=\epsilon} T'''^{\mu\nu}\kappa_{\nu}d\Omega - T'''^{\mu}(Q),$$

where $T'''^{\mu}(Q)$ is the integral over the two-dimensional surface of intersection of S and the tube Q. It can be directly inferred from this that the singularity in $\int T'''^{\mu\nu}v_{\nu}dV$ is independent of τ.

In the case of the electromagnetic and mesic particles it is usually assumed that such a motion is possible.* Therefore

$$c^{\mu} = 0 \tag{58}$$

for the corresponding fields.† Thus the integral of $T'''^{\mu\nu}$ over the light cone

$$\left(\int_{\kappa_1}^{\kappa_2} d\kappa \int_{\kappa} T'''^{\mu\nu} \frac{s_{\nu}}{\kappa} d\Omega\right)_{\kappa_1 \to 0}$$

is finite. From this it will be proved that the integral of $T'''^{\mu\nu}$ over any finite space-like surface is finite, irrespective of the manner in which the point of singularity is approached. One need consider only the 3rd order term $T^{\mu\sigma}_{(-3)} \frac{s_{\sigma}}{\kappa} v^{\nu}$, since this is the only one which can give rise to a singularity in the integral.

Let the space-like surface be

$$\zeta^0(x_{\mu}) = \text{constant}, \tag{59}$$

and let it intersect the world-line at c. Using the notation of A, the integral is

$$\int T^{\mu\sigma}_{(-3)} \frac{s_{\sigma}}{\kappa} v^{\nu} \frac{\partial \zeta^0}{\partial x^{\nu}} \,|\, D \,|^{-1} d\zeta^1 d\zeta^2 d\zeta^3. \tag{60}$$

Take $\zeta^1 = x^1$, $\zeta^2 = x^2$, $\zeta^3 = x^3$, where x^1, x^2, x^3 are measured in the rest system at c. Then $D = \partial\zeta^0/\partial x^0$. Now let c be surrounded by a two-dimensional closed surface lying in (59) and given by the equation

$$\zeta^1(x_{\mu}) = \text{constant} = \epsilon, \tag{61}$$

such that as $\epsilon \to 0$ this surface tends to the point c. It will be assumed that for sufficiently small values of ϵ (61) is always entirely enclosed between the two two-dimensional surfaces resulting from the intersection of the two tubes

$$\kappa = \alpha\epsilon, \tag{62}$$

$$\kappa = \beta\epsilon, \tag{63}$$

with (59). Here α and β are constants ($\alpha < \beta$) independent of ϵ. Now calculate the integral (60) taken over the portion of (59) lying between (61) and (63). For points lying in this region, it follows from the Taylor expansion of v^{ν} that for sufficiently small values of ϵ

$$| v^{\nu} - (v^{\nu})_0 | < \lambda\epsilon,$$

where $(v^{\nu})_0$ is the velocity at the point c and λ is a constant independent of ϵ. Therefore without affecting the singular terms v^{ν} can be replaced by $(v^{\nu})_0$ in (60). Remembering that $(v^{\nu})_0 = (1, 0, 0, 0)$ and $D = \partial\zeta^0/\partial x^0$ one gets the integral as

$$\int_{\zeta^1=\epsilon}^{\kappa=\beta\epsilon} T^{\mu\sigma}_{(-3)} \frac{s_{\sigma}}{\kappa} dx^1 dx^2 dx^3. \tag{64}$$

* In fact it may even be necessary to demand that only such motions are permissible (Dirac 1938).

† This can also be proved directly, from the form of the energy-momentum tensor of the electomagnetic and the meson fields.

Put
$$T^{\mu\sigma}_{(-3)}\frac{s_\sigma}{\kappa} = \frac{a^{\mu\sigma}_{(-3)}}{\kappa^3}\frac{s_\sigma}{\kappa} \equiv \frac{a^\mu}{\kappa^3},$$

so that a^μ is a finite function of τ and s^α/κ only. Let $|a^\mu|_m$ be the maximum value of $|a^\mu|$ in the region of integration of (64). Then

$$\left|\int_{\zeta^1=\epsilon}^{\kappa=\beta\epsilon}\frac{a^\mu}{\kappa^3}dx^1dx^2dx^3\right| < \int_{\zeta^1=\epsilon}^{\kappa=\beta\epsilon}\left|\frac{a^\mu}{\kappa^3}\right| dx^1dx^2dx^3 < \frac{|a^\mu|_m}{(\alpha\epsilon)^3}\int_{\kappa=\alpha\epsilon}^{\kappa=\beta\epsilon}dx^1dx^2dx^3. \qquad (65)$$

The spatial volume enclosed between $\kappa = \alpha\epsilon$ and $\kappa = \beta\epsilon$ is obviously of the order ϵ^3. The right side of (65) is thus finite and hence so also must be the left. Now it is seen from figure 1, where AB is the world-line, L the light cone and Q any finite tube, that since the integrals of $T'''^{\mu\nu}$ over P and L are non-singular the integral over the portion of S intercepted between P and Q must also be so. It has just now been shown that the integral taken over the portion of S enclosed between P and $\zeta^1 = \epsilon$ remains finite as $\epsilon \to 0$. Therefore the integral over the portion intercepted between $\zeta^1 = \epsilon$ and Q also remains finite as $\epsilon \to 0$. It has thus been proved that the integral over S is finite, no matter how the point of singularity be approached.*

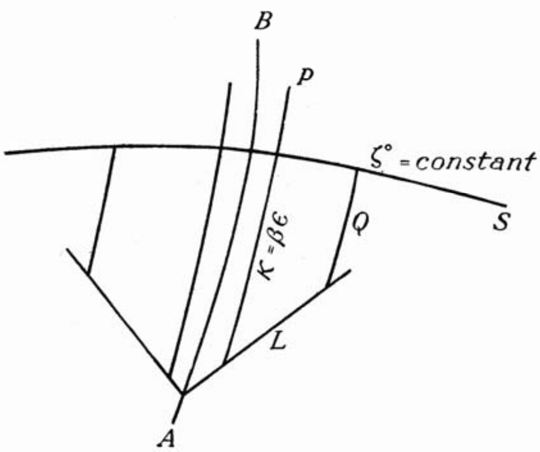

FIGURE 1

If $T'''^{\mu\nu}$ be integrated over the region of S enclosed between $\zeta^1 = \epsilon$ and $\zeta^1 = \eta$ ($\eta > \epsilon$), then from (52) and equation (20) of A it is seen that

$$\int_S T'''^{\mu\nu}dS_\nu = \int_S T^{\mu\nu}dS_\nu + \int_{\zeta^1=\eta}K^{\mu\nu\sigma}dS_{\nu\sigma} - \int_{\zeta^1=\epsilon}K^{\mu\nu\sigma}dS_{\nu\sigma}, \qquad (66)$$

where $dS_{\nu\sigma}$ is written for

$$\frac{\partial\zeta^0}{\partial x^\nu}\frac{\partial\zeta^1}{\partial x^\sigma}|D|^{-1}d\zeta^2d\zeta^3$$

* It is noteworthy that the energy-momentum integrals though finite are not unique. Their value may depend on the way the point of singularity is approached. The energy is therefore indeterminate to this extent.

and $K^{\mu\nu\sigma}$ for $K^{\mu\nu}s^{\sigma}-K^{\mu\sigma}s^{\nu}$. It has been shown that as $\epsilon\to 0$ the left side remains finite. Now $K^{\mu\nu\sigma}$ contains certain terms of the 2nd order, namely, those arising from the 3rd order term $T^{\mu\nu}_{(-3)}-T^{\mu\sigma}_{(-3)}\frac{s_\sigma}{\kappa}v^{\nu}$ in $K^{\mu\nu}$. These terms can be removed and form a tensor $\tilde{T}^{\mu\nu}$ given by

$$\tilde{T}^{\mu\nu} = T^{\mu\nu}+\frac{\partial}{\partial x^{\sigma}}[K'^{\mu\nu}s^{\sigma}-K'^{\mu\sigma}s^{\nu}], \tag{67}$$

where

$$K'^{\mu\nu} = \sum_{n=4}^{n=m}\left[\frac{T^{\mu\nu}_{(-n}}{n-2}+\frac{T^{\mu\sigma}_{(-n)}\frac{s_\sigma}{\kappa}v^{\nu}}{(n-2)(n-3)}\right]. \tag{68}$$

Since only 2nd order terms have been removed from $K^{\mu\nu\sigma}$ it is easily seen from (66) that $\tilde{T}^{\mu\nu}$ also gives a finite integral as $\epsilon\to 0$. But as $K'^{\mu\nu\sigma}$ now contains no term of an order less than the 3rd its integral over the surface $\zeta^1=\eta$ tends to zero as $\eta\to\infty$,* so that

$$\int_S \tilde{T}^{\mu\nu}dS_\nu = \int_S T^{\mu\nu}dS_\nu - \int_{\zeta^1=\epsilon} K'^{\mu\nu\sigma}dS_{\nu\sigma}, \tag{69}$$

where the three-dimensional integrals are to be taken over the entire space outside $\zeta^1=\epsilon$. Since the total energy and momentum obtained by making ϵ tend to zero remain finite, the inflow must also be finite.

For small values of κ when $T^{\mu\nu}$ can be expanded into terms of different orders with respect to κ, it can be proved without difficulty that

$$\tilde{T}^{\mu\nu} = T^{\mu\nu}_{-3)}+s^{\nu}\frac{\partial T^{\mu\sigma}_{-3)}}{\partial x^{\sigma}}, \tag{70}$$

where $T^{\mu\nu}_{-3)}$ is the part of $T^{\mu\nu}$ left over after omitting terms of order higher than $1/\kappa^3$.

5. All the above results except (58) can be applied to any arbitrary tensor $U^{\alpha\beta\ldots\nu}$ having any number of indices, provided it satisfies a conservation equation with respect to ν, and can be expanded for small values of κ in a series involving different powers of κ. The tensor $U'''^{\alpha\ldots\nu}$, and therefore $\tilde{U}^{\alpha\ldots\nu}$ derived from $U^{\alpha\ldots\nu}$ by the above process, may, in general, still contain a singularity of the order $1/\kappa$ in the integral

$$\int_\kappa \tilde{U}^{\alpha\ldots\nu}\frac{s_\nu}{\kappa}d\Omega. \tag{71}$$

This singularity, however, must be independent of τ. It is now possible to show that this singularity can be removed.

It is easily seen by actual calculation that

$$\left\{\frac{\partial}{\partial x^{\sigma}}\left[\frac{(v^{\nu}s^{\sigma}-v^{\sigma}s^{\nu})}{\kappa^3}e^{-b\kappa}\log h\kappa\right]\right\}\frac{s_\nu}{\kappa} = \frac{1}{\kappa^3}(1-b\kappa\log h\kappa)e^{-b\kappa}, \tag{72}$$

* Here it is assumed that for sufficiently large values of η, $\xi^1=\eta$ is completely contained in the space between two spheres of radius $\alpha\eta$ and $\beta\eta$ described about c in the rest system. (α and β are constants independent of η.) The total surface of $\xi^1=\eta$ is assumed to tend to infinity as η^2.

where b and h are positive constants. Therefore

$$\int_\kappa \frac{\partial}{\partial x^\sigma}\left[\frac{(v^\nu s^\sigma - v^\sigma s^\nu)}{\kappa^3} e^{-b\kappa}\log h\kappa\right]\frac{s_\nu}{\kappa}\, d\Omega = \frac{4\pi}{\kappa}(1 - b\kappa\log h\kappa)\, e^{-b\kappa}. \tag{73}$$

The singularity in (71) is of the form $c^{\alpha\cdots}/\kappa$, where $c^{\alpha\cdots}$ are certain constants. Therefore

$$\int_\kappa \underline{U}^{\alpha\ldots\nu}\frac{s_\nu}{\kappa}\, d\Omega = bc^{\alpha\cdots}\log h\kappa\, e^{-b\kappa}, \tag{74a}$$

where
$$\underline{U}^{\alpha\ldots\nu} = \tilde{U}^{\alpha\ldots\nu} - \frac{\partial}{\partial x^\sigma}\left[\frac{c^{\alpha\cdots}}{4\pi}\frac{(v^\nu s^\sigma - v^\sigma s^\nu)}{\kappa^3} e^{-b\kappa}\log h\kappa\right]. \tag{74b}$$

Even though $(74a)$ contains a singularity it can easily be seen that

$$\int_{\kappa=\epsilon}^{\kappa=\eta} d\kappa \int \underline{U}^{\alpha\ldots\nu}\frac{s_\nu}{\kappa}\, d\Omega$$

remains finite as $\epsilon \to 0$. Very much in the same manner as for $T'''^{\mu\nu}$ it can be shown that the integral of $\underline{U}^{\alpha\ldots\nu}$ over any space-like surface is finite no matter how the point of singularity be approached.

Further on account of $e^{-b\kappa}$ the surface integral of the term inside brackets in $(74b)$ vanishes on the infinite sphere, so that in an equation like (69) the integral may still be omitted on the surface at infinity. It is to be noted that b is entirely arbitrary except for the fact that it is greater than zero.

Now apply these considerations to the case of the angular momentum tensor $M^{\mu\nu,\sigma}$ defined by

$$M^{\mu\nu,\sigma} = x^\mu T^{\nu\sigma} - x^\nu T^{\mu\sigma},$$

which is conserved with respect to σ, due to symmetry of the original tensor $T^{\mu\nu}$. One can therefore build $\tilde{M}^{\mu\nu,\sigma}$, which for small values of κ can be written as

$$\tilde{M}^{\mu\nu,\sigma} = M^{\mu\nu,\sigma}_{-3)} + s^\sigma \frac{\partial}{\partial x^\rho}(M^{\mu\nu,\rho}_{-3)}). \tag{75}$$

Since $x^\mu = z^\mu(\tau) + \dfrac{s^\mu}{\kappa}\kappa$ it is found that

$$M^{\mu\nu,\sigma}_{-3)} = x^\mu T^{\nu\sigma}_{-3)} - x^\nu T^{\mu\sigma}_{-3)} + s^\mu T^{\nu\sigma}_{(-4)} - s^\nu T^{\mu\sigma}_{(-4)}. \tag{76}$$

Therefore, using (57),

$$\int_\kappa \tilde{M}^{\mu\nu,\sigma}_{(-3)}\frac{s_\sigma}{\kappa}\, d\Omega = \frac{z^\mu c^\nu - z^\nu c^\mu}{\kappa} + \int (s^\mu T^{\nu\sigma}_{(-4)} - s^\nu T^{\mu\sigma}_{(-4)})\frac{s_\sigma}{\kappa}\, d\Omega \equiv \frac{c^{\mu\nu}}{\kappa}.$$

Thus a tensor $\underline{M}^{\mu\nu,\sigma}$ can be constructed defined by

$$\underline{M}^{\mu\nu,\sigma} = M^{\mu\nu,\sigma} + \frac{\partial}{\partial x^\rho}[\Theta^{\mu\nu,\sigma}s^\rho - \Theta^{\mu\nu,\rho}s^\sigma], \tag{77a}$$

where $$\Theta^{\mu\nu,\sigma} = \sum_{n=4}^{n=m} \left[\frac{M^{\mu\nu,\sigma}_{(-n)}}{n-2} + \frac{M^{\mu\nu,\rho}_{(-n)} \frac{s_\rho}{\kappa} v^\sigma}{(n-2)(n-3)} \right] - \frac{c^{\mu\nu}}{4\pi} v^\sigma \frac{e^{-b\kappa} \log h\kappa}{\kappa^3} \tag{77b}$$

and $$c^{\mu\nu} = (z^\mu c^\nu - z^\nu c^\mu) + \int (s^\mu T^{\nu\sigma}_{(-4)} s_\sigma - s^\nu T^{\mu\sigma}_{(-4)} s_\sigma)\, d\Omega, \tag{77c}$$

such that the total angular momentum calculated from it is always finite.

Now return to the special case of the meson and the electromagnetic fields. Precisely as in the case of c^μ (equation (58)) it can be shown that since $c^{\mu\nu}$ is antisymmetric in μ, ν while $g^{\mu\nu}$ is symmetric

$$c^{\mu\nu} = 0. \tag{78}$$

However, it is important to note that owing to the fact that the meson fields contain a factor $e^{-\chi\kappa}$ at infinity (Appendix, equation (116a)),* the angular-momentum integral as calculated from $M^{\mu\nu,\sigma}$ does not contain a singularity at infinity. In the electromagnetic case, however, a logarithmic singularity would appear at infinity if $c^{\mu\nu}$ did not vanish. However, as it is, it is found that the total angular momentum calculated from $\tilde{M}^{\mu\nu,\sigma}$ is free from singularity both at zero and infinity.†

All the above considerations are immediately applicable to an assembly of more than one particle, because near any particle the field due to all others is regular and can be included in the ingoing field. We have only to take instead of (67) and (68)

$$\tilde{T}^{\mu\nu} = T^{\mu\nu} + \sum_p \frac{\partial}{\partial x^\sigma} K_p^{\mu\nu\sigma}, \tag{78a}$$

where p refers to a particular particle,

$$K'^{\mu\nu\sigma}_p = K'^{\mu\nu}_p s^\sigma_p - K'^{\mu\sigma}_p s^\nu_p, \tag{78b}$$

$$K'^{\mu\nu}_p = \sum_{n=4}^{n=m} \left\{ \frac{T^{\mu\nu}_{p(-n)}}{n-2} + \frac{T^{\mu\sigma}_{p(-n)}}{(n-2)(n-3)} \frac{s_{\sigma,p}}{\kappa_p} v^\nu_p \right\}. \tag{78c}$$

* For large spatial distances from the particle (measured in the present space of any Lorentz-frame) the corresponding retarded point goes far back in the past of the particle, so that for sufficiently large distances the retarded point lies in the region of the world-line corresponding to uniform motion of the particle. The value of the fields at infinity may therefore be taken to be the same as in the static case. This is also evident from the consideration that since potentials and therefore fields are propagated with a velocity at most equal to that of light, the value of these quantities at sufficiently large distances must still be unaffected by radiation from the particle. The external field of course is supposed to vanish sufficiently fast at infinity.

† In this connexion it may be mentioned that a misprint has occurred in the expression for $F^{(2)}_{\mu\nu}$ given in the paper by Bhabha & Corben (1941). In equation (114) of their paper, instead of the term

$$\left[-2v_\mu s^\rho \frac{S_{\rho\nu}}{\kappa^3} \right]_-$$

it should read $$\left[-2v_\mu s^\rho \frac{\dot{S}_{\rho\nu}}{\kappa^3} \right]_-.$$

Here s_p^σ and v_p^ν refer to the particular particle. $T^{\mu\nu}_{p(-n)}$ is the nth order term in $T^{\mu\nu}$ when it is expanded in the neighbourhood of the particle p (i.e. for small κ_p). Similarly, corresponding to (77)

$$\tilde{M}^{\mu\nu,\sigma} = M^{\mu\nu,\sigma} + \sum_p \frac{\partial}{\partial x^\rho}[\Theta_p^{\mu\nu,\sigma} s_p^\rho - \Theta_p^{\mu\nu,\rho} s_p^\sigma], \tag{79a}$$

$$\Theta_p^{\mu\nu,\sigma} = \sum_{n=4}^{n=m_p}\left[\frac{M^{\mu\nu,\sigma}_{p(-n)}}{n-2} + \frac{M^{\mu\nu,\sigma}_{p(-n)}}{(n-2)(n-3)}\frac{s_{\rho,p}}{\kappa_p}v_p^\sigma\right] \tag{79b}$$

can be written. Here $c_p^{\mu\nu} = 0$.

6. Some special remarks may now be made regarding the symmetrization of the modified tensor for the electromagnetic dipole. As already observed in § 1 this symmetrization is, however, of no particular importance.

As proved in the appendix (equation (124)) it is found that for the electromagnetic case

$$T^{\mu\nu\,\text{ret.}}_{(-3)} s_\nu = 0, \tag{80}$$

where $T^{\mu\nu\,\text{ret.}}$ has the same meaning as that in the paper of Bhabha & Corben (1941). Therefore

$$T'''^{\mu\nu\,\text{ret.}} = T^{\mu\nu\,\text{ret.}}_{-2)} \quad \text{from (50)}, \tag{81}$$

which is obviously symmetrical. Since in the static case $T^{\mu\nu}$ does not contain terms of order lower than $1/\kappa^4$ (appendix, equation (116b))* then

$$T^{\mu\nu}_{-2)\,\text{static}} = 0. \tag{82}$$

Therefore the integral of $T^{\mu\nu\,\text{ret.}}_{-2)}$ does not contain any singularities at infinity.† Using the symbols used by Bhabha & Corben (1941), we write

$$T^{\mu\nu} = T^{\mu\nu\,\text{ret.}} + T^{\mu\nu\,\text{mix.}} + T^{\mu\nu\,\text{in.}}. \tag{83}$$

Because both $T^{\mu\nu\,\text{ret.}}$ and $T^{\mu\nu\,\text{in.}}$ are separately conserved $T^{\mu\nu\,\text{mix.}}$ must also be conserved, and the procedure of the preceding pages can be applied to $T^{\mu\nu\,\text{mix.}}$. We construct

$$T'''^{\mu\nu\,\text{mix.}} = T^{\mu\nu\,\text{mix.}}_{-2)} + T^{\mu\sigma\,\text{mix.}}_{(-3)}\frac{s_\sigma}{\kappa}v^\nu. \tag{84}$$

Further

$$M'''^{\mu\nu,\sigma\,\text{mix.}} = (x^\mu T^{\nu\sigma\,\text{mix.}}_{-2)} - x^\nu T^{\mu\sigma\,\text{mix.}}_{-2)}) + (s^\mu T^{\nu\sigma\,\text{mix.}}_{(-3)} - s^\nu T^{\mu\sigma\,\text{mix.}}_{(-3)})$$
$$+ (z^\mu T^{\nu\rho\,\text{mix.}}_{(-3)} - z^\nu T^{\mu\rho\,\text{mix.}}_{(-3)})\frac{s_\rho}{\kappa}v^\sigma + (s^\mu T^{\nu\rho\,\text{mix.}}_{(-4)} - s^\nu T^{\mu\rho\,\text{mix.}}_{(-4)})\frac{s_\rho}{\kappa}v^\sigma. \tag{85}$$

From (84) and (85)

$$M'''^{\mu\nu,\sigma\,\text{mix.}} = x^\mu T'''^{\nu\sigma\,\text{mix.}} - x^\nu T'''^{\mu\sigma\,\text{mix.}} + \left[s^\mu\left\{T^{\nu\rho\,\text{mix.}}_{(-3)}\left(\delta^\sigma_\rho - \frac{s_\rho}{\kappa}v^\sigma\right) + T^{\nu\rho\,\text{mix.}}_{(-4)}\frac{s_\rho}{\kappa}v^\sigma\right\}\right]_-, \tag{86}$$

* The potentials contain terms of order $1/\kappa$ or higher. The fields therefore contain terms of order $1/\kappa^2$ or higher. Hence $T^{\mu\nu}$ does not contain terms of order lower than $1/\kappa^4$.

† Cf. footnote * on p. 159.

the minus sign at the end denoting the subtraction of terms got by exchange of μ, ν in the expression within the square brackets.

Since the highest singularity in the mixed tensor in the case of an electromagnetic dipole is of order $1/\kappa^3$, $T^{\nu\rho\,\text{mix.}}_{(-4)} = 0$. Using this fact and the conservation of $M'''^{\mu\nu,\sigma\,\text{mix.}}$ and $T'''^{\mu\sigma\,\text{mix.}}$, it is found from (86) that

$$0 = \frac{\partial M'''^{\mu\nu,\sigma\,\text{mix.}}}{\partial x^\sigma} = T'''^{\nu\mu\,\text{mix.}} - T'''^{\mu\nu\,\text{mix.}} + \frac{\partial}{\partial x^\sigma}\left[s^\mu T^{\nu\rho\,\text{mix.}}_{(-3)}\left(\delta^\sigma_\rho - \frac{s_\rho}{\kappa}v^\sigma\right)\right]_- . \tag{87}$$

Putting
$$L^{\mu\nu,\sigma} \equiv \left[s^\mu T^{\nu\rho\,\text{mix.}}_{(-3)}\left(\delta^\sigma_\rho - \frac{s_\rho}{\kappa}v^\sigma\right)\right]_- \tag{88}$$

it is found that
$$T'''^{\mu\nu\,\text{mix.}} - \frac{1}{2}\frac{\partial}{\partial x^\sigma}[L^{\mu\nu,\sigma} + L^{\sigma\mu,\nu} - L^{\nu\sigma,\mu}] \tag{89}$$

is symmetrical and is conserved (Pauli 1941). It is also in the form (1) as required. Further, since $L^{\mu\nu,\sigma}$ is of the order $1/\kappa^2$ the additional term does not introduce any singularity in the energy integral as is obvious from (66). Also, since $T^{\mu\nu\,\text{mix.}}_{(-3)}$ vanishes at infinity (due to the absence of the external field in the remote past),* the integral in (66) over the surface at infinity is zero.

Thus
$$\overline{T}^{\mu\nu} = T^{\mu\nu\,\text{ret.}}_{-2)} + T'''^{\mu\nu\,\text{mix.}} - \frac{1}{2}\frac{\partial}{\partial x^\sigma}[L^{\mu\nu,\sigma} + L^{\sigma\mu,\nu} - L^{\nu\sigma,\mu}] + T^{\mu\nu\,\text{in.}}$$

can be taken as the modified tensor. This tensor is symmetrical. That in the case of a point charge this symmetry is possible has already been shown by Pryce (1938). In this case

$$T^{\mu\nu\,\text{ret.}} = \frac{e^2}{4\pi}\left[-\frac{(1-\kappa')^2}{\kappa^6}s^\mu s^\nu + (1-\kappa')\frac{(v^\mu s^\nu + v^\nu s^\mu)}{\kappa^5} + \frac{(\dot{v}^\mu s^\nu + \dot{v}^\nu s^\mu) - (\dot{v})^2 s^\mu s^\nu - \frac{1}{2}g^{\mu\nu}}{\kappa^4}\right]. \tag{90}$$

Thus according to (51)

$$K^{\mu\nu} = \left(T^{\mu\nu}_{(-3)} - T^{\mu\sigma}_{(-3)}\frac{s_\sigma}{\kappa}v^\nu\right) + \frac{T^{\mu\nu}_{(-4)}}{2} + \frac{T^{\mu\sigma}_{(-4)}}{2\,.\,1}\frac{s_\sigma}{\kappa}v^\nu.$$

* For small κ the retarded potentials U_ν and the retarded field quantities $G_{\mu\nu}$ can be expanded in a series of positive and negative powers of κ, and the ingoing field in a Taylor's series with positive powers of κ

$$G^{\text{in.}}_{\mu\nu} = (G^{\text{in.}}_{\mu\nu})_{z_\rho(\tau_0)} + \left(\frac{\partial G^{\text{in.}}_{\mu\nu}}{\partial x^\sigma}\right)_{z_\rho(\tau_0)}\frac{s^\sigma}{\kappa}\kappa + \frac{1}{2}\left(\frac{\partial^2 G^{\text{in.}}_{\mu\nu}}{\partial x^\sigma \partial x^\lambda}\right)\frac{s^\sigma}{\kappa}\frac{s^\lambda}{\kappa}\kappa^2 + \ldots,$$

where the suffix $z_\rho(\tau_0)$ means that the corresponding quantities are to be evaluated at the retarded point. Therefore $T^{\mu\nu\,\text{mix.}}_{(-3)}$ consists of terms of the form $\frac{u\,.\,v}{\kappa^3}$, where u and v are both of order zero in κ and u comes from $G^{\text{ret.}}_{\mu\nu}$, while v comes from $G^{\text{in.}}_{\mu\nu}$. The factor in v having the suffix $z_\rho(\tau_0)$ is to be evaluated at the retarded point. At infinity $v = 0$ because the corresponding retarded point goes to the remote past and there the field quantities and their differentials vanish.

(In spite of the fact that $K^{\mu\nu}$ is taken instead of $K'^{\mu\nu}$ given by (68), the third term in (66) vanishes as $\eta \to \infty$ due to (82).*) So that

$$K^{\mu\nu\sigma} = K^{\mu\nu}s^{\sigma} - K^{\mu\sigma}s^{\nu}$$
$$= \frac{e^2}{4\pi}\left[(\tfrac{3}{4}-\kappa')\frac{s^{\mu}}{\kappa^5}(v^{\nu}s^{\sigma}-v^{\sigma}s^{\nu}) + s^{\mu}\frac{(\dot{v}^{\nu}s^{\sigma}-\dot{v}^{\sigma}s^{\nu})}{\kappa^4} - \frac{1}{4}\frac{(g^{\mu\nu}s^{\sigma}-g^{\mu\sigma}s^{\nu})}{\kappa^4}\right]. \tag{91}$$

It is interesting to see that this is different from Pryce's tensor which is more complicated. It is (Pryce 1938)

$$K^{\mu\nu,\sigma} = \frac{e^2}{4\pi}\left[-\tfrac{9}{4}\kappa'\frac{(v^{\nu}s^{\sigma}-v^{\sigma}s^{\nu})}{\kappa^5}s^{\mu} + \frac{(g^{\mu\nu}v^{\sigma}-g^{\mu\sigma}v^{\nu})}{4\kappa^3} - (1+2\kappa')\frac{(g^{\mu\nu}s^{\sigma}-g^{\mu\sigma}s^{\nu})}{4\kappa^4} + \frac{3(v^{\nu}s^{\sigma}-v^{\sigma}s^{\nu})}{4\kappa^4}v^{\mu} + \frac{3(\dot{v}^{\nu}s^{\sigma}-\dot{v}^{\sigma}s^{\nu})}{4\kappa^4}s^{\mu}\right].$$

The modified tensor obtained from both these expressions is symmetric. Our modified tensor is just equal to $T^{\mu\nu\,\mathrm{ret.}}_{(-2)}$ as is proved quite generally in (81).

In conclusion I wish to express my thanks to Professor H. J. Bhabha for valuable discussions, general guidance and suggested improvements in the writing of this paper.

Appendix

Consider now
$$\kappa\frac{d}{d\kappa}\int T^{\mu\nu}\kappa_{\nu}\,.\,\kappa^2 d\omega, \tag{92}$$

τ being kept constant during differentiation. Here $d\omega$ is an element of solid angle in the rest system and $d\Omega = \kappa^2 d\omega$. Since

$$\kappa = s_{\mu}v^{\mu} \equiv [x_{\mu} - z_{\mu}(\tau)]\,v^{\mu}(\tau),$$

on differentiation keeping τ constant

$$d\kappa = dx_{\mu}v^{\mu}. \tag{93a}$$

For constant τ
$$s^{\mu}dx_{\mu} = 0. \tag{93b}$$

These are the only two conditions which the infinitesimal vector which connects a point on the sphere κ to a corresponding point on the slightly larger sphere $\kappa + d\kappa$, need satisfy. The way in which the points on these two spheres may be put in correspondence is immaterial. Now take dx_{μ} along s_{μ} thus satisfying (93b) automatically. (93a) then gives

$$ds_{\mu} = dx_{\mu} = d\kappa\frac{s_{\mu}}{\kappa},$$

i.e.
$$\frac{ds_{\mu}}{d\kappa} = \frac{s_{\mu}}{\kappa}. \tag{94}$$

* Cf. footnote * on p. 159.

From (94) it is easily found that

$$\frac{d}{d\kappa}\left(\frac{s_\mu}{\kappa}\right) = 0, \tag{95}$$

from which it may be concluded that

$$\frac{d}{d\kappa}(d\omega) = 0 \tag{96}$$

as $d\omega$ is dependent on s_μ/κ only. Thus

$$\frac{d}{d\kappa}\int T^{\mu\nu}\kappa_\nu \,.\, \kappa^2 d\omega = \int \frac{\partial}{\partial x^\sigma}(T^{\mu\nu}\kappa_\nu)\frac{dx^\sigma}{d\kappa}\kappa^2 d\omega + 2\int T^{\mu\nu}\kappa_\nu \kappa\, d\omega.$$

Since $\quad \dfrac{\partial \kappa_\nu}{\partial x^\sigma}\dfrac{dx^\sigma}{d\kappa} = \dfrac{\kappa'}{\kappa}\dfrac{s_\nu}{\kappa}$ (from equations (12) of A, (94) and (95))

then

$$\kappa\frac{d}{d\kappa}\int T^{\mu\nu}\kappa_\nu\, d\Omega = \int\left[2T^{\mu\nu} + s^\sigma\frac{\partial T^{\mu\nu}}{\partial x^\sigma}\right]\kappa_\nu\, d\Omega + \int T^{\mu\nu}\frac{s_\nu}{\kappa}\kappa'\, d\Omega$$

$$= \int \frac{\partial}{\partial x^\sigma}[T^{\mu\nu}s^\sigma - T^{\mu\sigma}s^\nu]\,.\,\kappa_\nu\, d\Omega, \tag{97}$$

as $\quad \dfrac{\partial}{\partial x^\sigma}[T^{\mu\nu}s^\sigma - T^{\mu\sigma}s^\nu] = 2T^{\mu\nu} + s^\sigma\dfrac{\partial T^{\mu\nu}}{\partial x^\sigma} + T^{\mu\sigma}\dfrac{s_\sigma}{\kappa}v^\nu,$ (cf. equation (6*b*))

if $\dfrac{\partial T^{\mu\nu}}{\partial x^\nu} = 0$. Since $\dfrac{d}{d\kappa}\left(\dfrac{s_\mu}{\kappa}\right) = 0$ it is easy to prove further that

$$\kappa\frac{d}{d\kappa}\int T^{\mu\nu}\kappa_\nu\frac{s^\alpha}{\kappa}\cdots\frac{s^\lambda}{\kappa}d\Omega = \int\frac{\partial}{\partial x^\sigma}[T^{\mu\nu}s^\sigma - T^{\mu\sigma}s^\nu]\,.\,\kappa_\nu\frac{s^\alpha}{\kappa}\cdots\frac{s^\lambda}{\kappa}d\Omega, \tag{98}$$

as quoted in the text. Remembering (94) and (95) it can similarly be proved that

$$\frac{d}{d\kappa}\int T^{\mu\nu}s_\nu\frac{s^\alpha}{\kappa}\cdots\frac{s^\lambda}{\kappa}d\Omega = \frac{1}{\kappa}\int\left[3T^{\mu\nu} + s^\sigma\frac{\partial T^{\mu\nu}}{\partial x^\sigma}\right]s_\nu\frac{s^\alpha}{\kappa}\cdots\frac{s^\lambda}{\kappa}d\Omega, \tag{99}$$

so that

$$\kappa\frac{d}{d\kappa}\int T^{\mu\nu}s_\nu\frac{s^\alpha}{\kappa}\cdots\frac{s^\lambda}{\kappa}d\Omega = \int\left[2T^{\mu\nu} + s^\sigma\frac{\partial T^{\mu\nu}}{\partial x^\sigma} + T^{\mu\sigma}\frac{s_\sigma}{\kappa}v^\nu\right]s_\nu\frac{s^\alpha}{\kappa}\cdots\frac{s^\lambda}{\kappa}d\Omega$$

$$= \int\theta T^{\mu\nu}\,.\,s_\nu\frac{s^\alpha}{\kappa}\cdots\frac{s^\lambda}{\kappa}d\Omega. \tag{100}$$

In the same way it is found that

$$\frac{d}{d\kappa}\int T^{\mu\sigma}(\delta^\nu_\sigma - v_\sigma v^\nu)\frac{s^\alpha}{\kappa}\cdots\frac{s^\lambda}{\kappa}d\Omega = \frac{1}{\kappa}\int\left[2T^{\mu\sigma} + s^\rho\frac{\partial T^{\mu\sigma}}{\partial x^\rho}\right](\delta^\nu_\sigma - v_\sigma v^\nu)\frac{s^\alpha}{\kappa}\cdots\frac{s^\lambda}{\kappa}d\Omega, \tag{101}$$

because $\delta^\nu_\sigma - v_\sigma v^\nu$, which depends on τ alone, is to be treated as a constant here. Since $v^\sigma(\delta^\nu_\sigma - v_\sigma v^\nu) = 0$ then

$$\kappa\frac{d}{d\kappa}\int T^{\mu\sigma}(\delta^\nu_\sigma - v_\sigma v^\nu)\frac{s^\alpha}{\kappa}\cdots\frac{s^\lambda}{\kappa}d\Omega = \int\theta T^{\mu\sigma}\,.\,(\delta^\nu_\sigma - v_\sigma v^\nu)\frac{s^\alpha}{\kappa}\cdots\frac{s^\lambda}{\kappa}d\Omega. \tag{102}$$

The most general equation which the potentials U_ν of a meson field may have to satisfy is

$$\frac{\partial}{\partial x^\rho}\frac{\partial}{\partial x_\rho} U^\nu + \chi^2 U^\nu = 4\pi \frac{\partial}{\partial x^\alpha}\cdots\frac{\partial}{\partial x^\lambda}\Sigma^{\alpha\ldots\lambda,\nu}, \tag{103}$$

where $\Sigma^{\alpha\ldots\lambda,\nu}$ is a tensor describing the effect of a multipole on the field. In fact, in the most general case the right side is a sum of terms containing different numbers of differentiations, but it will be sufficient to consider one such term since the solution of this general case is just the sum of the solutions of each of the equations involving one term only. Since interest arises only in the case of point-particles we can write (cf. Bhabha 1941, p. 321)

$$\Sigma^{\alpha\ldots\lambda,\nu}_{(x_\rho)} = \int_{-\infty}^{\infty} d\tau\, S^{\alpha\ldots\lambda,\nu}_{(\tau)}\, \delta(x^0-z^0)\,\delta(x^1-z^1)\,\delta(x^2-z^2)\,\delta(x^3-z^3). \tag{104}$$

As is well known the solution for the retarded potentials is

$$U^{\nu\,\text{ret.}}_{(x_\rho)} = \frac{\partial}{\partial x^\alpha}\cdots\frac{\partial}{\partial x^\lambda}\left(\frac{S^{\alpha\ldots\lambda,\nu}}{\kappa}\right) - \chi\frac{\partial}{\partial x^\alpha}\cdots\frac{\partial}{\partial x^\lambda}\int_{-\infty}^{\tau_0} S^{\alpha\ldots\lambda,\nu}\frac{J_1(\chi u)}{u}\,d\tau, \tag{105}$$

where $u_\rho(\tau) = x_\rho - z_\rho(\tau)$, $u = (u_\rho u^\rho)^{\frac{1}{2}}$ and τ_0 is the retarded time given by $u(\tau_0) = 0$. Here the first term is just the one which alone appears for the electromagnetic field ($\chi = 0$). It is obviously expressible in a finite series involving terms of different orders in κ. The differentiations in the 2nd term act in two ways. First, on the limit of integration τ_0, and secondly, on the integrand. Bearing in mind that

$$\frac{\partial}{\partial x^\lambda}\left(\frac{J_n(\chi u)}{u^n}\right) = -\chi u_\lambda \frac{J_{n+1}(u)}{u^{n+1}} \tag{106}$$

and

$$\frac{\partial u_\lambda}{\partial x^\alpha} = g_{\lambda\alpha}, \tag{107}$$

then

$$\frac{\partial}{\partial x^\nu}\cdots\frac{\partial}{\partial x^\lambda}\left(\frac{J_1(\chi u)}{u}\right) \tag{108}$$

can be evaluated in terms of u^α and $\frac{J_n(\chi u)}{u^n}$. Also, since

$$u^\alpha(\tau_0) = s^\alpha \tag{109}$$

and

$$\left(\frac{J_n(\chi u)}{u^n}\right)_{\tau_0} = \left(\frac{J_n(\chi u)}{u^n}\right)_{u\to 0} = \frac{\chi^n}{2^n . n!}, \tag{110}$$

then the value of (108) at the retarded point can be expressed as a finite series in s^α involving terms of different orders in κ. It should be noted that

$$\frac{\partial}{\partial x^\mu}\int_{-\infty}^{\tau_0} S^{\alpha\ldots,\gamma}\frac{\partial}{\partial x^\nu}\cdots\frac{\partial}{\partial x^\lambda}\left(\frac{J_1(\chi u)}{u}\right)d\tau$$

$$= \frac{s_\mu}{\kappa}\left[S^{\alpha\ldots,\gamma}\frac{\partial}{\partial x^\nu}\cdots\frac{\partial}{\partial x^\lambda}\left(\frac{J_1(\chi u)}{u}\right)\right]_{\tau_0} + \int_{-\infty}^{\tau_0} S^{\alpha\ldots,\gamma}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\cdots\frac{\partial}{\partial x^\lambda}\left(\frac{J_1(\chi u)}{u}\right)d\tau. \tag{111}$$

Again the first term on the right and its differentials can be written as finite series in s^α containing terms of different orders in κ. Therefore the only term in $U^{\nu\,\mathrm{ret.}}$ which has to be considered is

$$\int_{-\infty}^{\tau_0} S^{\alpha\ldots\lambda,\nu}\frac{\partial}{\partial x^\alpha}\cdots\frac{\partial}{\partial x^\lambda}\left(\frac{J_1(\chi u)}{u}\right)d\tau. \tag{112}$$

As shown above this will consist of terms of the type

$$\int_{-\infty}^{\tau_0} S^{\alpha\ldots\lambda,\nu}u_\gamma\ldots u_\mu\frac{J_n(\chi u)}{u^n}\,d\tau. \tag{113}$$

It is noteworthy that all these terms are non-singular. They can be expanded in a series containing only terms of positive order in κ in the following way. Write $u_\rho \equiv x_\rho - z_\rho(\tau) = l_\rho - s_\rho$, where $l_\rho \equiv z_\rho(\tau)-z_\rho(\tau_0)$, so that $u^2 = l_\rho l^\rho - 2l^\rho s_\rho = l^2 - 2(ls)$, where $l = (l_\rho l^\rho)^{\frac{1}{2}}$ and $(ls) = (l_\rho s^\rho)$. Then

$$\frac{J_n(\chi u)}{u^n} = \frac{J_n(\chi\sqrt{[l^2-2(ls)]})}{(\sqrt{[l^2-2(ls)]})^n}. \tag{114}$$

Since $\dfrac{J_n(\chi l)}{l^n}$ is a continuous and differentiable function of l^2, (114) can be written in the form of a Taylor's series for sufficiently small values of s_ρ

$$\begin{aligned}\frac{J_n(\chi u)}{u^n} &= \frac{J_n(\chi l)}{l^n} + \sum_{p=1}^{\infty}\frac{[-2(ls)]^p}{p!}\left(\frac{d}{dl^2}\right)^p\frac{J_n(\chi l)}{l^n}\\ &= \frac{J_n(\chi l)}{l^n} + \sum_{p=1}^{\infty}\chi^p\frac{(ls)^p}{p!}\frac{J_{n+p}(\chi l)}{l^{n+p}}.\end{aligned} \tag{115}$$

It follows from (113) and (115) that the expansion of (105) in a series of s^α is possible. s^α are independent of τ and so can be put outside the integral sign. Since the integrands involve only $S^{\alpha\ldots\lambda,\nu}$ and l_λ which are functions of τ and τ_0 alone, the integral is a function of τ_0 alone. It is now possible to separate terms of different orders by writing s^α/κ instead of s^α and multiplying by a suitable power of κ to compensate it. Thus the expansion of $U^{\nu\,\mathrm{ret.}}$ in the required form is obtained. $G^{\mathrm{ret.}}_{\mu\nu}$, which involves one more differentiation, can be expanded in an entirely similar manner. Since the ingoing field and potentials are always expressible as such a series by Taylor's expansion, $T^{\mu\nu}$ can be written in the required form with different order terms separated. This result is used in (9) and (10) in the text.

As is well known, in the static case when $S^{\alpha\ldots\lambda,\nu}$ is constant we get

$$U^\nu = S^{ik\ldots m,\nu}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^k}\cdots\frac{\partial}{\partial x^m}\left(\frac{e^{-\chi\kappa}}{\kappa}\right). \tag{116a}$$

(Here the latin indices run from 1 to 3 only.) In the electromagnetic case, by putting $\chi = 0$, we get

$$U^\nu = S^{ik\ldots m,\nu}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^k}\cdots\frac{\partial}{\partial x^m}\left(\frac{1}{\kappa}\right). \tag{116b}$$

From (105) it is found that in the electromagnetic case the retarded potentials* ϕ^ν do not contain terms of order lower than $1/\kappa$. These satisfy the equation

$$\frac{\partial \phi^\nu}{\partial x^\nu} = 0. \tag{117}$$

Using the relations (44) and (54), it is found from (117) that

$$\frac{s_\nu}{\kappa} v^\sigma \frac{\partial \phi^\nu_{(-n)}}{\partial x^\sigma} + \left(\delta^\sigma_\nu - \frac{s_\nu}{\kappa} v^\sigma\right) \frac{\partial \phi^\nu_{(-n+1)}}{\partial x^\sigma} = 0 \quad \text{for} \quad n > 1 \tag{118a}$$

and

$$\frac{s_\nu}{\kappa} v^\sigma \frac{\partial \phi^\nu_{(-1)}}{\partial x^\sigma} = 0. \tag{118b}$$

Here $(-n)$ denotes the nth order term. Now calculate

$$[F_{\mu\nu(-n)} F^{\nu\rho}_{(-1)} + F_{\mu\nu(-1)} F^{\nu\rho}_{(-n)} + \tfrac{1}{2}\delta^\rho_\mu F_{\alpha\beta(-1)} F^{\alpha\beta}_{(-1)}]\, s_\rho, \tag{119}$$

where $F_{\mu\nu}$ are the retarded electromagnetic field strengths. From the relation

$$F_{\mu\nu} = \frac{\partial \phi_\nu}{\partial x^\mu} - \frac{\partial \phi_\mu}{\partial x^\nu} \tag{120}$$

it is found that

$$F_{\mu\nu(-n)} = \left[\frac{s_\mu}{\kappa} v^\sigma \frac{\partial \phi_{\nu(-n)}}{\partial x^\sigma} + \left(\delta^\sigma_\mu - \frac{s_\mu}{\kappa} v^\sigma\right) \frac{\partial \phi_{\nu(-n+1)}}{\partial x^\sigma}\right]_- \quad \text{for} \quad n > 1 \tag{121a}$$

and

$$F_{\mu\nu(-1)} = \left[\frac{s_\mu}{\kappa} v^\sigma \frac{\partial \phi_{\nu(-1)}}{\partial x^\sigma}\right]_- . \tag{121b}$$

Using (121b) and (118b) it is found that $F_{\mu\nu(-1)}\, s^\nu = 0$, so that the first term in (119) does not contribute anything. Now

$$\begin{aligned} F_{\mu\nu(-n)}\, s^\nu &= \left[\frac{s_\mu}{\kappa} s^\nu v^\sigma \frac{\partial \phi_{\nu(-n)}}{\partial x^\sigma} + s^\nu \left(\delta^\sigma_\mu - \frac{s_\mu}{\kappa} v^\sigma\right) \frac{\partial \phi_{\nu(-n+1)}}{\partial x^\sigma} - s^\sigma \frac{\partial \phi_{\mu(-n+1)}}{\partial x^\sigma}\right] \\ &= \left[(s^\nu \delta^\sigma_\mu - s_\mu g^{\sigma\nu}) \frac{\partial \phi_{\nu(-n+1)}}{\partial x^\sigma} - s^\sigma \frac{\partial \phi_{\mu(-n+1)}}{\partial x^\sigma}\right] \quad \text{from (118}a\text{)}, \end{aligned} \tag{122}$$

so that

$$F_{\mu\nu(-1)} F^{\nu\rho}_{(-n)}\, s_\rho = F_{\mu\nu(-1)} \left\{ s^\rho \frac{\partial \phi_{\rho(-n+1)}}{\partial x^\nu} - s^\rho \frac{\partial \phi^\nu{}_{(-n+1)}}{\partial x^\rho} \right\} \tag{123}$$

as $F_{\mu\nu(-1)}\, s^\nu = 0$. Also

$$\begin{aligned} F_{\alpha\beta(-n)} F^{\alpha\beta}_{(-1)} &= 2F^{\alpha\beta}_{(-1)} \left[\frac{s_\alpha}{\kappa} v^\sigma \frac{\partial \phi_{\beta(-n)}}{\partial x^\sigma} + \left(\delta^\sigma_\alpha - \frac{s_\alpha}{\kappa} v^\sigma\right) \frac{\partial \phi_{\beta(-n+1)}}{\partial x^\sigma}\right] \\ &= 2F^{\alpha\beta}_{(-1)} \frac{\partial \phi_{\beta(-n+1)}}{\partial x^\alpha}. \end{aligned}$$

* ϕ^ν is used to denote $U^{\nu\,\text{ret.}}$ for the electromagnetic field.

Thus (119) becomes

$$\left[\frac{s_\mu}{\kappa}v^\sigma\frac{\partial\phi_{\nu(-1)}}{\partial x^\sigma}\right]\left[s^\rho\frac{\partial\phi_{\rho(-n+1)}}{\partial x_\nu}-s^\rho\frac{\partial\phi^\nu_{(-n+1)}}{\partial x_\rho}\right]+s_\mu\left[\frac{s^\alpha}{\kappa}v^\sigma\frac{\partial\phi^\beta_{(-1)}}{\partial x^\sigma}\right]_-\frac{\partial\phi_{\beta(-n+1)}}{\partial x^\alpha}=0.$$

Thus the expression (119) vanishes. Since $4\pi T^{\mu\rho\,\text{ret.}}_{(-3)}\,s_\rho$ is just this expression with $n=2$, then

$$T^{\mu\nu\,\text{ret.}}_{(-3)}\,s_\nu=0. \tag{124}$$

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