ON A CLASS OF VISCOUS COMPRESSIBLE FLOWS

BY S. K. LAKSHMANA RAO
(*Section of Civil and Hydraulic Engineering, Indian Institute of Science, Bangalore–3*)

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INTRODUCTION

The equation of motion of a viscous compressible fluid can be put in the form[1]

$$\frac{d\vec{q}}{dt} = -\nabla\left(\Omega + \int\frac{dp}{\rho}\right) + \tfrac{1}{3}\nu\nabla(\nabla\vec{q}) + \nu\nabla^2\vec{q}$$

when the external force field is conservative. The equation can be integrated once if the kinematic viscosity ν is constant and the motion is irrotational, the integral being

$$-\frac{\partial\phi}{\partial t} + \tfrac{1}{2}q^2 + \Omega + \int\frac{dp}{\rho} - \frac{4}{3}\nu\nabla\vec{q} = F(t)$$

where ϕ is the velocity potential and $\vec{q} = -\nabla\phi$. We can integrate the equation of motion also when the flow is steady and the fluid is inviscid, the result being the well-known Bernoulli's Theorem. Inviscid fluid motion is further characterized by the constancy of circulation (Kelvin's Theorem) and the permanence of rotational or irrotational nature of the flow. In a viscous incompressible fluid, however, the circulation in a circuit varies at a rate depending on the kinematic viscosity and the space rates of change of the vorticity components. The vorticity itself experiences a decay on account of viscosity.

In the present note we consider the class of flows of viscous compressible fluids for which the vector $\nabla^2\vec{q}$ is irrotational and has a scalar potential function H; equivalently the vorticity components are harmonic functions. It is shown here that such flows are similar to an inviscid flow in general characteristics. The kinematic viscosity is assumed constant.

(1) For the class of flows under consideration

$$\nabla^2\vec{q} = -\nabla H.$$

The equation of motion can therefore be written as

$$\frac{\partial\vec{q}}{\partial t} - \vec{q}\times\vec{\zeta} = -\nabla\left(\Omega + \int\frac{dp}{\rho} + \tfrac{1}{2}q^2 - \tfrac{1}{3}\nu\nabla\vec{q} + \nu H\right) \qquad (1)$$

where $\vec{\zeta} = \text{curl}\ \vec{q}$. For steady flow $\frac{\partial \vec{q}}{\partial t} = 0$ and

$$\vec{q} \times \vec{\zeta} = \nabla \left(\Omega + \int \frac{dp}{\rho} + \tfrac{1}{2} q^2 - \tfrac{1}{3} \nu\, \nabla \vec{q} + \nu\, \mathrm{H} \right) \tag{1 a}$$

Scalar multiplication of the two sides by $\frac{\vec{q}}{|\vec{q}|}$ and $\frac{\vec{\zeta}}{|\vec{\zeta}|}$ gives

$$\frac{\partial}{\partial s} \left(\Omega + \int \frac{dp}{\rho} + \tfrac{1}{2} q^2 - \tfrac{1}{3} \nu\, \nabla \vec{q} + \nu\, \mathrm{H} \right) = 0$$

and

$$\frac{\partial}{\partial \sigma} \left(\Omega + \int \frac{dp}{\rho} + \tfrac{1}{2} q^2 - \tfrac{1}{3} \nu\, \nabla \vec{q} + \nu\, \mathrm{H} \right) = 0$$

where ds, $d\sigma$ are arc elements along stream lines and vortex lines respectively. We have, therefore,

$$\Omega + \int \frac{dp}{\rho} + \tfrac{1}{2} q^2 - \tfrac{1}{3} \nu\, \nabla \vec{q} + \nu\, \mathrm{H} = \text{const.}, \tag{2}$$

along streamlines as well as vortex lines.

This result, in so far as it relates to streamlines, provides an extension of Bernoulli's Theorem and its analogue for plane incompressible flows has been noticed by Görtler and Wieghardt.[2]

(2) The circulation in a closed circuit in the fluid is

$$\mathrm{C} = \int \vec{q}.d\vec{r}$$

$$\frac{d\mathrm{C}}{dt} = \int \frac{d\vec{q}}{dt}.d\vec{r} + \int \vec{q}.\frac{d}{dt}\left(d\vec{r}\right)$$

$$= -\int \nabla\left(\Omega + \int \frac{dp}{\rho} - \tfrac{1}{3} \nu\, \nabla \vec{q} + \nu\, \mathrm{H} \right).d\vec{r} + \int \vec{q}.d\vec{q}$$

$$= -\int \frac{\partial}{\partial s} \left(\Omega + \int \frac{dp}{\rho} - \tfrac{1}{3} \nu\, \nabla \vec{q} + \nu\, \mathrm{H} \right) ds + \int \frac{\partial}{\partial s}\left(\tfrac{1}{2} q^2\right) ds$$

$$= 0.$$

The circulation in any circuit moving with the fluid is constant as in inviscid fluid motion.

(3) The curl of equation (1) gives

$$\frac{\partial}{\partial t}\left(\text{curl}\,\vec{q}\right) - \text{curl}\left(\vec{q}\times\vec{\zeta}\right) = 0$$

or

$$\frac{\partial}{\partial t}\left(\text{curl}\,\vec{q}\right) - \left\{\left(\text{curl}\,\vec{q}\,.\,\nabla\right)\vec{q} - \left(\vec{q}\,.\,\nabla\right)\text{curl}\,\vec{q} - \text{curl}\,\vec{q}\,\text{div}\,\vec{q} + \vec{q}\,\text{div}\left(\text{curl}\,\vec{q}\right)\right\} = 0$$

i.e.,

$$\frac{d}{dt}\left(\text{curl}\,\vec{q}\right) = \left(\text{curl}\,\vec{q}\,.\,\nabla\right)\vec{q} - \text{curl}\,\vec{q}\,\text{div}\,\vec{q} \qquad (3)$$

Helmholtz's Theorem in inviscid fluids can be put in the form[3]

$$\frac{d}{dt}\left(\frac{\vec{\zeta}}{\rho}\right) = \frac{1}{\rho}\left(\vec{\zeta}\,.\,\nabla\right)\vec{q}$$

and (3) is equivalent to it. Thus (3) provides an extension of Helmholtz's Theorem. A consequence of (3) is that the rotational or irrotational nature of any portion of the fluid is permanent as in inviscid fluids. This is evident otherwise for in the class of flows considered here, there is no decay of vorticity in spite of viscosity as $\nabla^2\vec{\zeta} = 0$.

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REFERENCES

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