TRANSIENTS IN NEGATIVE CONSTANT SERIES CIRCUITS.

By Lal C. Verman.

In a previous paper (Verman, *Proc. I.R.E.*, 1931, **19**, 676) the author has shown that the negative constants obtained by the Bartlett type circuits (A. C. Bartlett, *J.I.E.E.*, 1927, **65**, 373) act as negative constants not only under steady state conditions as described by Bartlett and others (Van der Pol, *Proc. I.R.E.*, 1930, **18**, 221), but also retain their negative character under transient conditions. George Crisson (*Bell System Tech. J.*, 1931, **10**, 485) has described two other schemes by means of which a negative impedance can be obtained provided its positive can be constructed. These are ascribed to R. C. Mathes and H. W. Dudley, and depend on the use of a distortionless amplifier of infinite input impedance. These too can be shown to be truly negative for transient states provided the amplifier used is absolutely distortionless; that is, its output voltage is at every instance proportional to its input voltage. (See appendix.)

Hence it may be asserted that in general for the study of transient as well as steady state currents the circuits containing the negative constants of either type may be treated like the conventional positive constants except for the change of the algebraic sign.

In the following are presented transient solutions of some simple series circuits containing combinations of positive and negative constants.

Let $\mathscr{R}$, $\mathscr{L}$, and $\mathscr{C}$ designate respectively resistance, inductance and capacitance—positive or negative—and let R, L, and C be the absolute values of the same respectively. Then for a simple series circuit containing $\mathscr{R}$, $\mathscr{L}$, and $\mathscr{C}$, we get the usual differential equations, one for q the charge and one for i the current:

$$\frac{q}{\mathscr{C}}+\mathscr{R}\frac{dq}{dt}+\mathscr{L}\frac{d^2q}{dt^2} = f(t).$$

$$\frac{i}{\mathscr{C}}+\mathscr{R}\frac{di}{dt}+\mathscr{L}\frac{d^2i}{dt^2} = \frac{d}{dt}f(t) = f'(t).$$

where $f(t)$ is the applied voltage expressed as a function of time. The solutions of these are

$$q = \frac{\mathscr{C}}{\sqrt{\mathscr{R}^2\mathscr{C}^2-4\mathscr{L}\mathscr{C}}}\left\{\epsilon^{\frac{-t}{T_1}}\int\epsilon^{\frac{t}{T_1}}f(t)dt-\epsilon^{\frac{-t}{T_2}}\int\epsilon^{\frac{t}{T_2}}f(t)dt\right\}+c_1\epsilon^{\frac{-t}{T_1}}+c_2\epsilon^{\frac{-t}{T_2}};\ldots\ldots\ldots(1)$$

and

$$i = \frac{\mathscr{C}}{\sqrt{\mathscr{R}^2\mathscr{C}^2-4\mathscr{L}\mathscr{C}}}\left\{\epsilon^{\frac{-t}{T_1}}\int\epsilon^{\frac{t}{T_1}}f'(t)dt-\epsilon^{\frac{-t}{T_2}}\int\epsilon^{\frac{t}{T_2}}f'(t)dt\right\}+c_3\epsilon^{\frac{-t}{T_1}}+c_4\epsilon^{\frac{-t}{T_2}};\ldots\ldots\ldots(2)$$

the upper left-hand corner of each compartment in the second and fourth columns. Equational form of solutions are given for the first and third columns, whereas graphical solutions are plotted for the second and fourth columns.

It will be observed that the circuits containing all the positive or all the negative elements, given in the first and second columns, give rise to transients that exponentially die out after a short space of time; whereas the circuits containing one positive and one negative element, given in the third and fourth columns, produce constantly increasing transients. In cases No. (11), (12), (23) and (24), the rates of increase depend, among other things, on the instant the circuit is closed with reference to the voltage wave. This is so because the arbitrary constant that multiplies the exponential term is determined by the instant at which the circuit is closed. This constant can therefore be made zero by the proper choice of the closing instant, thus eliminating the transient term altogether. The current obtained in these cases will then be a simple harmonic function of time. To obtain this condition in practice, however, would be extremely difficult.

Chart II gives solutions of circuits containing three elements. Here each circuit, for each type of applied voltage and each set of initial conditions, has to be studied three times according to the relative values of the constants involved, *viz.*,

$$\mathcal{R}^2\mathcal{C}^2 \lesseqqgtr 4\mathcal{L}\mathcal{C}$$

Certain cases, however, become inconsistent due to both negative and positive elements being involved, as will be clearly seen from the chart. Again we find that the solutions of adjacent column pairs are related to each other by a factor of plus or minus one. The same general method of showing these relations, plotting curves and giving solutions in algebraic form is adopted in this case as in Chart I.

We observe again that, when all the elements of a circuit are either positive or negative (columns one and two), the transients have a decreasing character, *i.e.*, they die out after a short interval of time. In all the other cases, where both positive and negative elements are involved, we get increasing transients. Hence such circuits and similar circuits of Chart I cannot be employed for steady state applications. However, they furnish means of obtaining increasing currents of various forms. Since it is physically impossible for the currents to increase indefinitely, the rising currents can occur only for short intervals of time. In case such circuits were left alone after the voltage is applied, the negative elements would cease to function as such after a certain limiting value of current is reached, since negative constants are only obtainable within definite current limits. What would happen beyond these limits, would depend on the particular circuit employed to obtain the negative constant.

As an example, consider circuit 11 of Chart I, which is composed of a nesistance and an inductance. Let the nesistance be of the dynatron type (A. W. Hull, *Proc. I.R.E.*, 1918, 6, 5). The complete circuit is shown in where

$$T_1 = \frac{2\mathscr{L}\mathscr{C}}{\mathscr{R}\mathscr{C} - \sqrt{\mathscr{R}^2\mathscr{C}^2 - 4\mathscr{L}\mathscr{C}}}; \qquad T_2 = \frac{2\mathscr{L}\mathscr{C}}{\mathscr{R}\mathscr{C} + \sqrt{\mathscr{R}^2\mathscr{C}^2 - 4\mathscr{L}\mathscr{C}}};$$

ϵ is the Naperian base, and c_1, c_2, c_3, c_4 are arbitrary constants depending upon initial conditions.

For the purpose of detailed study, all the possible series combinations of the six circuit constants, *viz.*, resistance, inductance, capacitance, nesistance, ninductance, and napacitance[1] have been taken. In practice it is extremely difficult, though not theoretically impossible, to obtain a circuit without a positive or negative resistance: hence all the circuits considered contain either one or other of these elements. Each circuit is studied under three forms of applied e.m.f., $f(t)$, *viz.*, zero, a constant e.m.f. E, and a simple harmonic e.m.f. E $\sin\omega t$. Typical initial conditions are assumed. The results are shown in Charts I and II, in which the following symbols are used for brevity:

$$T_1 = \frac{2LC}{RC - \sqrt{R^2C^2 - 4LC}} \qquad T_2 = \frac{2LC}{RC + \sqrt{R^2C^2 - 4LC}}$$

$$T_3 = \frac{-2LC}{RC - \sqrt{R^2C^2 + 4LC}} \qquad T_4 = \frac{2LC}{RC + \sqrt{R^2C^2 + 4LC}}$$

$$\omega_0 = \frac{\sqrt{4LC - R^2C^2}}{2LC} \qquad \theta = \tan^{-1}\frac{2\omega_0 L}{R} = \tan^{-1}\frac{\sqrt{4LC - R^2C^2}}{RC}$$

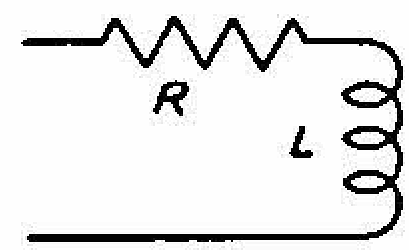
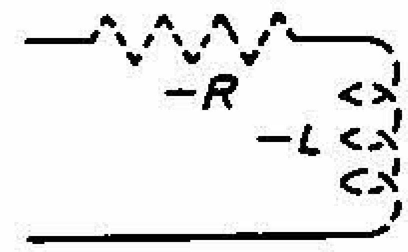
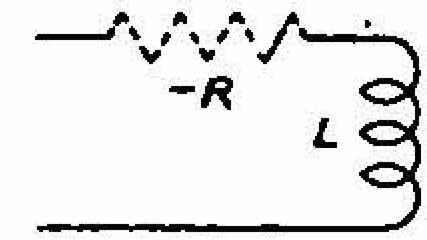
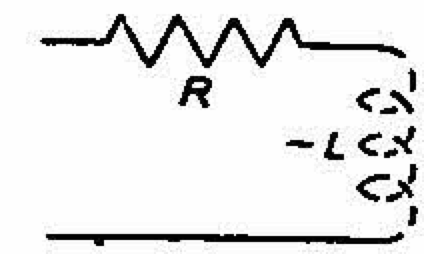
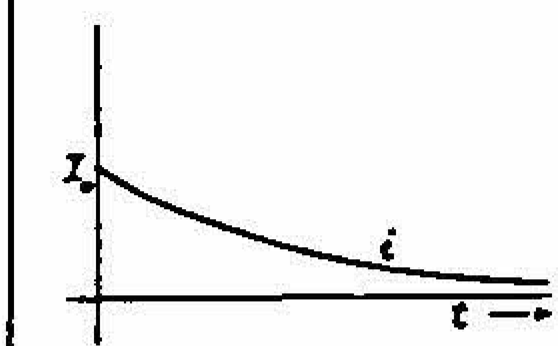
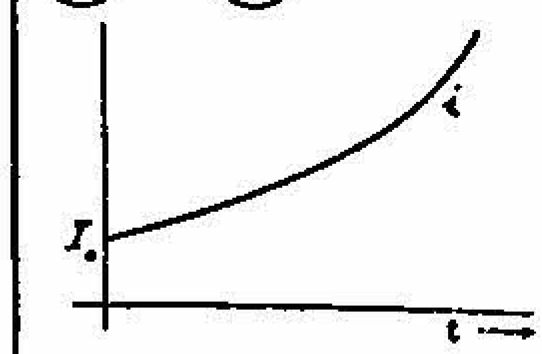
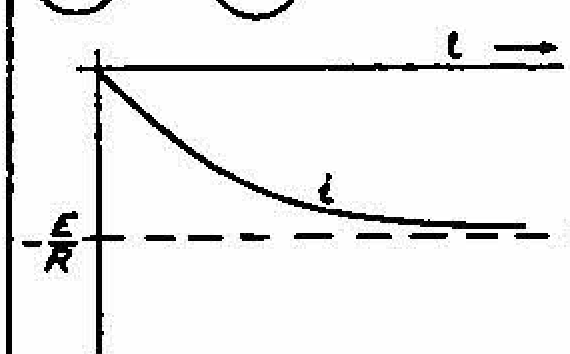
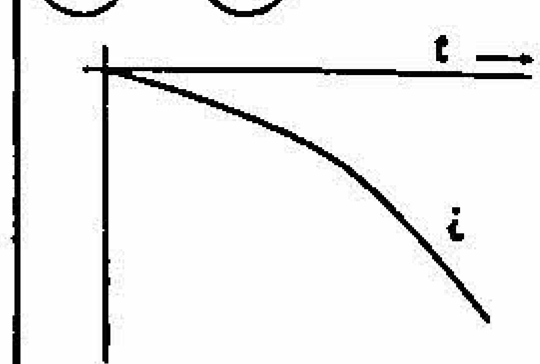
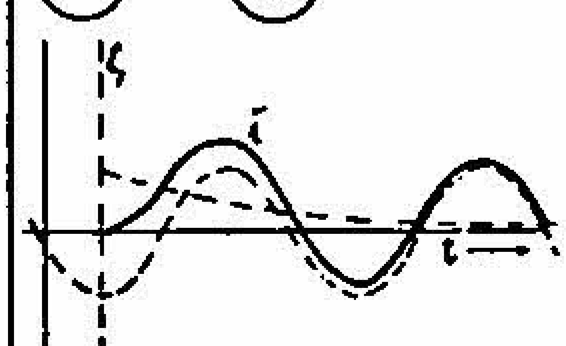
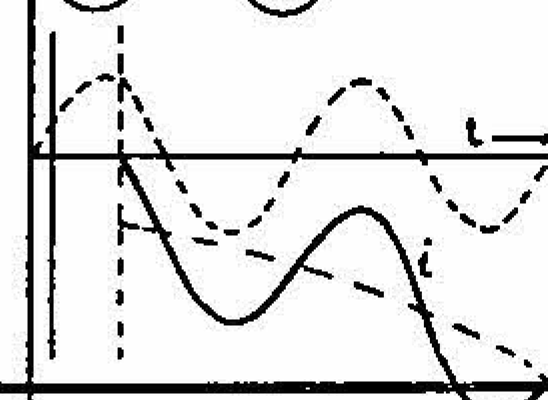
$$Z_1 = \sqrt{R^2 + \left(L\omega - \frac{1}{\omega C}\right)^2} \qquad Z_2 = \sqrt{R^2 + \left(L\omega + \frac{1}{\omega C}\right)^2}$$

$$X_1 = L\omega - \frac{1}{\omega C} \qquad X_2 = L\omega + \frac{1}{\omega C}$$

In Chart I are given circuits containing only two elements, one of which is always a resistance or a nesistance. Each solution obtained by the application of Equations (1) and (2) is given on the chart and designated by a number. It is found that the solutions of circuits contained in the adjacent column pairs are similar, differing from each other only by a factor of plus or minus one. To avoid repetition, this relation is indicated by an identity in

[1] Nesistance, ninductance and napacitance are terms applied to negative resistance, negative inductance, and negative capacitance respectively: see author's earlier paper referred to above.

CHART I

E.M.F. AND INITIAL CONDITIONS				
$e=0$. When $t=0$, $i=I_0$	(1) $i=I_0\epsilon^{-\frac{R}{L}t}$	(2) ≡ (1) 	(3) $i=I_0\epsilon^{\frac{R}{L}t}$	(4) ≡ (3) 
$e=E$. When $t=0$. $i=0$	(5) $i=\frac{E}{R}(1-\epsilon^{-\frac{R}{L}t})$	(6) ≡ −(5) 	(7) $i=\frac{E}{R}(\epsilon^{\frac{R}{L}t}-1)$	(8) ≡ −(7) 
$e=E\sin\omega t$. When $t=t_1$, $i=0$	(9) $i=\frac{E}{\sqrt{R^2+L^2\omega^2}}\sin\{\omega t-\tan^{-1}\frac{L\omega}{R}\}+c\epsilon^{-\frac{R}{L}t}$	(10) ≡ −(9) 	(11) $i=\frac{-E}{\sqrt{R^2+L^2\omega^2}}\sin\{\omega t+\tan^{-1}\frac{L\omega}{R}\}+c\epsilon^{\frac{R}{L}t}$	(12) ≡ −(11) 

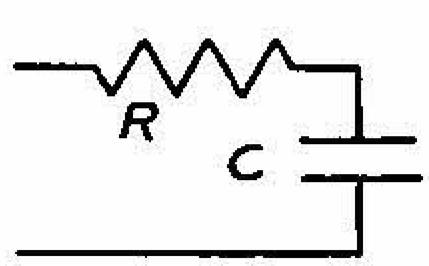
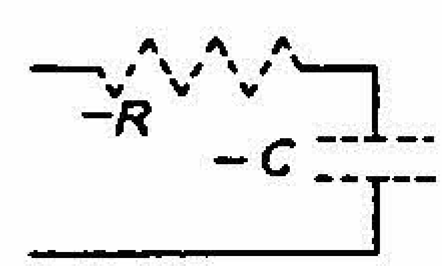
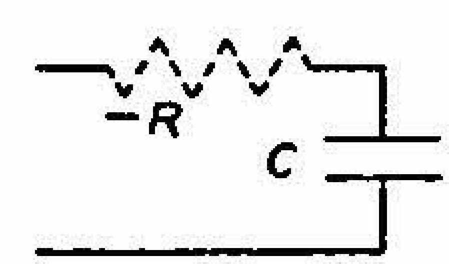
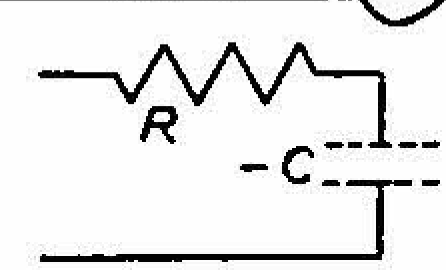
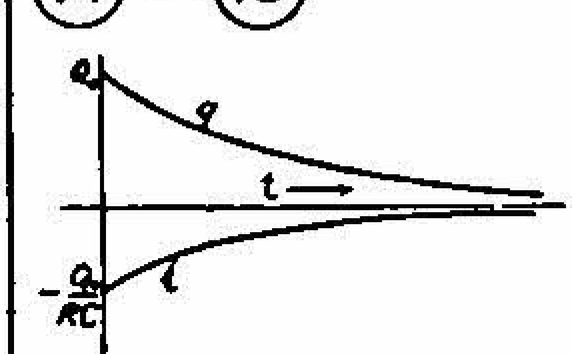
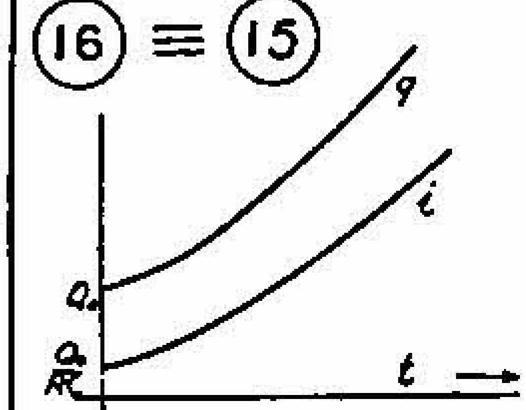
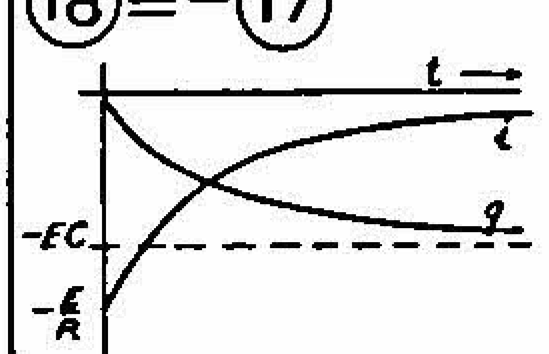
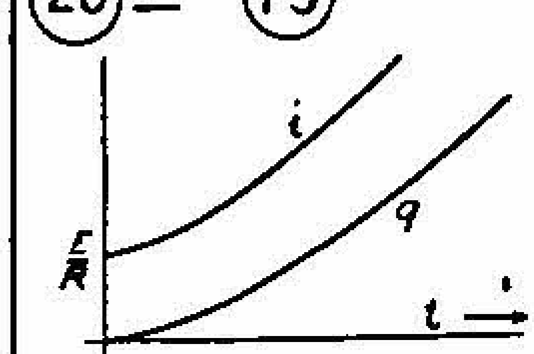
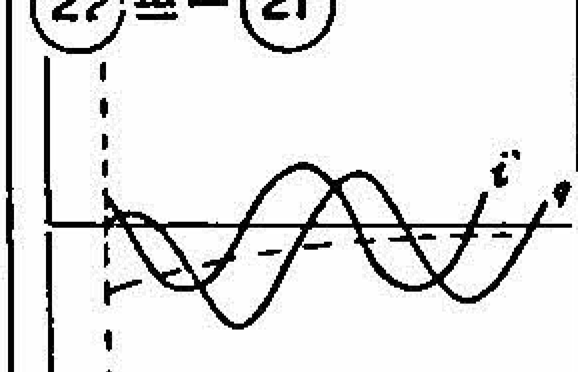
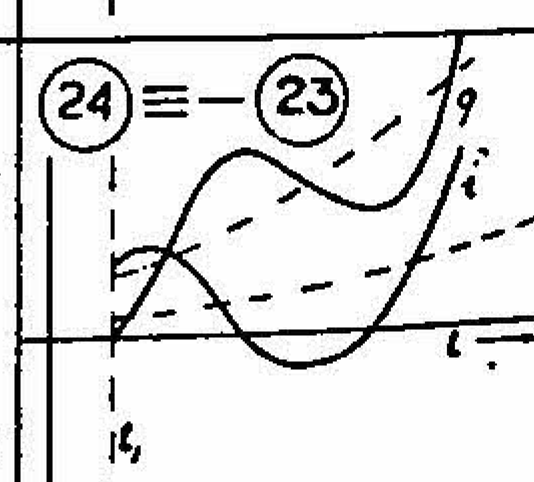
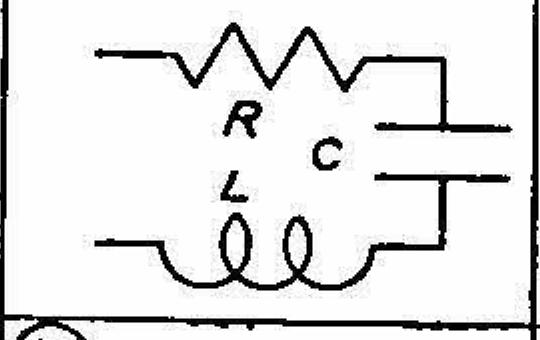
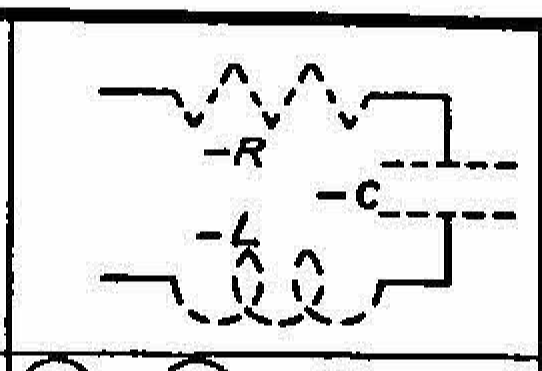
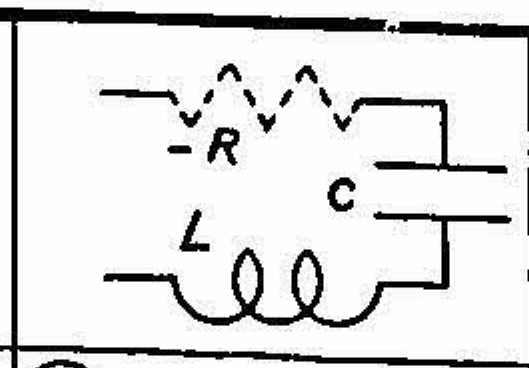
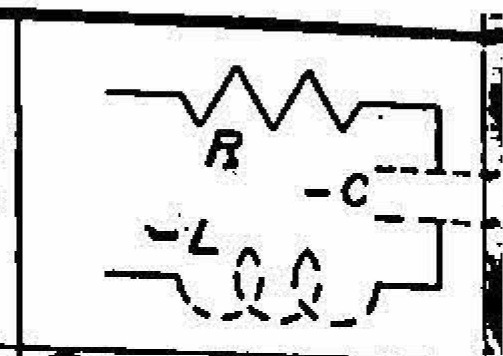
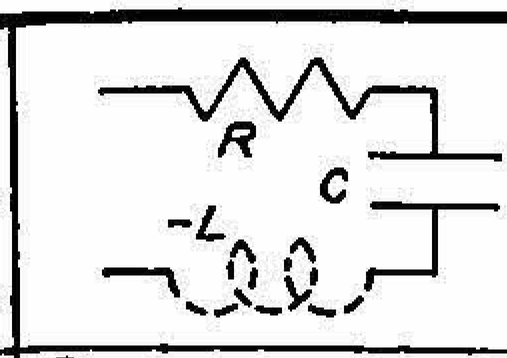
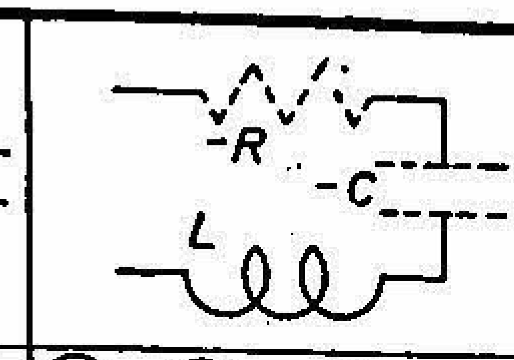
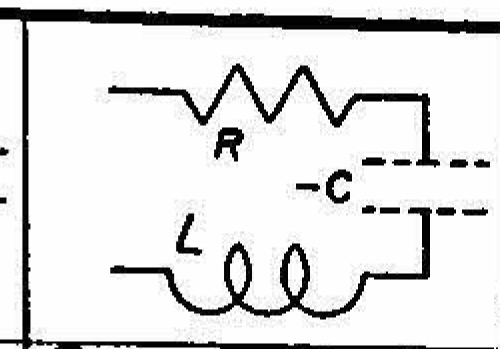
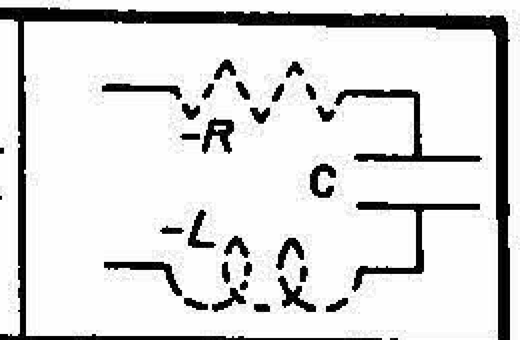
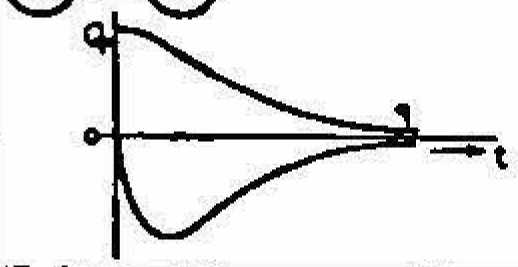
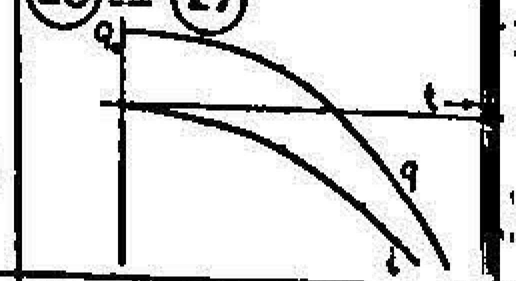
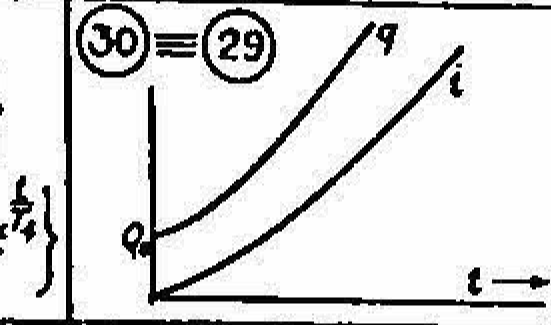
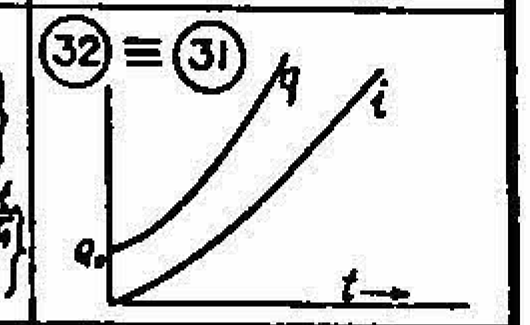
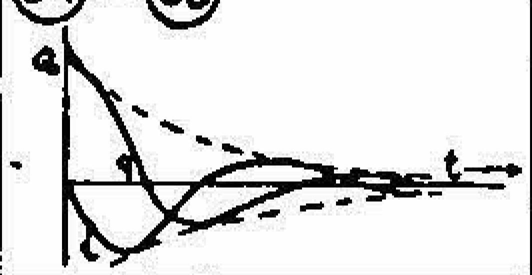
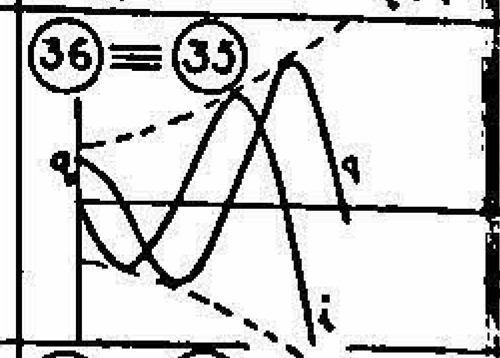
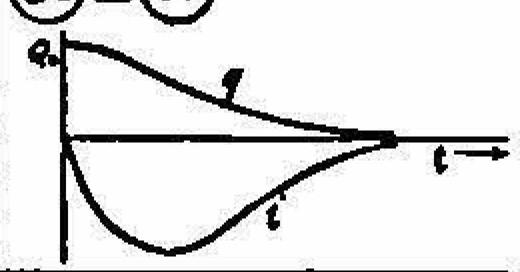
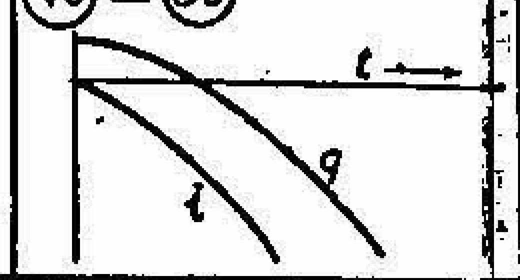
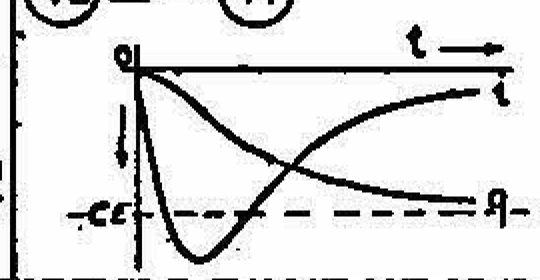
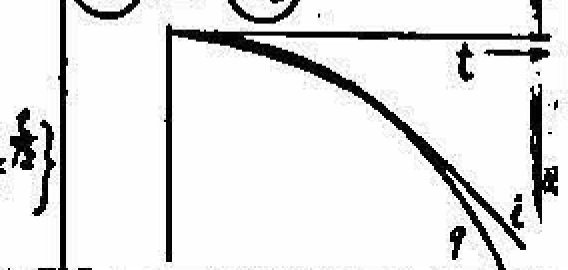
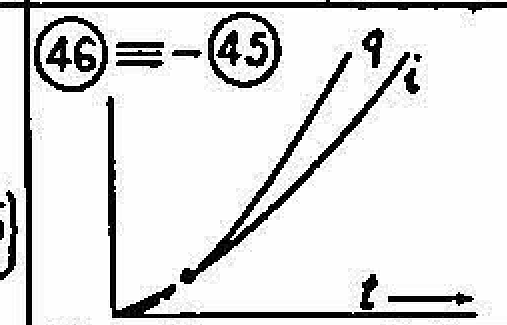
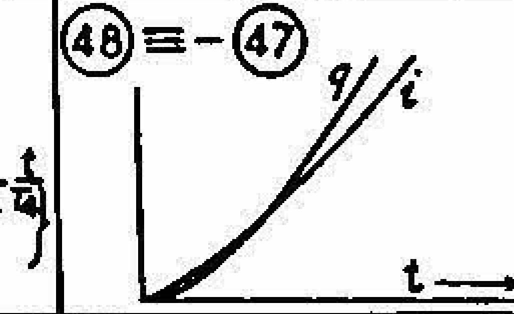
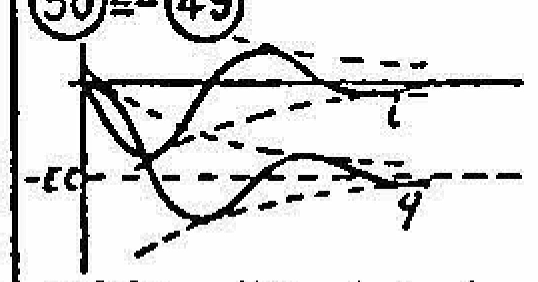
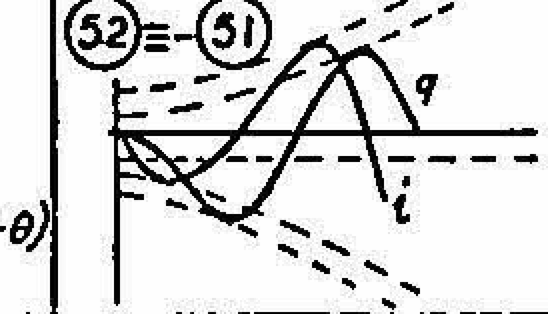
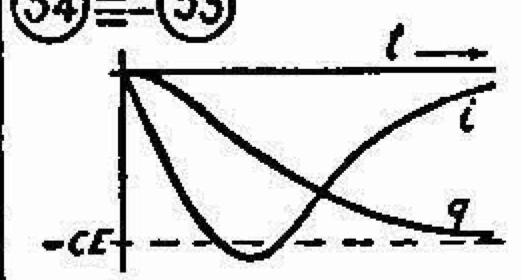
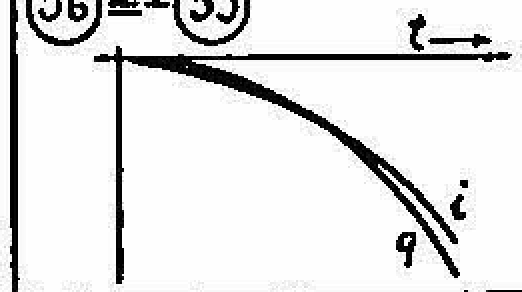
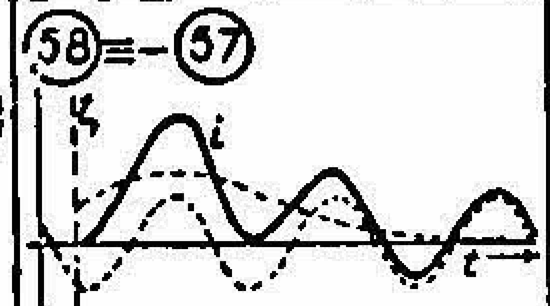
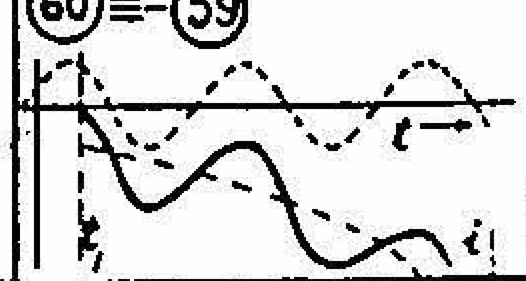
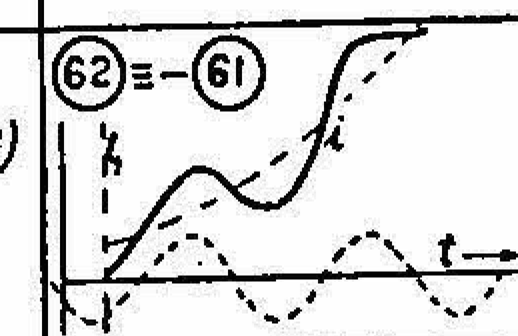
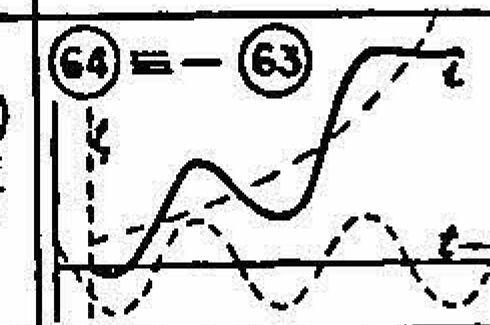
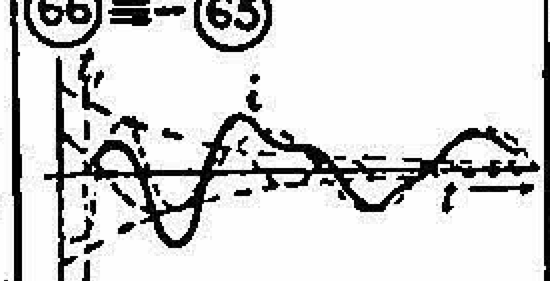
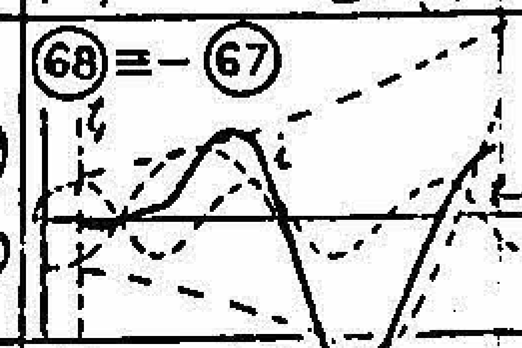
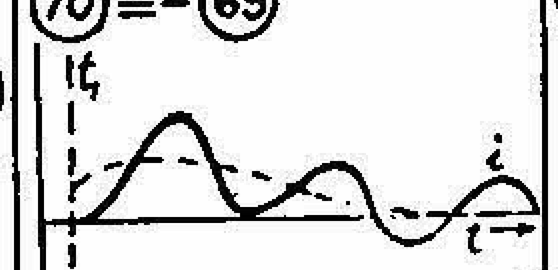
E.M.F. AND INITIAL CONDITIONS				
$e=0$. When $t=0$, $q=Q_0$	(13) $q=Q_0\epsilon^{-\frac{t}{RC}}$ $i=-\frac{Q_0}{RC}\epsilon^{-\frac{t}{RC}}$	(14) ≡ (13) 	(15) $q=Q_0\epsilon^{\frac{t}{RC}}$ $i=\frac{Q_0}{RC}\epsilon^{\frac{t}{RC}}$	(16) ≡ (15) 
$e=E$. When $t=0$, $q=0$	(17) $q=EC(1-\epsilon^{-\frac{t}{RC}})$ $i=\frac{E}{R}\epsilon^{-\frac{t}{RC}}$	(18) ≡ −(17) 	(19) $q=EC(1-\epsilon^{\frac{t}{RC}})$ $i=-\frac{E}{R}\epsilon^{\frac{t}{RC}}$	(20) ≡ −(19) 
$e=E\sin\omega t$. When $t=t_1$, $q=0$	(21) $q=\frac{-EC}{\sqrt{R^2C^2\omega^2+1}}\cos\{\omega t+\tan^{-1}\frac{1}{CR\omega}\}+C_1\epsilon^{-\frac{t}{RC}}$ $i=\frac{EC\omega}{\sqrt{R^2C^2\omega^2+1}}\sin\{\omega t+\tan^{-1}\frac{1}{CR\omega}\}+C_2\epsilon^{-\frac{t}{RC}}$	(22) ≡ −(21) 	(23) $q=\frac{EC}{\sqrt{R^2C^2\omega^2+1}}\cos\{\omega t-\tan^{-1}\frac{1}{CR\omega}\}+C_1\epsilon^{\frac{t}{RC}}$ $i=\frac{-EC\omega}{\sqrt{R^2C^2\omega^2+1}}\sin\{\omega t-\tan^{-1}\frac{1}{RC\omega}\}+C_2\epsilon^{\frac{t}{RC}}$	(24) ≡ −(23) 

CHART II

E.M.F. & INITIAL CONDITIONS	RELATION BETWEEN CONSTANTS								
$e = 0.$ When $t = 0$, $i = 0$, $q = Q_0$	$R^2C^2 > 4LC$	(25) $i = \frac{-Q_0}{T_1 - T_2}\left\{\epsilon^{-\frac{t}{T_1}} - \epsilon^{-\frac{t}{T_2}}\right\}$ $q = \frac{Q_0}{T_1 - T_2}\left\{T_1\epsilon^{-\frac{t}{T_1}} - T_2\epsilon^{-\frac{t}{T_2}}\right\}$	(26) ≡ (25) 	(27) $i = \frac{Q_0}{T_1 - T_2}\left\{\epsilon^{\frac{t}{T_1}} - \epsilon^{\frac{t}{T_2}}\right\}$ $q = \frac{Q_0}{T_1 - T_2}\left\{T_1\epsilon^{\frac{t}{T_1}} - T_2\epsilon^{\frac{t}{T_2}}\right\}$	(28) ≡ (27) 	(29) $i = \frac{Q_0}{T_3 + T_4}\left\{\epsilon^{\frac{t}{T_4}} - \epsilon^{-\frac{t}{T_3}}\right\}$ $q = \frac{Q_0}{T_3 + T_4}\left\{T_3\epsilon^{-\frac{t}{T_3}} + T_4\epsilon^{\frac{t}{T_4}}\right\}$	(30) ≡ (29) 	(31) $i = \frac{Q_0}{T_3 + T_4}\left\{\epsilon^{\frac{t}{T_3}} - \epsilon^{-\frac{t}{T_4}}\right\}$ $q = \frac{Q_0}{T_3 + T_4}\left\{T_3\epsilon^{\frac{t}{T_3}} + T_4\epsilon^{-\frac{t}{T_4}}\right\}$	(32) ≡ (31) 
	$R^2C^2 < 4LC$	(33) $i = \frac{-Q_0}{\omega_0 LC}\epsilon^{-\frac{Rt}{2L}}\sin\omega_0 t$ $q = \frac{Q_0}{\omega_0\sqrt{LC}}\epsilon^{-\frac{Rt}{2L}}\sin(\omega_0 t + \theta)$	(34) ≡ (33) 	(35) $i = \frac{-Q_0}{\omega_0 LC}\epsilon^{\frac{Rt}{2L}}\sin\omega_0 t$ $q = \frac{-Q_0}{\omega_0\sqrt{LC}}\epsilon^{\frac{Rt}{2L}}\sin(\omega_0 t - \theta)$	(36) ≡ (35) 	INCONSISTENT	INCONSISTENT	INCONSISTENT	INCONSISTENT
	$R^2C^2 = 4LC$	(37) $i = -\frac{Q_0}{LC}t\epsilon^{-\frac{Rt}{2L}}$ $q = (1 + \frac{Rt}{2L})Q_0\epsilon^{-\frac{Rt}{2L}}$	(38) ≡ (37) 	(39) $i = -\frac{Q_0}{LC}t\epsilon^{\frac{Rt}{2L}}$ $q = (1 - \frac{Rt}{2L})Q_0\epsilon^{\frac{Rt}{2L}}$	(40) ≡ (39) 	//	//	//	//
$e = E.$ When $t = 0$, $i = 0$, $q = 0$	$R^2C^2 > 4LC$	(41) $i = \frac{CE}{T_1 - T_2}\left\{\epsilon^{-\frac{t}{T_1}} - \epsilon^{-\frac{t}{T_2}}\right\}$ $q = CE - \frac{CE}{T_1 - T_2}\left\{T_1\epsilon^{-\frac{t}{T_1}} - T_2\epsilon^{-\frac{t}{T_2}}\right\}$	(42) ≡ − (41) 	(43) $i = \frac{CE}{T_1 - T_2}\left\{\epsilon^{\frac{t}{T_2}} - \epsilon^{\frac{t}{T_1}}\right\}$ $q = CE - \frac{CE}{T_1 - T_2}\left\{T_1\epsilon^{\frac{t}{T_1}} - T_2\epsilon^{\frac{t}{T_2}}\right\}$	(44) ≡ − (43) 	(45) $i = \frac{EC}{T_3 + T_4}\left\{\epsilon^{-\frac{t}{T_3}} - \epsilon^{\frac{t}{T_4}}\right\}$ $q = EC - \frac{CE}{T_3 + T_4}\left\{T_3\epsilon^{-\frac{t}{T_3}} + T_4\epsilon^{\frac{t}{T_4}}\right\}$	(46) ≡ − (45) 	(47) $i = \frac{EC}{T_3 + T_4}\left\{\epsilon^{\frac{t}{T_3}} - \epsilon^{-\frac{t}{T_4}}\right\}$ $q = -EC + \frac{EC}{T_3 + T_4}\left\{T_3\epsilon^{\frac{t}{T_3}} + T_4\epsilon^{-\frac{t}{T_4}}\right\}$	(48) ≡ − (47) 
	$R^2C^2 < 4LC$	(49) $i = \frac{E}{\omega_0 L}\epsilon^{-\frac{Rt}{2L}}\sin\omega_0 t$ $q = EC - \frac{E}{\omega_0}\sqrt{\frac{C}{L}}\epsilon^{-\frac{Rt}{2L}}\sin(\omega_0 t + \theta)$	(50) ≡ − (49) 	(51) $i = \frac{E}{\omega_0 L}\epsilon^{\frac{Rt}{2L}}\sin\omega_0 t$ $q = EC + \frac{E}{\omega_0}\sqrt{\frac{C}{L}}\epsilon^{\frac{Rt}{2L}}\sin(\omega_0 t - \theta)$	(52) ≡ − (51) 	INCONSISTENT	INCONSISTENT	INCONSISTENT	INCONSISTENT
	$R^2C^2 = 4LC$	(53) $i = \frac{E}{L}t\epsilon^{-\frac{Rt}{2L}}$ $q = EC - EC(1 + \frac{Rt}{2L})\epsilon^{-\frac{Rt}{2L}}$	(54) ≡ − (53) 	(55) $i = \frac{E}{L}t\epsilon^{\frac{Rt}{2L}}$ $q = EC - EC(1 - \frac{Rt}{2L})\epsilon^{\frac{Rt}{2L}}$	(56) ≡ − (55) 	//	//	//	//
$e = E\sin\omega t.$ When $t = t_1$, $i = 0$, $q = 0$	$R^2C^2 > 4LC$	(57) $i = \frac{E}{Z_1}\sin(\omega t - \tan^{-1}\frac{X_1}{R}) + C_1\epsilon^{-\frac{t}{T_1}} + C_2\epsilon^{-\frac{t}{T_2}}$	(58) ≡ − (57) 	(59) $i = \frac{-E}{Z_1}\sin(\omega t + \tan^{-1}\frac{X_1}{R}) + C_1\epsilon^{\frac{t}{T_1}} + C_2\epsilon^{\frac{t}{T_2}}$	(60) ≡ − (59) 	(61) $i = \frac{E}{Z_2}\sin(\omega t + \tan^{-1}\frac{X_2}{R}) + C_1\epsilon^{-\frac{t}{T_3}} + C_2\epsilon^{\frac{t}{T_4}}$	(62) ≡ − (61) 	(63) $i = \frac{E}{Z_2}\sin(\omega t - \tan^{-1}\frac{X_2}{R}) + C_1\epsilon^{\frac{t}{T_3}} + C_2\epsilon^{-\frac{t}{T_4}}$	(64) ≡ − (63) 
	$R^2C^2 < 4LC$	(65) $i = \frac{E}{Z_1}\sin(\omega t - \tan^{-1}\frac{X_1}{R}) + A\epsilon^{-\frac{Rt}{2L}}\sin(\omega_0 t + \theta)$	(66) ≡ − (65) 	(67) $i = \frac{-E}{Z_1}\sin(\omega t + \tan^{-1}\frac{X_1}{R}) + A\epsilon^{\frac{Rt}{2L}}\sin(\omega_0 t + \theta)$	(68) ≡ − (67) 	INCONSISTENT	INCONSISTENT	INCONSISTENT	INCONSISTENT
	$R^2C^2 = 4LC$	(69) $i = \frac{E}{Z_1}\sin(\omega t - \tan^{-1}\frac{X_1}{R}) + C_1\epsilon^{-\frac{Rt}{2L}} + C_2 t\epsilon^{-\frac{Rt}{2L}}$	(70) ≡ − (69) 	(71) $i = \frac{-E}{Z_1}\sin(\omega t + \tan^{-1}\frac{X_1}{R}) + C_1\epsilon^{\frac{Rt}{2L}} + C_2 t\epsilon^{\frac{Rt}{2L}}$	(72) ≡ − (71) 	//	//	//	//

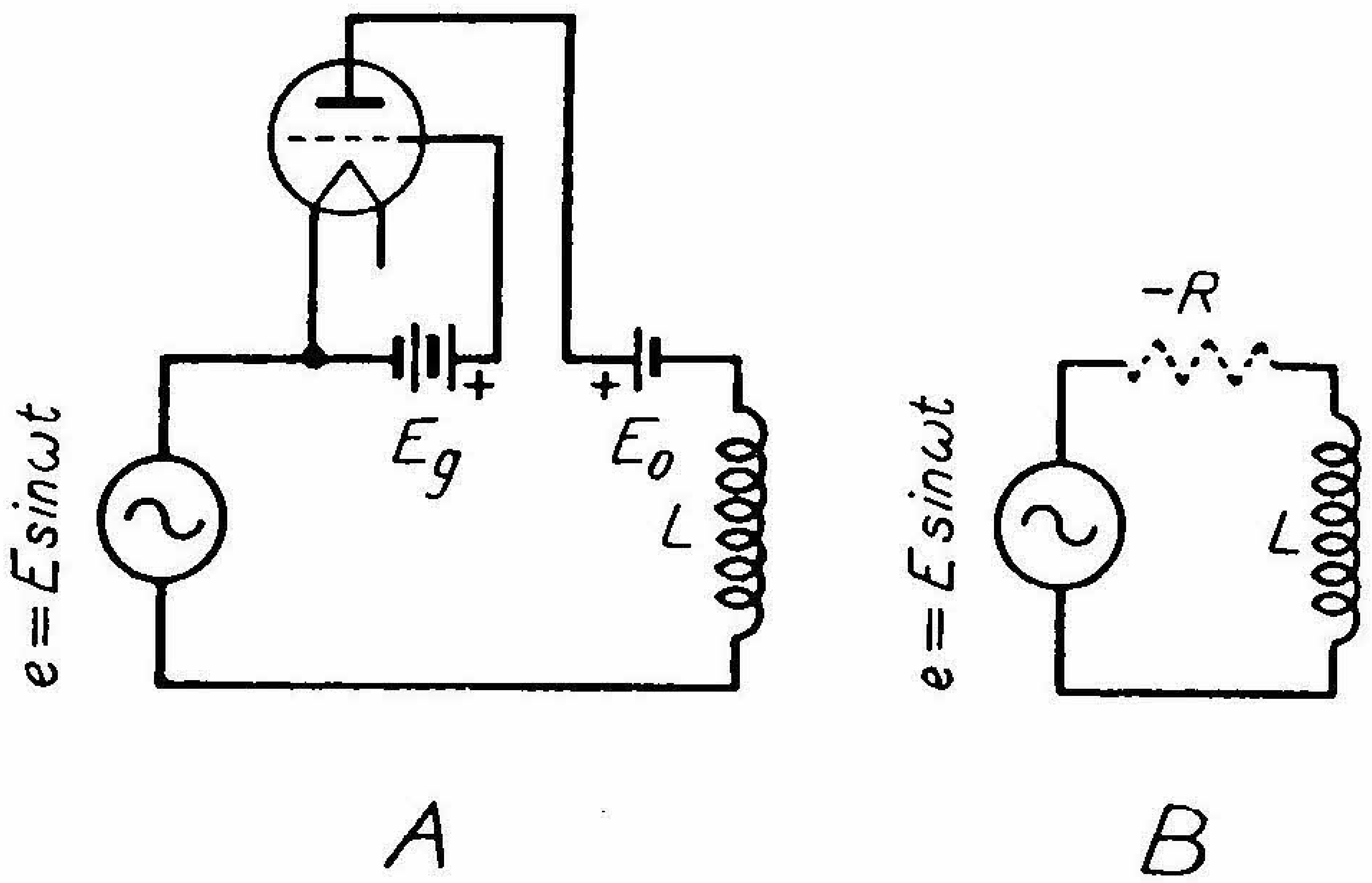


Fig. 1.

Fig 1A, which is equivalent to the circuit shown in Fig. 1B. The dynatron tube comprising the resistance has a characteristic of the form shown in

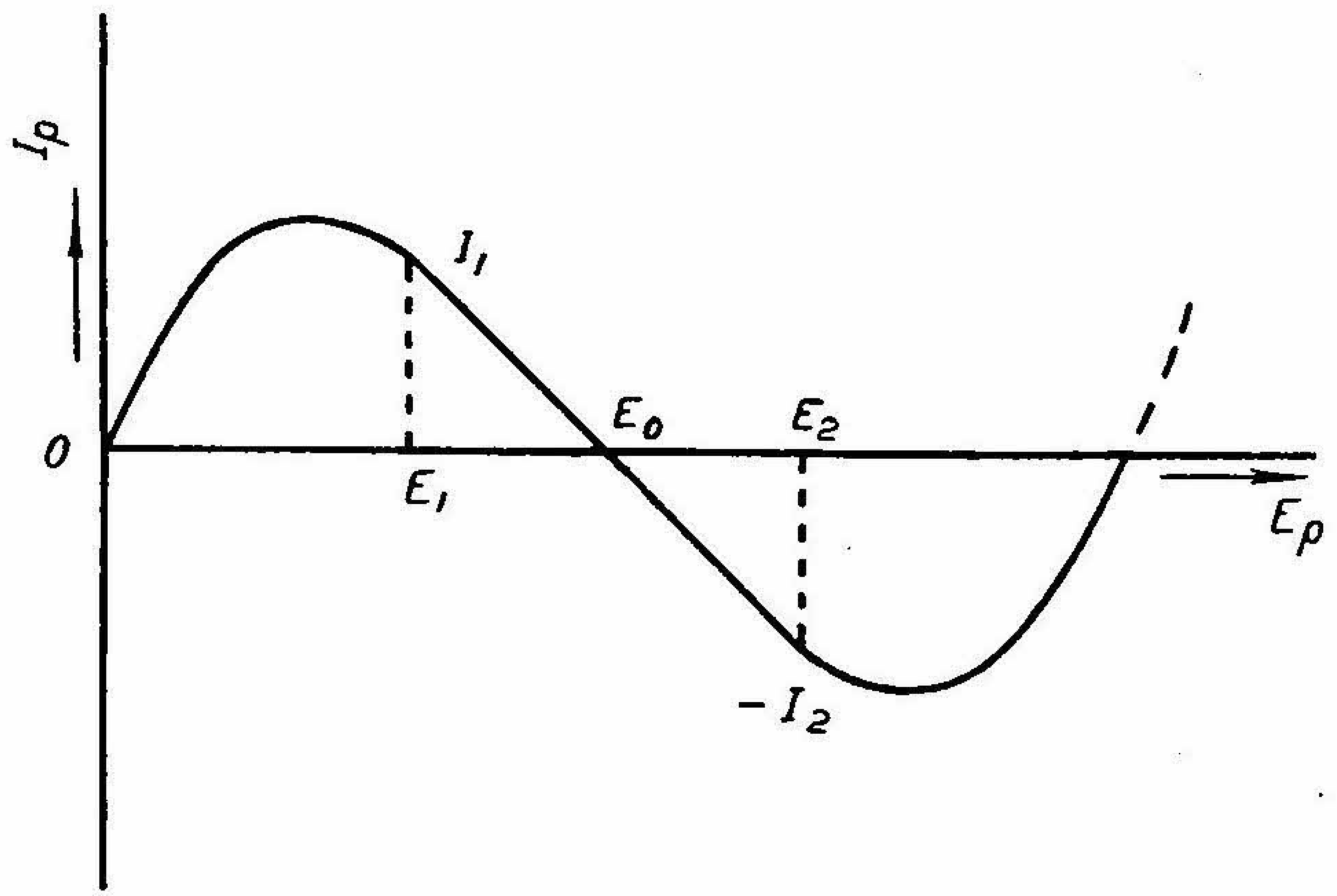
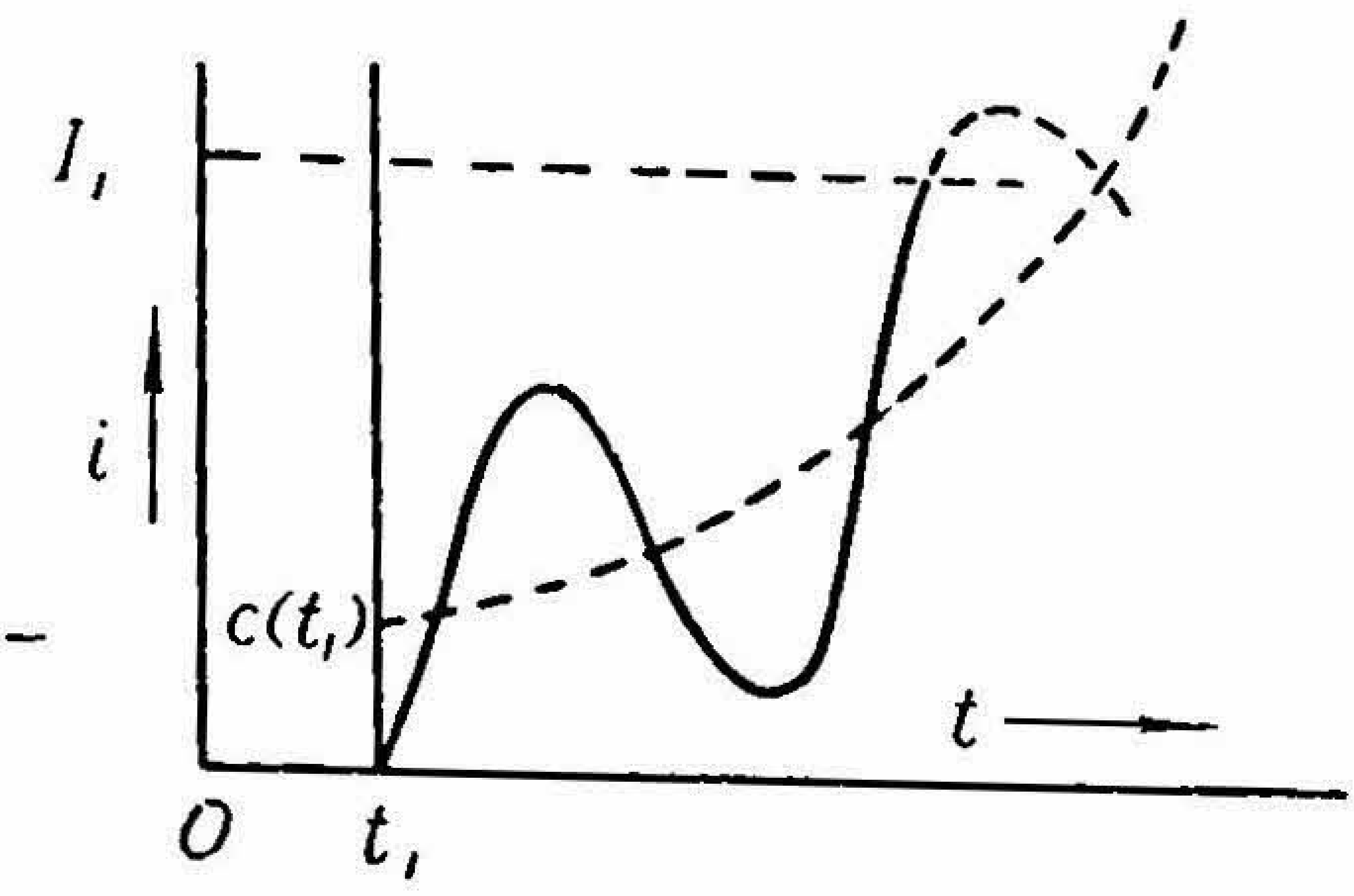


Fig. 2.

Fig. 2, so that the negative resistance characteristic exists over the range of current from I_1 to $-I_2$, corresponding to the voltage from E_1 to E_2. The transient solution for the circuit is given in full line in Fig. 3, for an applied voltage of the form $e=E \sin \omega t$ and the initial condition $i=0$ when $t=t_1$.



$$i = \frac{-E}{\sqrt{R^2+\omega^2 L^2}} \sin\left\{\omega t + \tan^{-1}\frac{\omega L}{R}\right\} + c(t_1)\,\epsilon^{\frac{R}{L}t}$$

Fig. 3.

The positive limiting current I_1 is also shown in Fig. 3. After the transient current has reached this limit the dynatron ceases to be a pure negative resistance, because its characteristic begins to deviate from the straight line (Fig. 2). Hence the transient ceases to follow the full line curve of Fig. 3 beyond I_1.

The exact analysis of the current form beyond this limit is outside the scope of this paper and will be discussed in a future paper along with the experimental verifications.

APPENDIX.

In Fig. 4 is shown the circuit devised by R. C. Mathes, in which R_2 is the output resistance of the amplifier, the input resistance of which is infinite.

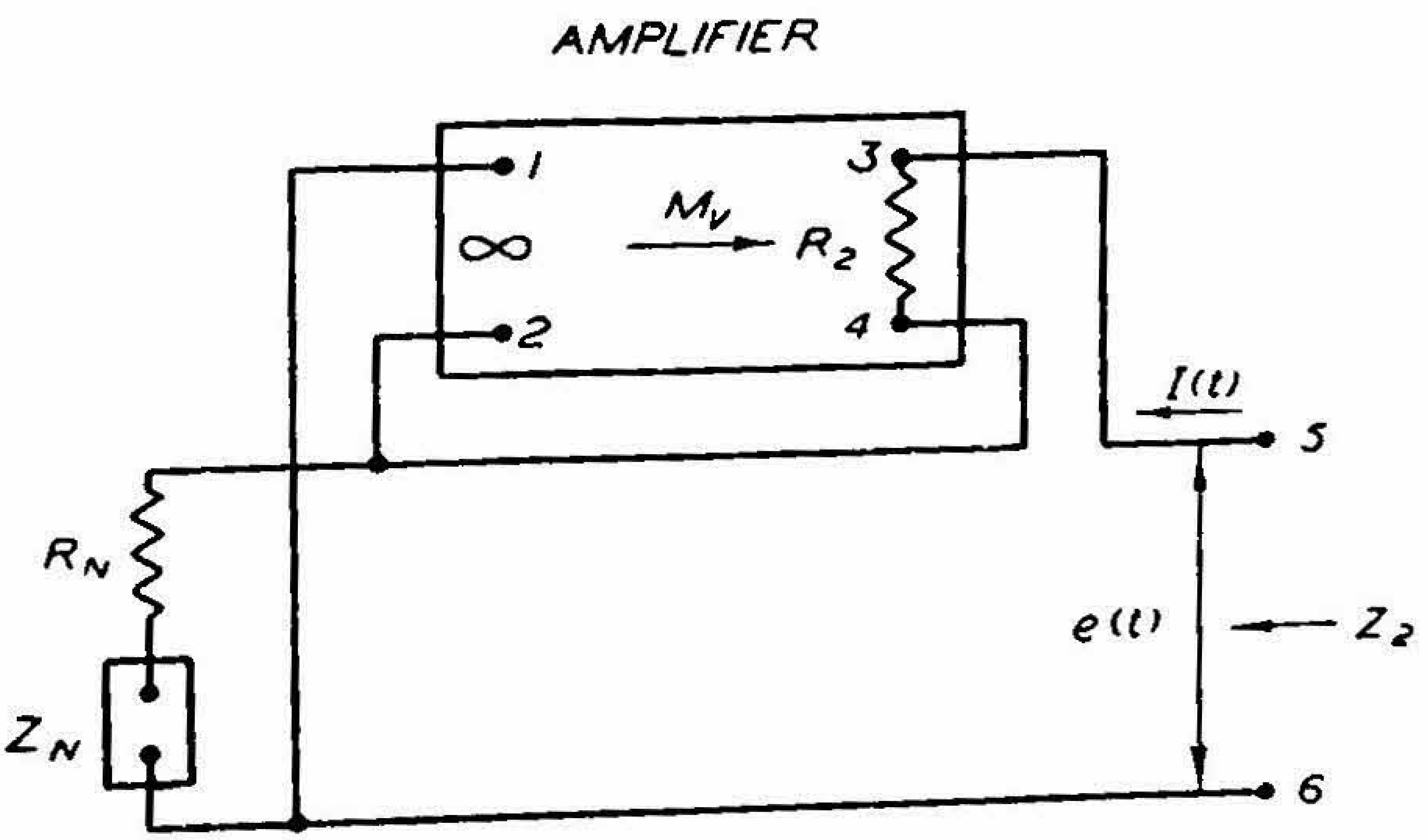


Fig. 4.

The amplifier is perfectly distortionless with regard to amplitude as well as phase. Amplification is expressed as M_v, *i.e.* the ratio of the voltage generated in the output resistance R_2 to the voltage impressed on the input terminals 1, 2. M_v is hence a real number greater than unity and may be adjusted by some suitable means such as a potentiometer.

Crisson has shown that if Z be the positive of any negative impedance that is desired, then choosing

$$Z_N = \frac{Z}{M_v - 1}; \qquad \text{and} \qquad R_N = \frac{R_2}{M_v - 1};$$

we obtain the equivalent impedance Z_2 of the whole circuit as being equal to $-Z$. In our case, however, the circuit has to be treated for transient currents and hence the general concept of impedance cannot be employed. Working with differential equations, we have to show that with certain choice of constants comprising the network Z_N and the resistance R_N, the whole circuit behaves as if it were a network composed of negative elements of the desired value arranged in the desired way.

Now, assuming that it is required to produce a network Z of n circuits composed of negative elements—nesistances, ninductances and napacitances, let the individual circuits of the network be coupled to one another by any desired means, *viz.*, nesistive, ninductive, or napacitive coupling. Then using Bush's operational notation (V. Bush, *Operational Circuit Analysis*, chapters II and III), we can write the differential equations for the desired network in the following form:—

$$\left.\begin{aligned} &Z_{11}(p)\, i_1 + Z_{12}(p)\, i_2 + \ldots\ldots\ldots\ldots + Z_{1n}(p)\, i_n = e_1(t) \\ &Z_{21}(p)\, i_1 + Z_{22}(p)\, i_2 + \ldots\ldots\ldots\ldots + Z_{2n}(p)\, i_n = 0. \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &Z_{n1}(p)\, i_1 + Z_{n2}(p)\, i_2 + \ldots\ldots\ldots\ldots + Z_{nn}(p)\, i_n = 0. \end{aligned}\right\} \ldots\ldots(3)$$

where

$$Z_{11}(p) = -R_{11} - L_{11}p - \frac{1}{C_{11}p}$$

$$Z_{12}(p) = -R_{12} - L_{12}p - \frac{1}{C_{12}p}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

etc.,

$$p = \frac{d(\;)}{dt}, \qquad \frac{1}{p} = \int (\;)\, dt,$$

$-R_{11}$, $-L_{11}$, $-C_{11}$ being the elements in circuit 1,

$-R_{12}$, $-L_{12}$, $-C_{12}$ being the elements common to circuits 1 and 2,

....

and so on,

i_1, i_2, ...etc., being currents in circuits 1, 2, etc.; $e_1(t)$ being the known voltage applied to the input terminals of the network, which are supposed to be located in circuit 1.

Then the current flowing from the source can be expressed as

$$i_1 = \frac{M_{11}(p)}{D(p)} e_1(t) ; \quad \ldots\ldots (4)$$

where D(p) is the determinant

$$\begin{vmatrix} Z_{11}(p) & Z_{12}(p) \ldots\ldots & Z_{1n}(p) \\ Z_{21}(p) & Z_{22}(p) \ldots\ldots & Z_{2n}(p) \\ \ldots & \ldots & \ldots \\ Z_{n1}(p) & Z_{n2}(p) \ldots\ldots & Z_{nn}(p) \end{vmatrix}$$

and $M_{11}(p)$ is the minor of the first row and first column. The equation (4) represents the differential equation of the desired network Z composed of the desired negative elements arranged in a desired manner.

Now construct a network Z_N similar to the above network Z, but with each element of Z_N positive and in absolute value equal to $\frac{1}{M_v - 1}$ times the corresponding element in Z. This can be constructed out of physically available parts. Also choose a resistance

$$R_N = \frac{R_2}{M_v - 1} ;$$

Insert Z_N and R_N in the circuit as shown in Fig. 4. Then if a current $I(t)$ flows in this circuit at the terminals 5, 6 due to an applied e.m.f. $e(t)$, then the voltage drop across the network Z_N will be given by

$$e_N(t) = \frac{-1}{M_v - 1} \frac{D(p)}{M_{11}(p)} I(t) \ldots\ldots (5)$$

This relation is obtained by writing down equations of the form (3) for the network Z_N and solving for the current in the input circuit 1, just as we did to obtain equation (4). The voltage applied to the input of the amplifier is then

$$e_i(t) = \left[R_N - \frac{1}{M_v - 1} \frac{D(p)}{M_{11}(p)} \right] I(t).$$

Hence the output voltage of the amplifier is

$$e_0(t) = M_v \left[R_N - \frac{1}{M_v - 1} \frac{D(p)}{M_{11}(p)} \right] I(t).$$

Let $e_0(t)$ act in such a direction as to aid $e(t)$ in pushing the current through the circuit of Fig. 4. Then the Kirchhoff's equation for the whole circuit can be written as:

$$e(t) + e_0(t) = I(t) \left[R_2 + R_N - \frac{1}{M_v - 1} \frac{D(p)}{M_{11}(p)} \right]$$

or substituting the value of $e_0(t)$,

$$e(t) = I(t) \left[R_2 + R_N - \frac{1}{M_v - 1} \frac{D(p)}{M_{11}(p)} - R_N M_v + \frac{M_v}{M_v - 1} \frac{D(p)}{M_{11}(p)} \right]$$

Substituting the value of R_N and simplifying, we get

$$e(t) = \frac{D(p)}{M_{11}(p)} I(t) \ldots\ldots (6)$$

Equation (6) is identical with the differential equation (4). Hence the circuit of Fig. 4 acts, under transient conditions, just like the hypothetical network Z composed of negative constants alone, as we desired.

Let us now consider the circuit of H. W. Dudley shown in Fig. 5. Again we shall assume that it is desired to obtain a perfectly general network Z of n circuits composed of negative elements. The input voltage and current, therefore, satisfy again the differential equation (4). Next construct a corresponding network Z_N composed of all positive elements such that each element of the network Z_N is, in absolute value, equal to (M_v-1) times the corresponding element of Z. Also assume

$$R_N = \frac{R_2}{M_v - 1}.$$

Now, if the current through the circuit of Fig. 5 be $I(t)$ due to an applied

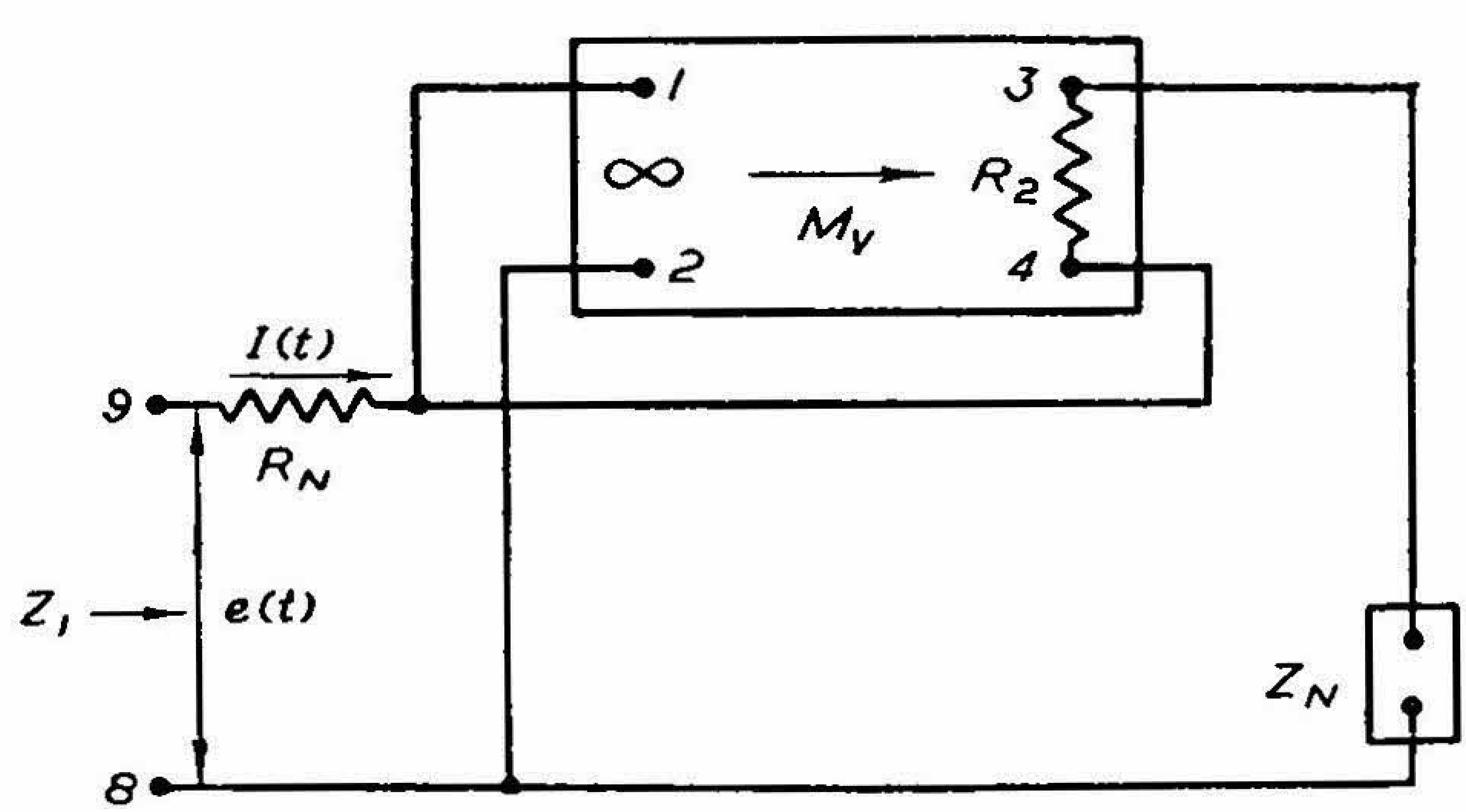


Fig. 5.

e.m.f. $e(t)$, then the drop across Z_N becomes

$$e_N(t) = (1 - M_v)\frac{D(p)}{M_{11}(p)} I(t) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(7)}$$

The voltage applied to the input terminals of the amplifier is

$$e_i(t) = e(t) - R_N I(t)$$

Therefore the output voltage

$$e_0(t) = M_v\,[e(t) - R_N I(t)]$$

Let this act in opposition to the applied e.m.f. Then the Kirchhoff's Law for the circuit of Fig. 5 assumes the form

$$e(t) - e_0(t) = I(t)\left[R_N + R_2 + (1 - M_v)\frac{D(p)}{M_{11}(p)}\right]$$

Substituting the value of $e_0(t)$

$$(1-M_V)e(t)=I(t)\left[R_N+R_2+(1-M_V)\frac{D(p)}{M_{11}(p)}-M_V R_N\right]$$

Substituting the value of R_N and simplifying, we obtain

$$e(t)=\frac{D(p)}{M_{11}(p)}I(t)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

Equation (8) again is identical with the differential equation (4). Hence the circuit of Fig. 5 is proved to be identical with the desired network Z under transient conditions.

It may be pointed out that it is not necessary that the desired network Z be composed of entirely negative elements. It may be any combination of negative and positive elements. Then the circuits of Figs. 4 and 5 will yield such a desired network, provided each element of Z_N is so chosen that it is negative of the corresponding element in Z and bears the necessary ratio in absolute magnitude. It is clear that the above proofs will hold unaltered for such a general case. However, Z_N will now contain certain negative elements, which may be constructed by any of the known methods.

Summary.

It is shown (in the appendix) that Mathes and Dudley types of circuits can be treated as negative impedances for transient as well as steady state currents just like the Bartlett type of circuits. Transient solutions of series combinations of positive and negative circuit constants are given in charts showing equations and curves for currents under typical applied voltages and initial conditions. For circuits containing only negative constants, it is found that the transient currents die out after a certain period of time, as is well known in the case of circuits containing only positive constants. However, the transients in circuits containing combinations of positive and negative constants increase with time. It is pointed out that these currents cannot increase indefinitely, for a limit is reached beyond which the negative constants lose their pure negative character.

Department of Electrical Technology,
Indian Institute of Science,
Bangalore.

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