APPLICATION OF MACLAURIN SERIES IN STRUCTURAL ANALYSIS

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Summary

In the simple theory of flexure of beams, the slope, bending moment, shearing force, load and other quantities are functions of a derivative of y with respect to x. It is shown that the elastic curve of a transversely loaded beam can be represented by the Maclaurin series. Substitution of the values of the derivatives gives a direct solution of beam problems. In this paper the method is applied to derive the theorem of three moments and slope deflection equations. The method is extended to the solution of a rigid portal frame. Finally the method is applied to deduce three results on which the moment distribution method of analysing rigid frames is based.

1. Introduction

The elementary theory of bending of beams based on the assumption that plane cross-sections of a beam remain plane after deformation gives the following equation for the bending moment at a point in a beam.

$$EI\, d^2y/dx^2 = -M^1 \qquad (1.1)$$

By successive differentiation, we have

$$EI\, d^3y/dx^3 = -V \qquad (1.2)$$

$$EI\, d^4y/dx^4 = q \qquad (1.3)$$

where E = Modulus of elasticity of the material.

I = Moment of inertia of the section about the neutral axis.

M = Bending moment at any point.

V = Shear at the point.

q = The intensity of the distributed loading.

Equations 1.1 to 1.3 represent the 'biography' of the bent beam as shown in Fig. 1.

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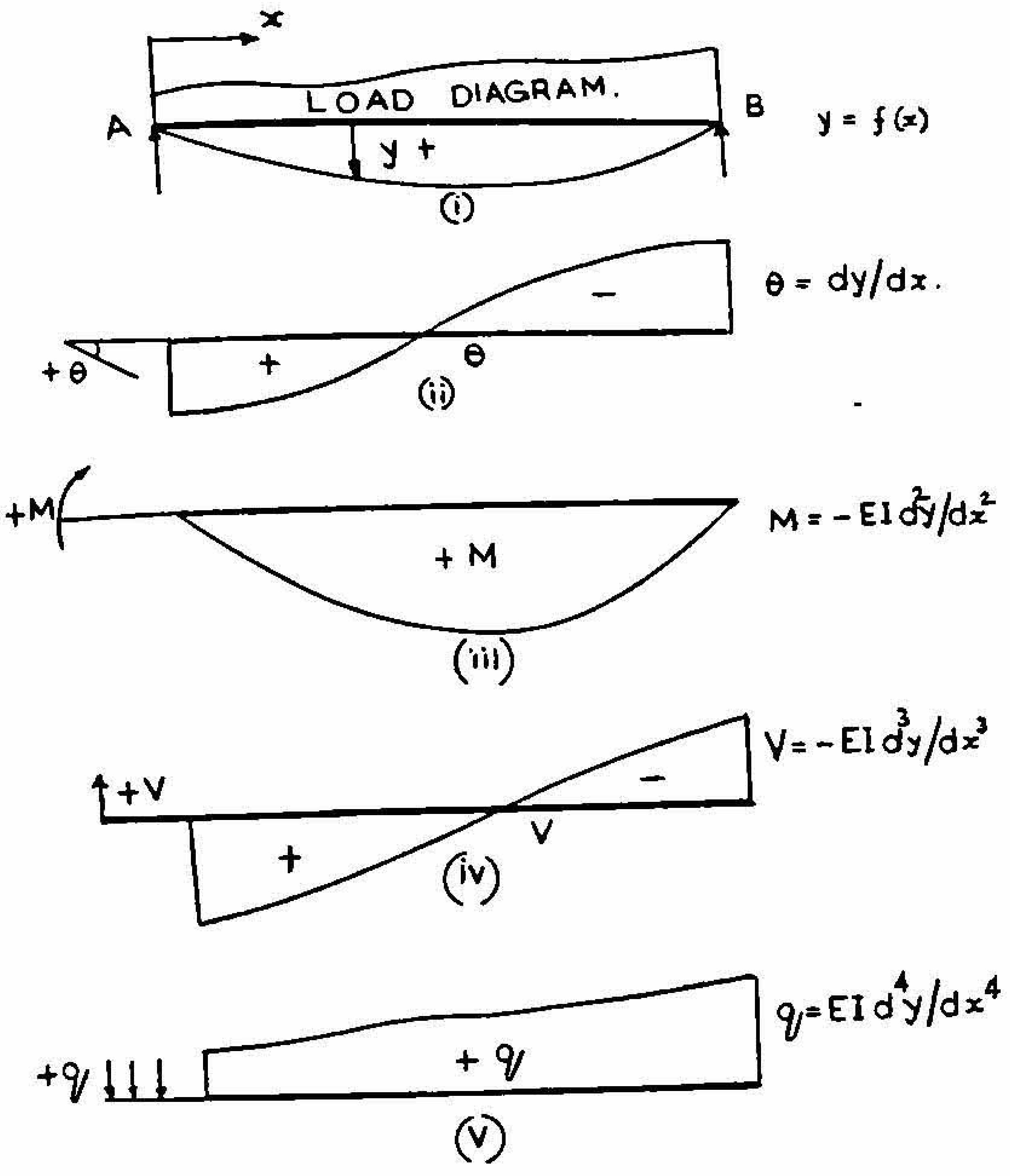

FIG. 1

In an analytical solution, the orthodox method of solving the beam problem is by successive integration of equation (1.1), the constants of integration being given by the end conditions of the beam.

2. Representation of the Deflection Curve by Maclaurin Series

If $y = f(x)$ represents the elastic curve for the beam under consideration, the values of y at any point distant 'x' from the origin may be given by the Maclaurin Series:

$$y = f(x) = f(0) + x/1!\cdot f^{i}(0) + x^2/2!\cdot f^{ii}(0) + x^3/3!\cdot f^{iii}(0) + \dots + x^n/n!\cdot f^{n}(0) + \dots \quad (2\cdot 1)$$

where $f(0)$, $f^{i}(0)$, ... $f^{n}(0)$, etc., denote the values of $f(x)$ and its derivatives at the origin, $x = 0$. Thus we have,

$$\left.\begin{aligned} f\ (0) &= y_0 \text{ (deflection at the origin)} \\ f^{i}\ (0) &= \theta_0 \text{ (slope at the origin)} \\ f^{ii}\ (0) &= -\ M_0/EI \\ f^{iii}\ (0) &= -\ V_0/EI \\ f^{iv}\ (0) &= q_0/EI \end{aligned}\right\} \quad (2.2)$$

$$\therefore \quad y = y_0 + \theta_0 x - \tfrac{1}{2!}\, M_0/EI\cdot x^2 - \tfrac{1}{3!}\cdot V_0/EI\cdot x^3 + \tfrac{1}{4!}\, q_0/EI\cdot x^4 + \ldots \quad (2.3)$$

The great advantage of using this equation for the elastic curve lies in the fact that most often, the physical nature of the problem or simple requirements of statics, define the values of y_0, θ_0, M_0, V_0 and q_0. *e.g.*, In a simply supported beam as shown in Fig. 2 (i), if A is taken as the origin, the end

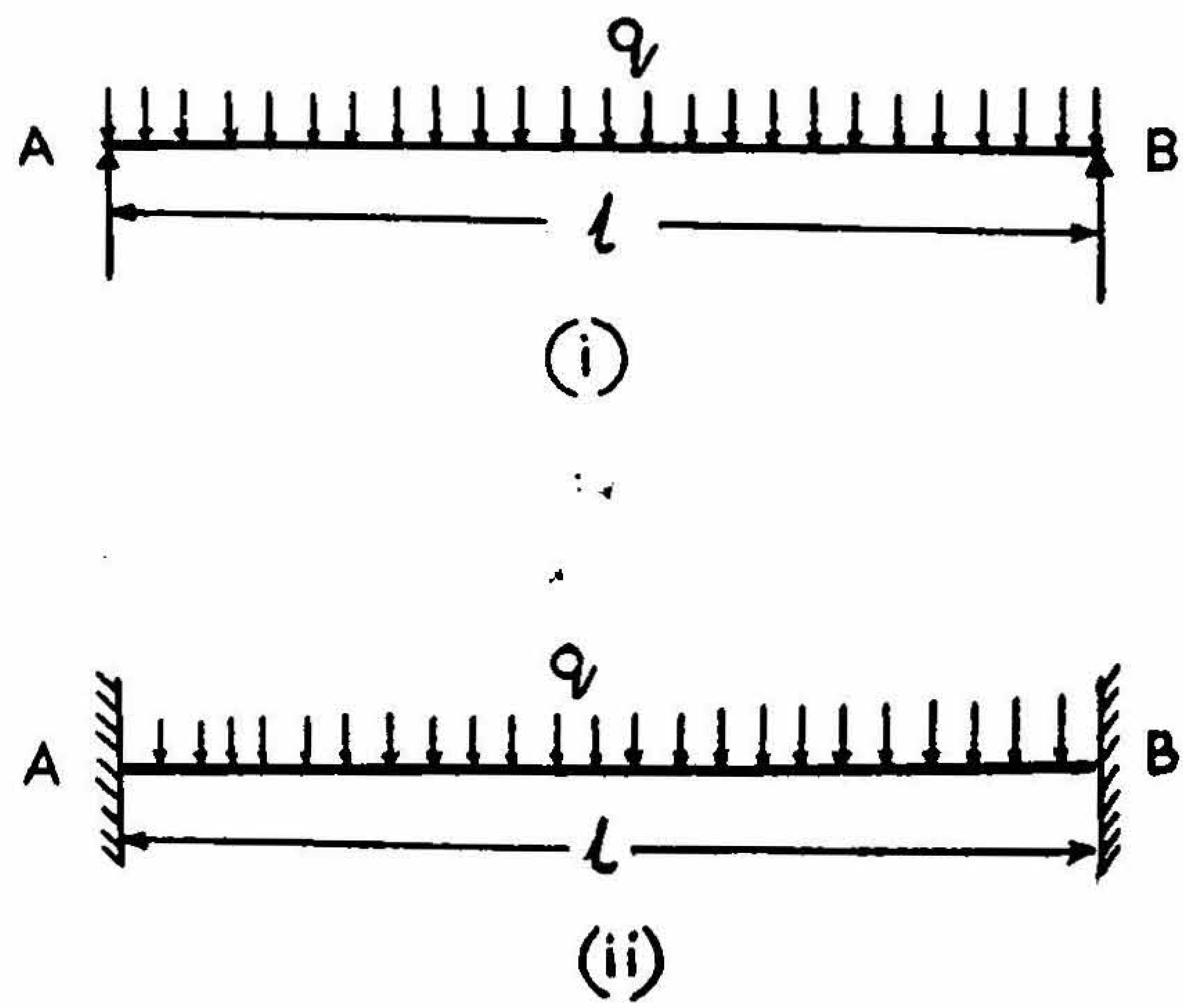

FIG. 2

conditions are $y_A = y_B = 0$ and $M_A = M_B = 0$. V_A and V_B are known from statics. Hence Equation (2.3) can be completely written and the required quantity can be calculated at any point. The series form for the deflection curve will be

$$y = \theta_0 x - \tfrac{1}{3!}\, ql/2\, EI\cdot x^3 + \tfrac{1}{4!}\, qx^4/EI \quad (2.4)$$

Substituting $x = l$, we have

$$\theta_0 = 1/24\cdot ql^3/EI \quad (2.5)$$

$$\therefore \quad y = 1/24\cdot ql^3/EI - 1/12\cdot ql\, x^3/EI + 1/24\, qx^4/EI \quad (2.6)$$

Substituting $x = l/2$ in (2·6), we get the maximum deflection.

$$y_{max} = 5/384 \cdot ql^4/\mathrm{EI} \tag{2·7}$$

In the case of a fixed beam [Fig. 2 (ii)] $y_A = y_B = 0$ and $\theta_A = \theta_B = 0$, V_A and V_B can be calculated from statics. Hence the series for y can be written down easily.

3. Case of an Arbitrary Loading

Representation of the elastic curve $y = f(x)$ by the Maclaurin series, assumes the continuity of the function representing the deflection curve, and all its derivatives. However, in the case of a concentrated loading, there occurs a discontinuity in the shear force at the point of application of the load, *i.e.*, a discontinuity in the 3rd derivative of the deflection curve. Similarly a couple acting at a point produces a discontinuity in the bending moment curve at that point, *i.e.*, in the 2nd derivative of the function. To include all such above cases, the equation (2.3) for the elastic curve can be modified as an extension of Macaulay's method.

Referring to Fig. 3 the modified equation will be

$$y = \theta_0 x - 1/2!\, \mathrm{G/EI}\,(x-b)^2 - 1/3!\, \mathrm{V_0/EI}\, x^3 + \tfrac{1}{3}!\, \mathrm{W/EI}\,(x-a)^3 + 1/24\, qx^4/\mathrm{EI} \tag{3.1}$$

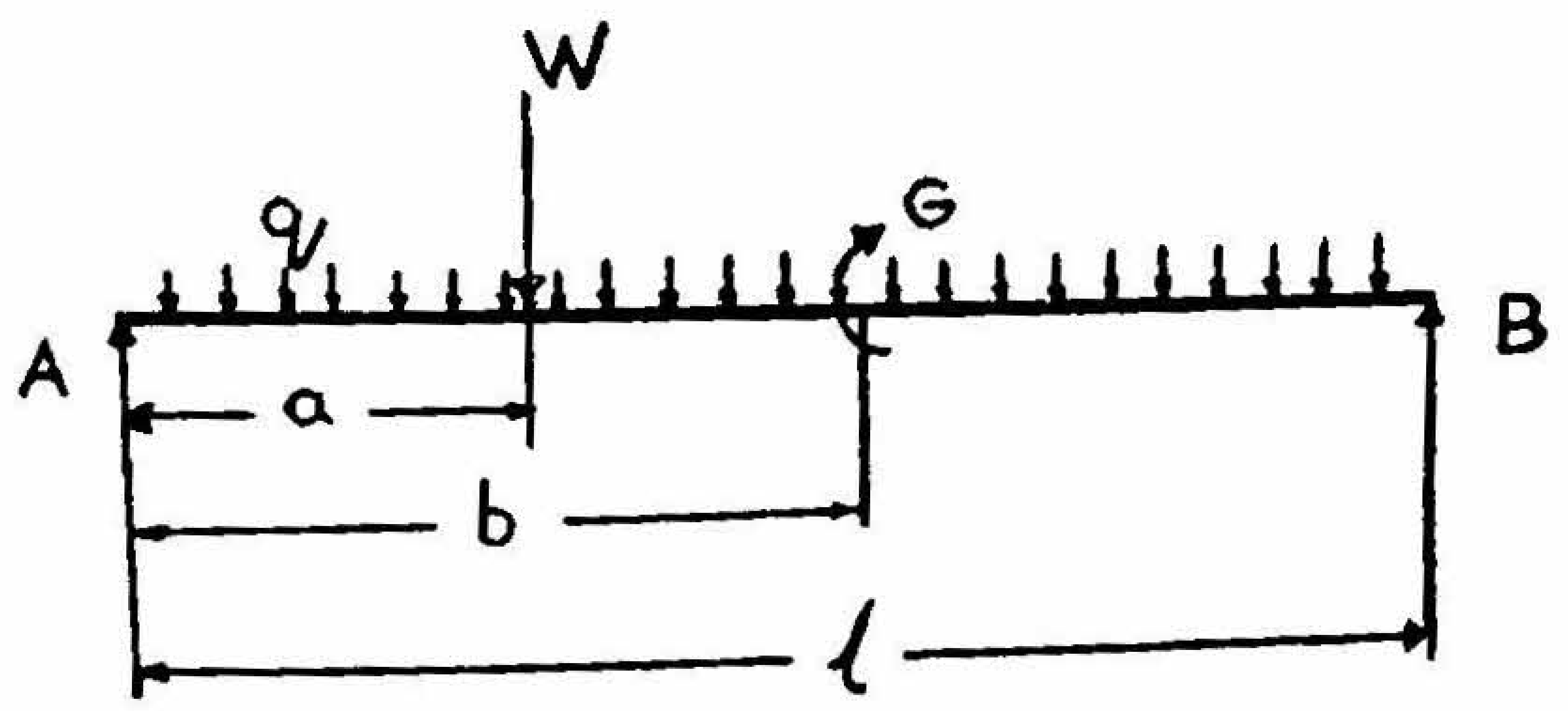

Fig. 3

The terms in the brackets should be interpreted as below:

$(z)^n = z^n$ if z is positive,

$(z)^n = 0$ if z is negative.

With such modifications, the method is capable of generalisation for discontinuities in y, θ or in the derivative of the distributed loading. In all the above cases, the flexural rigidity EI is assumed to be constant. However, the method is readily applicable to the solution of beams of varying cross-sections as shown by Hetenyi.[2]

4. THEOREM OF THREE MOMENTS FOR A CONTINUOUS BEAM

The series form of the deflection curve can be applied to derive the theorem of three moments for a continuous beam shown in Fig. 4.

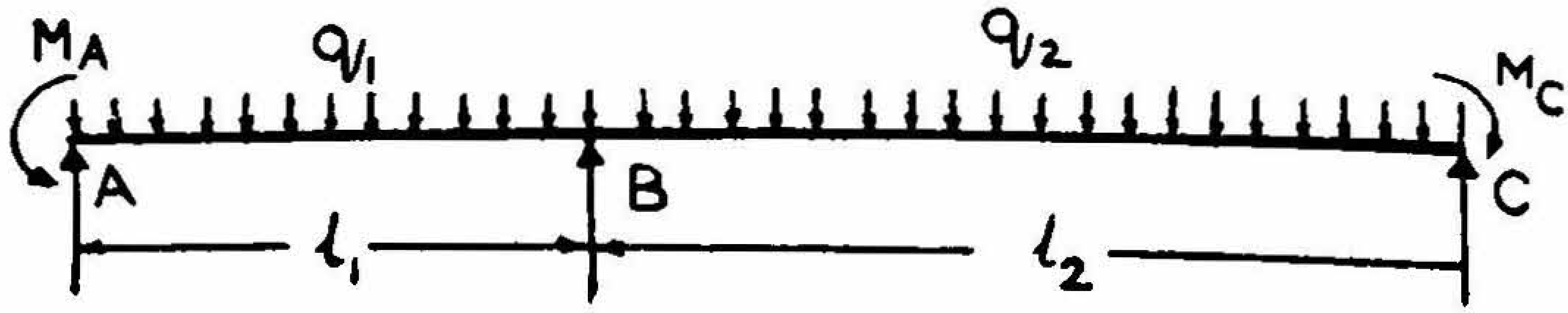

FIG. 4

Consider the span BC. Taking B as the origin, applying equation (2.3) for the span BC,

$$y = \theta_B x - 1/2 \cdot M_B/EI \cdot x^2 - 1/6 \cdot V_B/EI \cdot x^3 + 1/24 \cdot q_2/EI \cdot x^4 \qquad (4.1)$$

Putting $x = l_2$,

$$y_C = \theta_B l_2 - 1/2 \cdot M_B/EI \cdot l_2^2 - 1/6 \cdot V_B/EI \cdot l_2^3 + 1/24\, q_2/EI\, l_2^4 \qquad (4.2)$$

Differentiating twice equation (4.1)

$$d^2y/dx^2 = - M_B/EI - V_B/EI \cdot x + 1/2\, q_2/EI \cdot x^2 \qquad (4.3)$$

Putting $x = l_2$.

$$- M_C = - M_B - V_B l_2 + q_2 l_2^2/2 \qquad (4.4)$$

Eliminating V_B from equations (4.2) and (4.4)

$$- M_C l_2^2/6 - EI\, y_C = - EI\, \theta_B l_2 + M_B l_2^2/3 + q_2 l_2^4/24 \qquad (4.5)$$

There are no concentrated loads between A and C except the reaction at B. Hence the same equation holds good for the span BA (θ_B becomes negative).

$$- M_A l_1^2/6 - EI y_A = EI \theta_B l_1 + M_B\, l_1^2/3 + q_1\, l_1^4/24 \qquad (4.6)$$

Eliminating θ_B we have,

$$M_A l_1 + 2\, M_B (l_1 + l_2) + M_C l_2 = - 6\, EI \left(\frac{y_A}{l_1} + \frac{y_C}{l_2}\right) - q_1 l_1^3/4 - q_2 l_2^3/4 \qquad (4 \cdot 7)$$

which is the Theorem of Three Moments for the continuous beam in Fig. 4. In a similar way any type of loading can be treated.

5. SLOPE-DEFLECTION EQUATIONS IN STRUCTURAL ANALYSIS

The Slope-deflection equations can be conveniently derived for a member of a framework, using the series form of the deflection curve. Let AB e a member of a framework rigidly connected at A and B to other members of the frame. Let it carry a uniformly distributed load of intensity q (Fig. 5).

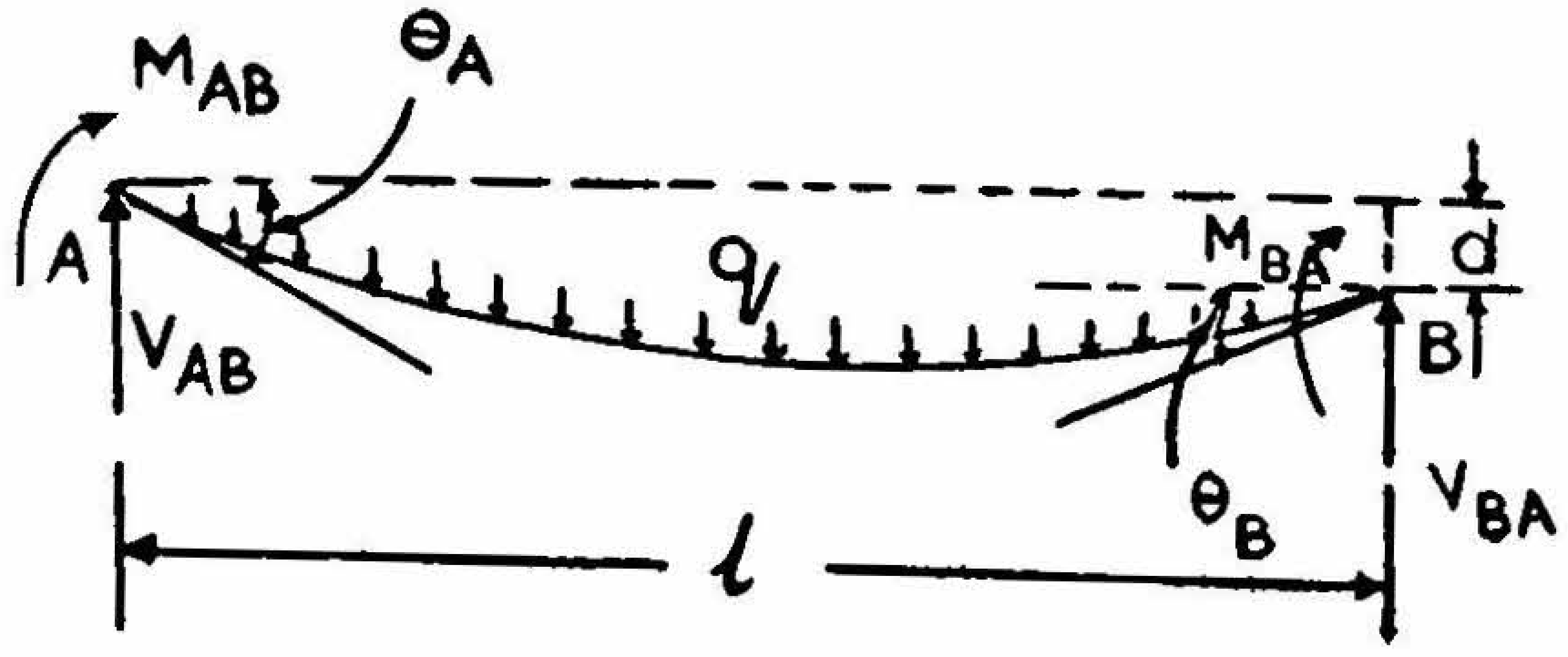

FIG. 5

The restraining moments at the ends A and B are M_{AB} and M_{BA}.

Taking A as origin, applying equation (2.3) for the elastic curve of the beam AB,

$$y = \theta_A \cdot x - 1/2!\, M_{AB}/EI \cdot x^2 - 1/6 \cdot V_{AB}/EI \cdot x^3 + 1/24 \cdot qx^4/EI \qquad (5.1)$$

$$dy/dx = \vartheta_A - M_{AB}/EI \cdot x - 1/2 \cdot V_{AB}/EI \cdot x^2 + 1/6 \cdot qx^3/EI \qquad (5.2)$$

and

$$d^2y/dx^2 = - M_{AB}/EI - V_{AB}/EI \cdot x + 1/2 \cdot qx^2/EI \qquad (5.3)$$

Substituting $x = l$ in the above equations, we have

$$d = \theta_A \cdot l - 1/2 \cdot M_{AB}/EI \cdot l^2 - 1/6 \cdot V_{AB}/EI \cdot l^3 + 1/24 \cdot ql^4/EI \qquad (5.4)$$

$$\theta_B = \theta_A - M_{AB}/EI \cdot l - 1/2 \cdot V_{AB}/EI \cdot l^2 + 1/6\, ql^3/EI \qquad (5.5)$$

$$M_{BA} = - M_{AB} - V_{AB} \cdot l + 1/2\, ql^2 \qquad (5.6)$$

Eliminating V_{AB} from these equations, we obtain the standard slope-deflection equations for the beam AB.[3]

$$M_{AB} = 2\, EI/l \cdot [2\, \theta_A + \theta_B - 3d/l] - ql^2/12 \qquad (5.7)$$

$$M_{BA} = 2\, EI/l \cdot [\theta_A + 2\, \theta_B - 3d/l] + ql^2/12 \qquad (5.8)$$

6. APPLICATION TO RIGID FRAME

Consider a hinged portal frame with a single concentrated load on the horizontal member at its centre, Fig. 6 (i).

The bent shape of the frame is shown in dotted lines. In the analysis, we neglect the change in length of the bars and the effect of axial forces on the bending of bars. The frame can be considered as made up of three beams as shown in Fig. 6 (ii).

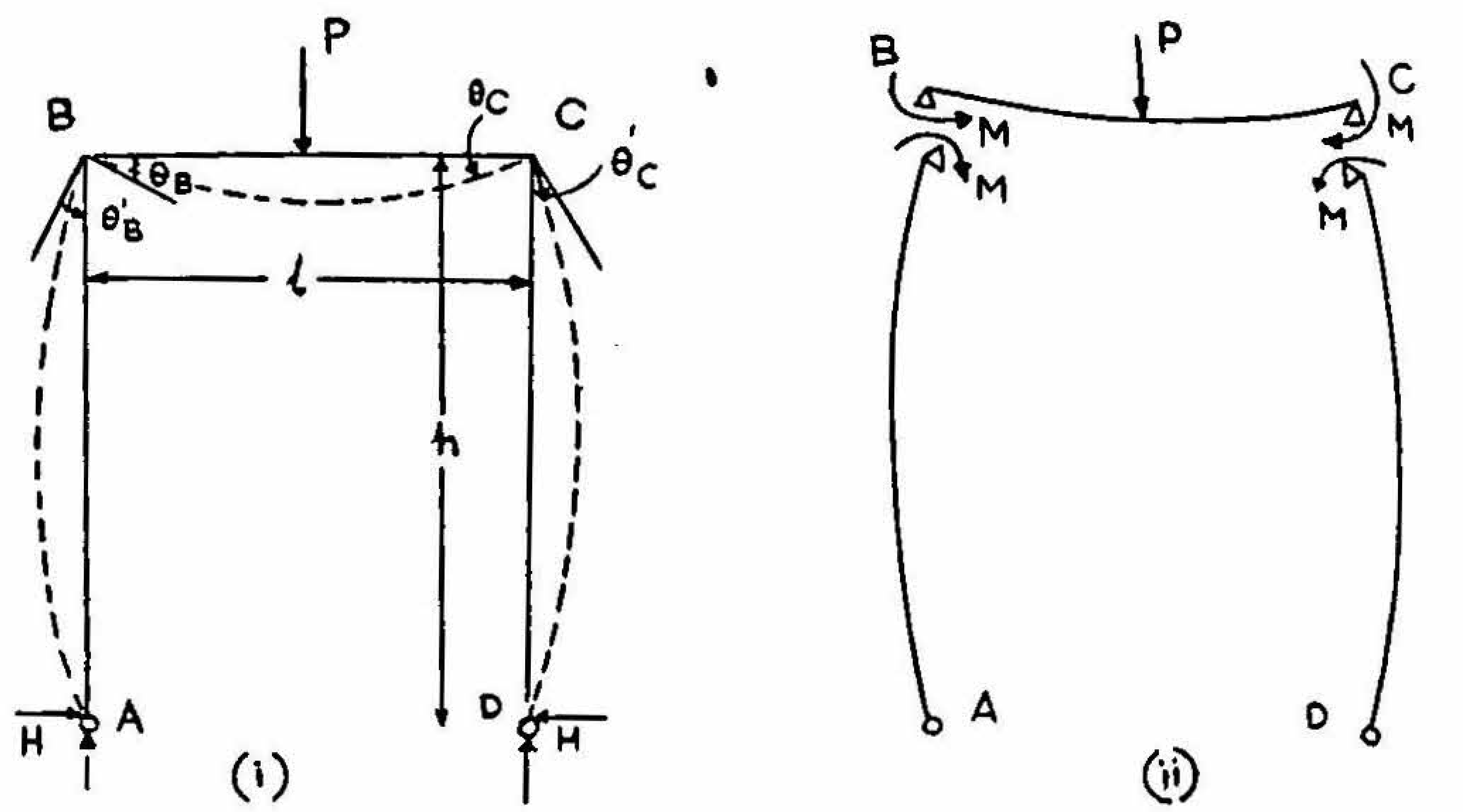

FIG. 6

Consider the horizontal beam BC. Taking B as the initial point, writing the deflection curve in the series form,

$$y = \theta_B \cdot x + 1/2\ M/EI \cdot x^2 - 1/6\ V_B/EI \cdot x^3 + 1/6\ P/EI\,(x - l/2)^3 \qquad (6.1)$$

$$dy/dx = \theta_B + M/EI \cdot x - 1/2 \cdot V_B/EI \cdot x^2 + 1/2\ P/EI\,(x - l/2)^2 \qquad (6.2)$$

$$d^2y/dx^2 = M/EI - V_B \cdot x/EI + P/EI.\ (x - l/2) \qquad (6.3)$$

Substituting $x = l$ and eliminating θ_B and V_B

$$\theta_C = Ml/2\ EI - 1/16\ Pl^2/EI \qquad (6 \cdot 4)$$

Applying similar equations to the member DC, taking D as the initial point and assuming uniform EI throughout the frame, we obtain $\theta'_C = - Mh/3\ EI$. Since C is a rigid joint, $\theta_C = \theta'_C$

$$\therefore \quad Ml/2\ EI - 1/16\ Pl^2/EI = - Mh/3\ EI \qquad (6.5)$$

$$\therefore \quad M = Pl/8 \cdot 1/1 + \tfrac{2}{3}\, h/l \qquad (6.6)$$

The above equations can be modified for different types of loading symmetrical or asymmetrical.

Application of this method to multistoreyed frames, including sway effects, will be considered in a subsequent paper.

7. Using the method of series expansion for the deflection curve, three results will be deduced on which is based the Moment Distribution Method of analysing rigid frames.

Problem I.—Given AB, a member of a frame or a section of a beam under any system of transverse loading, it is required to find the external couple required to produce an additional rotation θ at B such that there is no additional deflection at B and additional slope and deflection at A are zero.

Taking A as the origin, considering only additional loads and deflections, the series expansion for the deflection curve will be

$$y = y_A + \theta_A \cdot x - 1/2\, M_A/EI \cdot x^2 - 1/6\, V_A/EI \cdot x^3 \tag{7.1}$$

$$dy/dx = \theta_A - M_A \cdot x/EI - 1/2\, V_A/EI\, x^2 \tag{7.2}$$

$$d^2y/dx^2 = - M_A/EI - V_A/EI \cdot x \tag{7.3}$$

From the conditions of the problem, we have

$$y_A = y_B = 0. \quad \theta_A = 0 \text{ and } \theta_B = \theta$$

from which we obtain the external couple required at B

$$M_B = 2\, M_A = 4\, EI\theta/l \cdot\ = 4\, EK\theta \text{ where } I/l = k$$

and the couple required at A will be $M_A = 1/2 \cdot M_B$.

Thus we have the first result: To produce a rotation θ at B of a beam AB without change of deflection at B and the end A being fixed in direction and position, the external moment required is proportional to $Ek\theta$ and the external moment required at the end A is half the above moment in the same direction.

Problem II.—It is required to find the external moment required to produce an additional rotation at B and no deflection, the end A being fixed in position but not in direction.

In this case we have $y_A = 0$, $M_A = 0$. The deflection curve for the beam AB takes the form

$$y_x = \theta_A \cdot x - V_A/6\, EI \cdot x^3 \tag{7.4}$$

$$\therefore\ \theta_x = \theta_A - V_A/2\, EI \cdot x^2 \tag{7.5}$$

when $x = l$, $y = 0$, $\theta_B = \theta$

$$\therefore\ V_A = 6\, EI/l^2 \cdot \theta_A \tag{7.6}$$

and

$$M_B = - V_A \cdot l = - 6\, EI/l \cdot \theta_A = 3\, EI\, \theta/l = 3\, E\, k\, \theta \tag{7.7}$$

Hence the external couple required at B, $M_B = 3\, Ek\theta$.

Problem III.—It is required to find the external moment required to produce an additional deflection Δ at B without rotation. A is fixed in position and direction.

The equation for the deflection curve will be

$$y_x = - M_A/2\ EI \cdot x^2 - V_A/6\ EI \cdot x^3 \tag{7.8}$$

$$\therefore\ \theta_x = - M_A/EI^x - V_A/2\ EI \cdot x^2 \tag{7.9}$$

and

$$M_x = - M_A - V_A x \tag{7.10}$$

When $x = l$, $y = \Delta$ and $\theta = 0$.

These conditions give

$$EI\ \Delta = - M_A l^2/2 - V_A l^3/6 \tag{7.11}$$

$$0 = - M_A l/EI - V_A l^2/2\ EI \tag{7.12}$$

$$M_B = - M_A - V_A l \tag{7.13}$$

from which $M_B = - 6\ EI\ \Delta/l^2 = M_A$.

Hence, the external moment required to produce an additional deflection Δ at B without rotation, the end A being fixed in position and direction is $- 6\ Ek\Delta/l$ and the couple required at A is also the same.

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