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THE TORSION OF SOLID REGULAR HEXAGONAL SHAFT BY RELAXATION METHODS

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INTRODUCTION

Torsion problems of solid and hollow prismatical bars of uniform cross-section have been solved by various workers. Ōkubo[1] recently developed an approach for the torsion problem of a prismatic cylinder. The technique of the relaxational approach to these problems has been used by Southwell.[2] The solution for the torsion of the equilateral triangle problem has been given by him. Similar problems have been solved by Shaw[3] and others.

The determination of stress distribution in a prismatic bar with uniform cross-section subjected to torsion consists in finding the stress function ψ. The stress function ψ satisfies the differential equation

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = -2 \tag{1}$$

at every internal point of the shaft and satisfies the boundary condition that ψ has a constant value on the boundary. Without loss of generality we assume ψ to be zero on the boundary.

The corresponding stress components are

$$Y_z = -\mu\tau\frac{\partial\psi}{\partial x}, \qquad X_z = \mu\tau\frac{\partial\psi}{\partial y}, \tag{2}$$

where τ denotes the angle of twist per unit length and μ, the modulus of rigidity.

To work with non-dimensional units we put $\psi = D^2\chi$, where D is some representative length in the cross-section so that χ is purely numerical. Writing

$$x = Dx', \; y = Dy' \tag{3}$$

so that x', y' are numbers, and substituting in equation (1) we obtain

$$\nabla'^2\chi = \frac{\partial^2\chi}{\partial x'^2} + \frac{\partial^2\chi}{\partial y'^2} = -2 \tag{4}$$

an equation involving purely numerical quantities. This equation is now replaced by approximate relations involving finite differences. We next proceed to the evaluation of the wanted function χ, at a large number of nodal points within the specified domain.

To start with we employ triangular net so that there is only one internal node O, at the centre of the hexagon. Let D denote the length of a side and a, the mesh side. The partial differential equation (4) is now replaced by equations of the type

$$\chi_1 + \chi_2 + \chi_3 + \chi_4 + \chi_5 + \chi_6 - 6\chi_0 = -\frac{3}{2} \cdot 2a^2 \tag{5}$$

where $\chi_i \; (i = 1, 2, \ldots, 6)$ denote the values of χ at the six nodal points A, B, C, D, E, F around O.

Here $\chi_i = 0 \; (i = 1, 2, \ldots, 6)$, since all these points are on the boundary. Also $a = 1$. Hence χ_0, the displacement at O is given by

$$\chi_0 = \frac{1}{6} \cdot \frac{3}{2} \cdot 2 = \frac{1}{2} \tag{i}$$

The equation (5) of finite differences written as

$$\chi_1 + \chi_2 + \chi_3 + \chi_4 + \chi_5 + \chi_6 - 6\chi_0 + \tfrac{1}{2}\,\mathrm{N}a^2 = 0 \tag{6}$$

(N denoting the number of meshes adjoining each node, N = 6 in the case of triangular nets) represents the equation of equilibrium at the node O.

In general, equation (6) is not true since the values of χ around any node are only assumed values and hence will not be very close to correct values. So we have to replace equation (6) by the equation

$$\sum_{i=1}^{6} \chi_i - 6\chi_0 + \tfrac{1}{2}\mathrm{N}a^2 = \mathbf{F}_0 \tag{7}$$

where $\mathbf{F}_0$ denotes the residual force at O. The problem can be considered to be solved when the residual force at each nodal point is reduced to zero or a negligibly small number.

Equation (7) can be rewritten in the form

$$\mathbf{F}_0 = F_0 + \mathrm{F}_0 \tag{8}$$

where $F_0 = \sum_{i=1}^{6} \chi_i - 6\chi_0$, is the force at O due to the displacements and $\mathrm{F}_0 = \frac{1}{2}\mathrm{N}\,a^2$, is the external force applied at O.

We now proceed to finer and finer nets. The second advance is to hexagonal nets with the centroids of these six triangles of the original triangular net as nodes. Now the mesh side $a = \dfrac{1}{\sqrt{3}}$ and $\mathrm{N} = 3$. Denoting these new nodes by A_i $(i = 1, 2, \ldots, 6)$ the values of the trial displacements at each nodal point are calculated as follows. The external force at A_1 is

$$\mathrm{F}_{\mathrm{A}_1} = \tfrac{1}{2}\mathrm{N}a^2 = \tfrac{1}{2}\cdot 3\cdot\left(\frac{1}{\sqrt{3}}\right)^2 = \tfrac{1}{2}.$$

The force at A_1 due to displacements is

$$\begin{aligned} F_{\mathrm{A}_1} &= \chi_{\mathrm{O}} + \chi_{\mathrm{A}} + \chi_{\mathrm{B}} - 3\chi_{\mathrm{A}_1} \\ &= \tfrac{1}{2} + 0 + 0 - 3\chi_{\mathrm{A}_1} \end{aligned}$$

The residual force at A_1 is

$$\mathbf{F}_{\mathrm{A}_1} = \tfrac{1}{2} + \tfrac{1}{2} - 3\chi_{\mathrm{A}_1} \text{ by eqn. (8)}$$

and it will vanish if $\chi_{\mathrm{A}_1} = \frac{1}{3}$.

From consideration of symmetry,

$$\chi_{\mathrm{A}_i} = \tfrac{1}{3}\ (i = 1, 2, \ldots, 6) \tag{ii}$$

The third advance is to triangular nets in which $a = \frac{1}{3}$; since six surrounding nodes round each nodal point of this third net at which the values of χ are known cannot be found, we take $\mathrm{N} = 3$, in the calculation of the trial displacements.

Let $\mathrm{B}_1, \mathrm{B}_2, \ldots, \mathrm{B}_6$ denote the centroids of triangles $\mathrm{OA_1A_2}$, $\mathrm{OA_2A_3}$, $\ldots$, $\mathrm{OA_6A_1}$ and $\mathrm{C}_1, \mathrm{C}_2, \ldots, \mathrm{C}_6$, the centroids of triangles $\mathrm{BA_1A_2}$, $\mathrm{CA_2A_3}$, $\ldots$, $\mathrm{AA_1A_6}$. Then

$$\begin{aligned} \mathrm{F}_{\mathrm{B}_1} &= \tfrac{1}{2}\mathrm{N}a^2 = \tfrac{1}{2}\cdot 3\cdot(\tfrac{1}{3})^2 = \tfrac{1}{6}. \\ F_{\mathrm{B}_1} &= \chi_0 + \chi_{\mathrm{A}_1} + \chi_{\mathrm{A}_2} - 3\chi_{\mathrm{B}_1} \\ &= \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{3} - 3\chi_{\mathrm{B}_1} \\ \therefore\quad \mathbf{F}_{\mathrm{B}_1} &= \tfrac{1}{6} + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{3} - 3\chi_{\mathrm{B}_1} \end{aligned}$$

$\mathbf{F}_{\mathrm{B}_1}$ will vanish if $\chi_{\mathrm{B}_1} = \frac{4}{9}$.

By symmetry $\chi_{B_i} = \frac{1}{6} \; (i = 1, 2, \ldots, 6)$ (iii)

Similarly,

$$F_{C_1} = \tfrac{1}{6}$$

$$F_{C_1} = \chi_B + \chi_{A_1} + \chi_{A_2} - 3\chi_{C_1}$$

$$= 0 + \tfrac{1}{3} + \tfrac{1}{3} - 3\chi_{C_1}$$

$$F_{C_1} = \tfrac{1}{6} + \tfrac{1}{3} + \tfrac{1}{3} - 3\chi_{C_1}$$

$$\therefore \quad F_{C_1} = 0 \text{ if } \chi_{C_1} = \tfrac{5}{18}.$$

Again by symmetry

$$\chi_{C_i} = \tfrac{5}{18} \; (i = 1, 2, \ldots, 6) \qquad \text{(vi)}$$

If $D_1, D_2, \ldots, D_{12}$ denote the nodal points on the boundary then

$$\chi_{D_i} = 0 \; (i = 1, 2, \ldots, 12) \qquad \text{(v)}$$

The fourth advance is to hexagonal nets in which $a = \dfrac{1}{3\sqrt{3}}$. The nodal points of this net are the centroids $E_i \; (i = 1, 2, \ldots, 6)$ of the six triangles surrounding O, the centroids F_i, G_{i_1}, G_{i_2}, H_{i_1}, H_{i_2}, I_{i_2}, $(i = 1, 2, \ldots, 6)$ of the six triangles surrounding each A_i and the centroids I_{i_1}, I_{i_3} $(i = 1, 2, \ldots, 6)$ of the two triangles surrounding each nodal point of the boundary belonging to the first triangular net. The values of the trial displacements at each nodal point of this net are calculated just as before in terms of the displacements of the three surrounding nodes of the previous net.

The fifth and the final advance is to triangular nets in which $a = \frac{1}{9}$. The new nodal points of this net on the sides of the first triangular net are J_i, K_i, L_i, M_i, N_i and $P_i \; (i = 1, 2, \ldots, 6)$. $J_1, J_2, \ldots, J_6$ are the centroids of the triangles $OE_1E_2, OE_2E_3, \ldots, OE_6E_1$; $K_1, \ldots, K_6$ are the centroids of the triangles $B_1E_1E_2, \ldots, B_6E_6E_1$; $L_1, \ldots, L_6$ are of triangles $B_1G_{12}G_{21}, \ldots, B_6G_{62}G_{11}$; $M_1, \ldots, M_6$ are of triangles $C_1G_{12}G_{21}, \ldots, C_6G_{62}G_{11}$; $N_1, \ldots, N_6$ of triangles $C_1I_{13}I_{21}, \ldots, C_6I_{63}I_{11}$, and $P_1, \ldots, P_6$ of triangles $BI_{13}I_{21}, \ldots, AI_{63}I_{11}$. The values of χ at these points are calculated just as before by taking $N = 3$.

On account of six axes of symmetry of the regular hexagon, it is possible to consider only 1/12th of the section. The values of χ at the nodal points within this section are obtained similarly in terms of the known values of χ at three surrounding nodes of the previous net.

Now the residual force at each nodal point is calculated by taking $N = 6$. The residual at O is given by

$$\mathbf{F}_0 = \mathrm{F}_0 + F_0$$

$$= \tfrac{1}{2}\,\mathrm{N}a^2 + \chi_{J_1} + \chi_{J_2} + \chi_{J_3} + \chi_{J_4} + \chi_{J_5} + \chi_{J_6} - 6\chi_0$$

$$= \frac{1}{2}\cdot 6\cdot\left(\frac{1}{9}\right)^2 + \frac{40}{81} + \frac{40}{81} + \frac{40}{81} + \frac{40}{81} + \frac{40}{81} + \frac{40}{81} - 6\cdot\tfrac{1}{2}$$

$$= \frac{3}{81} + \frac{240}{81} - 3$$

$$= 0.$$

Similarly the residual at each point is calculated. Before relaxing the trial displacements and residuals are multiplied by 162 to avoid fractions.

Fig. 1 shows the regular hexagon with only four advances. Fig. 2 shows 1/12th of this section with all the nodal points of the finest net. The values of the trial displacements and the consequent residûals are marked to the left and right of each nodal point.

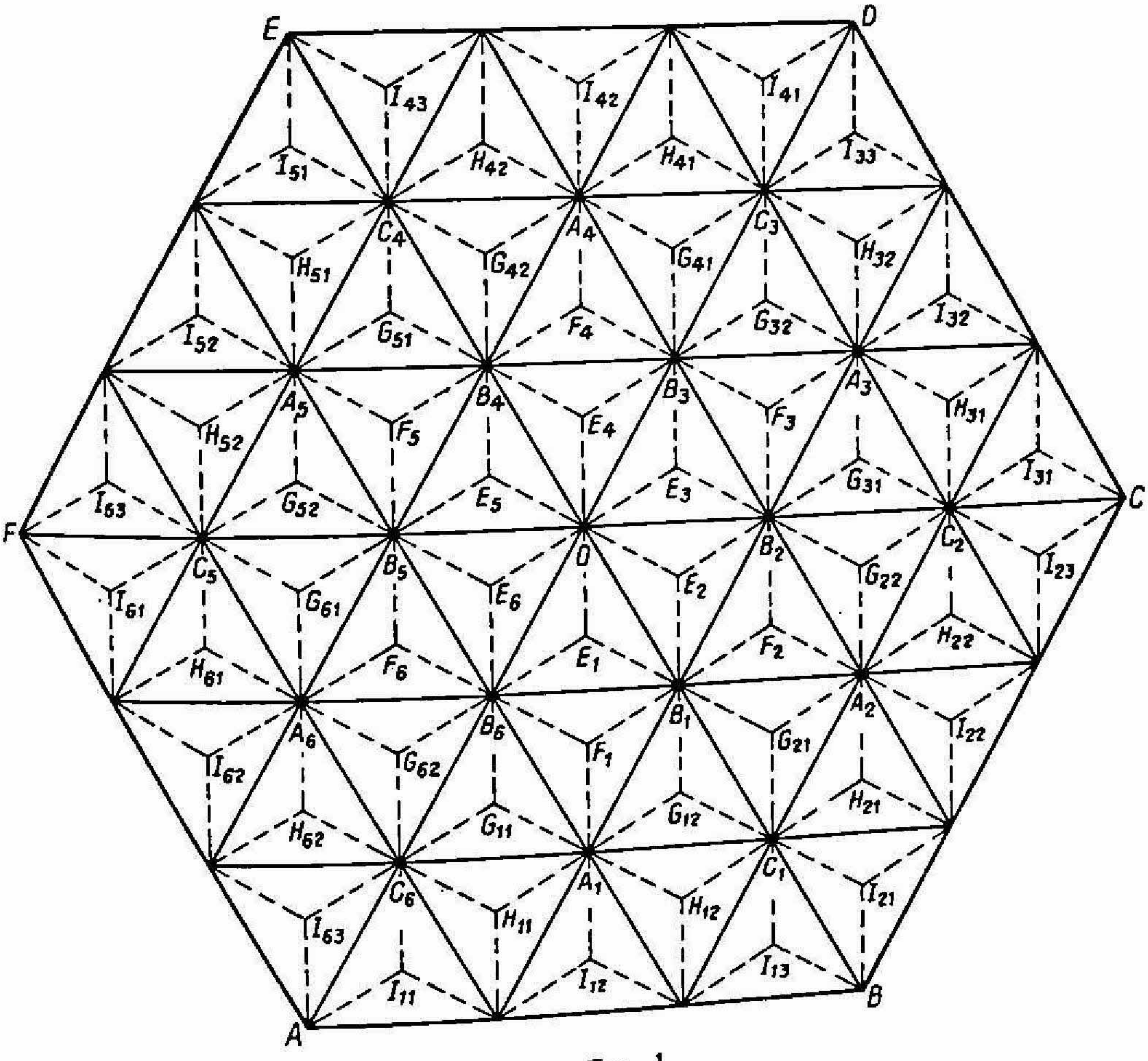

FIG. 1

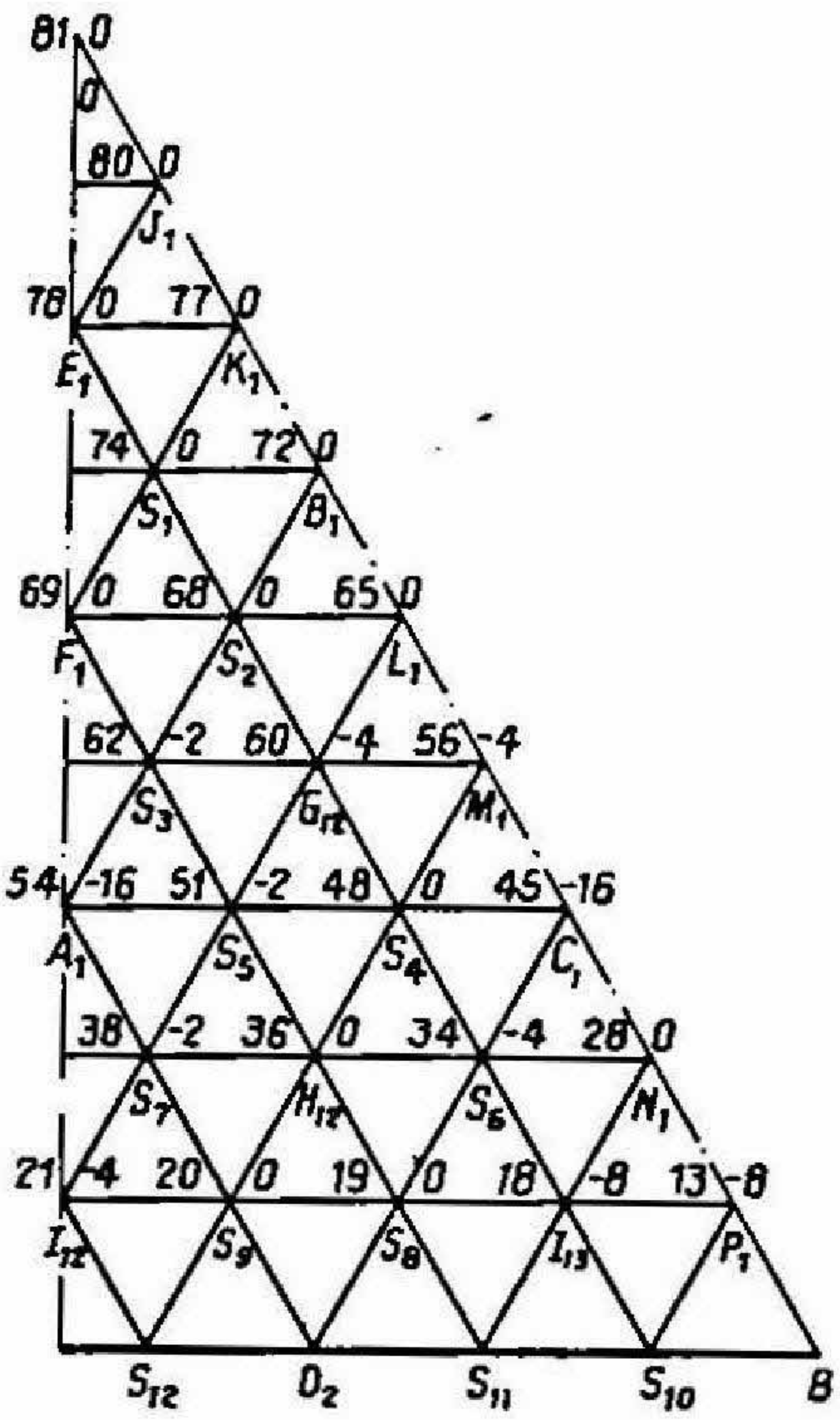

FIG. 2

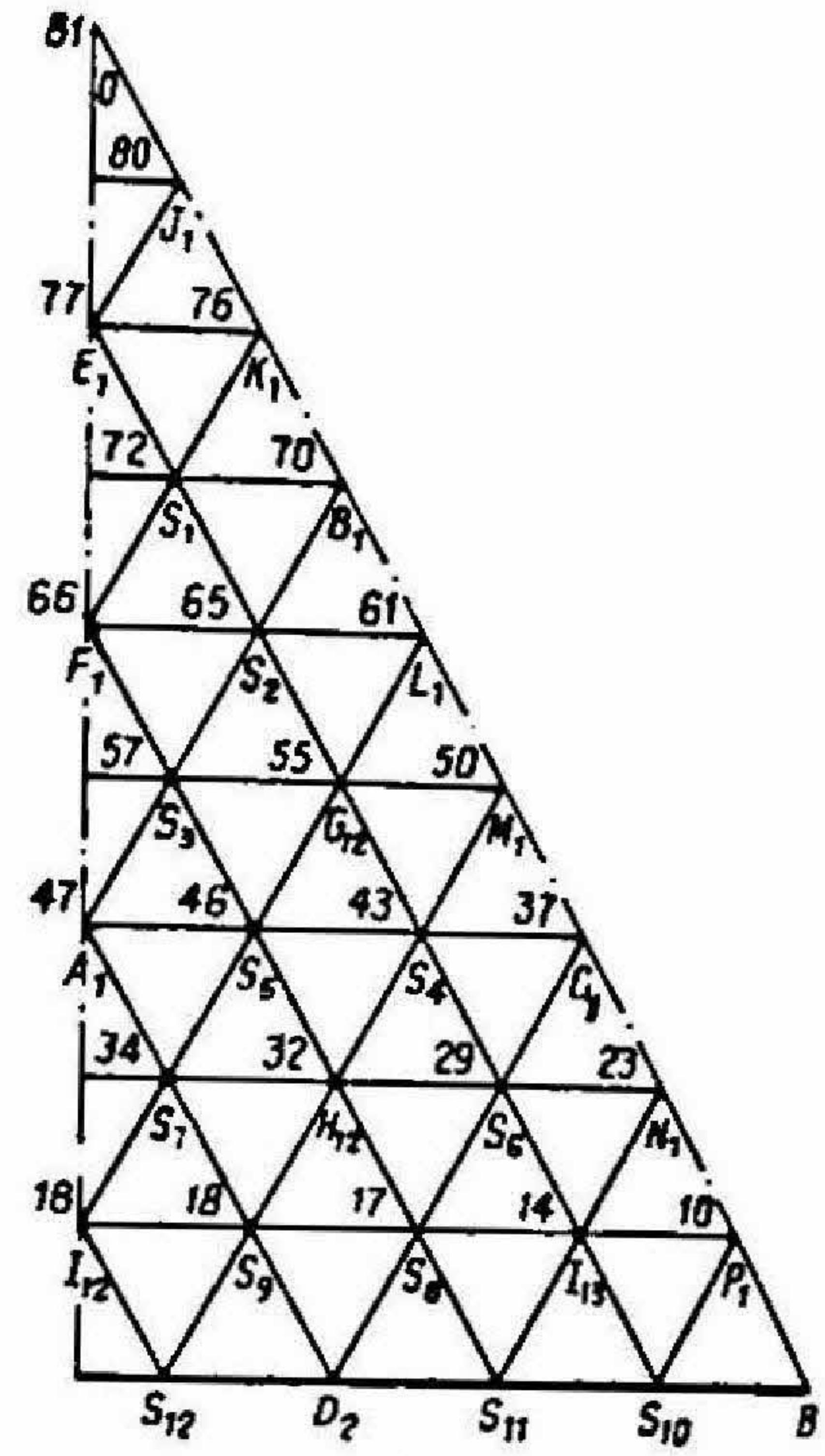

FIG. 3

The final values of the displacements (values of X) after relaxation are shown in Fig. 3. The shear stress at any point can be calculated from equations (2).

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