

ROTATING DISC WITH A STRESS-VARIATION GIVEN BY UNIT FUNCTION

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Problems in Rotating Discs can be classified broadly as those in which the disc shape is specified, *i.e.*, $Y=f(x)$ is given and the stresses are required, and the converse problem where the stress variation is prescribed and the disc shape is required.

When a stress variation is prescribed, the procedure for determining the shape is simple. Since the tangential and the radial stresses are not independent of each other, when one of the stresses is prescribed the other stress has got to be computed from the compatibility relation. Having known both the tangential and radial stress variation the disc thickness can be computed by integrating the equilibrium equation for the given boundary conditions.

The disc shape thus determined satisfies:

- (a) the equilibrium equation;
- (b) the compatibility equation; and
- (c) the boundary conditions,

which is a fundamental requirement of disc design.

Since in general the radial stress is usually greater, let us assume its variation as given by the unit function

$$\sigma_r = \sigma_{r \max} (1 - e^{-kx}), \quad x > 0 \quad (1)$$
$$k > 0$$

where x is the radius and k is a positive parameter. When $x=0$, *i.e.*, at the origin $\sigma_r=0$. The variation of σ_r for different values of k has been plotted in Fig. 1 (a). From the figure it is clear that as k increases from 0.5 to 5.0 σ_r rises very rapidly from zero to its maximum value. In fact by choosing k very large σ_r may be made to attain its maximum in as small an interval

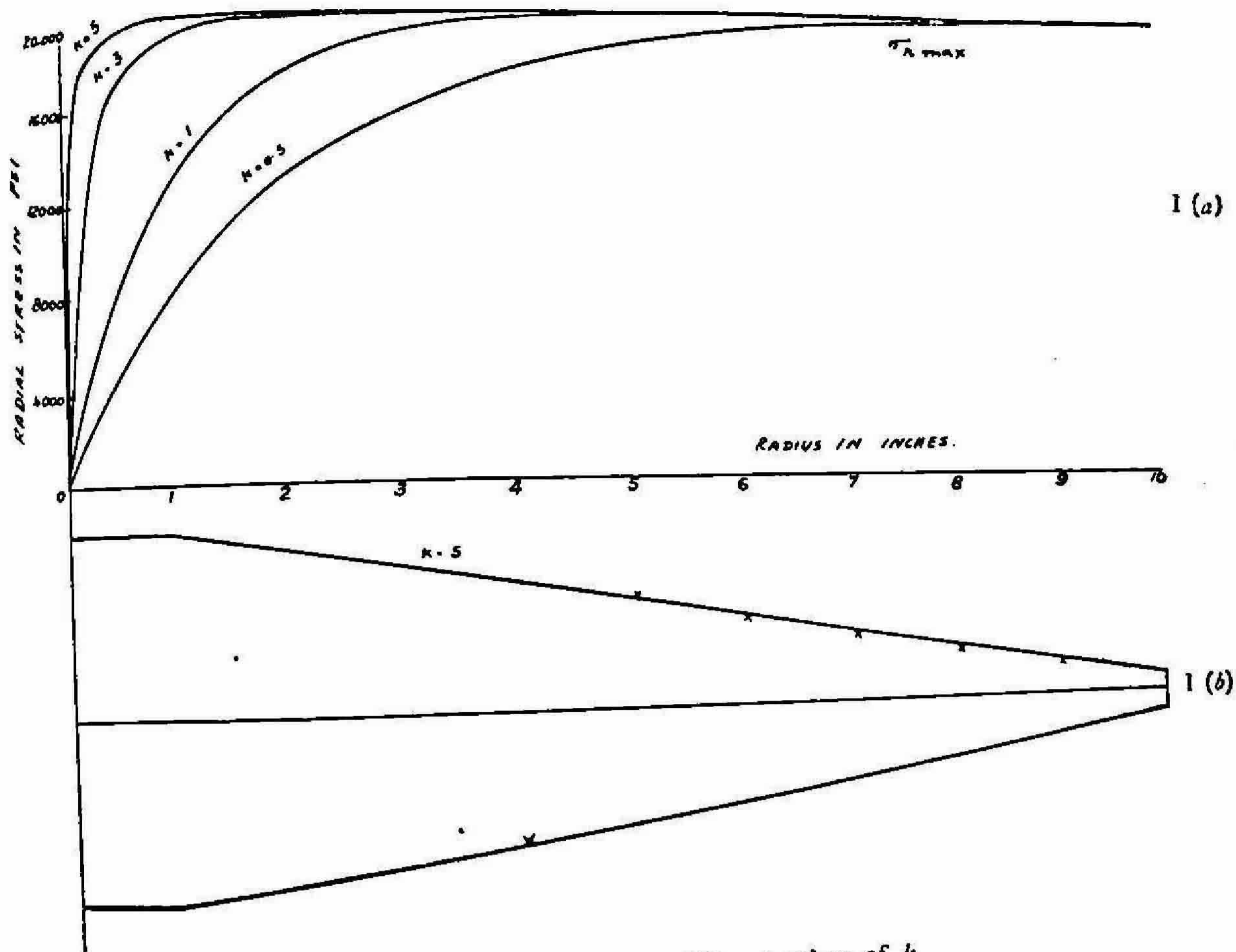


FIG. 1 (a) Shows variation of radial stress for different values of k .
 (b) Shows disc shape for $k = 5$.

as desired. To determine σ_t the tangential stress, we make use of the compatibility equation,

$$\frac{d\sigma_t}{dx} + \frac{(1 + \nu)\sigma_t}{x} = \frac{\nu d\sigma_r}{dx} + \frac{(1 + \nu)\sigma_r}{x} = f(x). \quad (2)$$

On the basis of equation (1), function $f(x)$ is known and equation (2) allows the determination of σ_t . For this purpose, we use the expression

$$\sigma_t = uv$$

Denoting derivatives by primes, we get

$$vu' + u \left[v' + (1 + \nu) \frac{v}{x} \right] = f(x)$$

v is so chosen that

$$v' + (1 + \nu) \frac{v}{x} = 0.$$

Therefore, we get

$$v = k_1 x^{-(1+\nu)}$$

and for u we have

$$vu' = f(x)$$

so that we get

$$u = \int \frac{f(x) dx}{v} + k_2$$

We have therefore if $k_1 k_2$ is replaced by c_1

$$\sigma_t = x^{-(1+\nu)} \left[\int f(x) x^{(1+\nu)} dx + c_1 \right] \quad (3)$$

where

$$f(x) = \nu \frac{d\sigma_r}{dx} + (1 + \nu) \frac{\sigma_r}{x}$$

Making use of equation (1), we get

$$f(x) = \nu \sigma_{r \max} k e^{-kx} + \frac{(1 + \nu) \sigma_{r \max} (1 - e^{-kx})}{x} \quad (4)$$

and substituting (4) in (3), we have

$$\sigma_t = x^{-(1+\nu)} \left[\int x^{(1+\nu)} \left\{ \nu \sigma_{r \max} k e^{-kx} + \frac{(1 + \nu) \sigma_{r \max} (1 - e^{-kx})}{x} \right\} dx + c_1 \right]$$

Integrating term by term

$$\begin{aligned} \sigma_t &= x^{-(1+\nu)} \cdot \nu k \sigma_{r \max} \int x^{(1+\nu)} \cdot e^{-kx} dx + x^{-(1+\nu)} (1 + \nu) \sigma_{r \max} \int x^\nu dx \\ &\quad - x^{-(1+\nu)} (1 + \nu) \sigma_{r \max} \int x^\nu e^{-kx} dx + c_1 x^{-(1+\nu)} \\ &= -\nu \sigma_{r \max} e^{-kx} + \sigma_{r \max} + \frac{e^{-kx}}{kx} (1 + \nu) \sigma_{r \max} \\ &\quad + x^{-(1+\nu)} [c_1 + c_2 \nu k \sigma_{r \max} + c_3 (1 + \nu) \sigma_{r \max} + c_4 (1 + \nu) \sigma_{r \max}] \end{aligned}$$

Denoting the constant under the bracket by c

$$\sigma_t = \sigma_{r \max} \left[1 - \nu e^{-kx} + \frac{(1 + \nu) e^{-kx}}{kx} \right] + c x^{-(1+\nu)} \quad (5)$$

The constant c has to be suitably chosen. In particular, if $c = 0$

$$\sigma_t = \sigma_{r \max} \left[1 - \nu e^{-kx} + \frac{(1 + \nu) e^{-kx}}{kx} \right]$$

We hold to equations (1) and (5) and find the disc thickness by means of the conditions of equilibrium, viz.,

$$\frac{d}{dx} (xy\sigma_r) - y\sigma_t + \mu\omega^2 x^2 y = 0$$

which can be written as

$$\frac{y'}{y} = \frac{\sigma_t - \mu\omega^2 x^2}{x\sigma_r} - \frac{d(x\sigma_r)}{dx} \cdot \frac{1}{x\sigma_r} \quad (6)$$

where y' stands for $\left(\frac{dy}{dx}\right)$.

From (1) and (5), we know σ_r and σ_t and integration of (6) therefore yields the disc thickness.

Let us specify that at $x = 1$, the disc has a thickness y_0 and at any radius x , the thickness is given by y . Also σ_r at $x = 1$ is given by

$$\sigma_{r_1} = \sigma_{r \max} (1 - e^{-k})$$

and at x

$$\sigma_r = \sigma_{r \max} (1 - e^{-kx}).$$

Substituting for σ_r and σ_t and integrating between the limits 1 and x , we get

$$\begin{aligned} \int_{y_0}^y \frac{y'}{y} = & \int_1^x \frac{dx}{x(1 - e^{-kx})} - \nu \int_1^x \frac{e^{-kx} \cdot dx}{x(1 - e^{-kx})} \\ & + \frac{(1 + \nu)}{k} \int_1^x \frac{e^{-kx} \cdot dx}{x^2(1 - e^{-kx})} - \frac{\mu\omega^2}{\sigma_{r \max}} \int_1^x \frac{x dx}{(1 - e^{-kx})} \\ & + \frac{c}{\sigma_{r \max}} \int_1^x \frac{x^{-(2+\nu)} \cdot dx}{(1 - e^{-kx})} - \int_1^x \frac{d(x\sigma_r)}{dx} \cdot \frac{1}{x\sigma_r} \end{aligned}$$

which gives

$$\begin{aligned} \log_e \frac{y}{y_0} = & \log x + \frac{(\nu - 1)}{k} \cdot \left\{ \frac{e^{-kx}}{x} - e^{-k} \right\} + \frac{\mu\omega^2}{\sigma_{r \max}} \cdot \left\{ \frac{e^{-kx}}{kx} - \frac{e^{-k}}{k} \right. \\ & \left. - \frac{x^2}{2} + \frac{1}{2} \right\} + \frac{c}{\sigma_{r \max}} \cdot \left\{ \frac{e^{-kx}}{kx^{2.3}} - \frac{e^{-k}}{k} - \frac{1}{1.3x^{1.3}} - 1 \right\} \\ & - \log_e \left\{ \frac{x(1 - e^{-kx})}{1 - e^{-k}} \right\} \end{aligned}$$

Taking $c = 0$, we get

$$\begin{aligned} \log_e \frac{y}{y_0} = & \log x + \frac{(\nu - 1)}{k} \cdot \left\{ \frac{e^{-kx}}{x} - e^{-k} \right\} + \frac{\mu\omega^2}{\sigma_{r \max}} \cdot \left\{ \frac{e^{-kx}}{kx} - \frac{e^{-k}}{k} \right. \\ & \left. - \frac{x^2}{2} + \frac{1}{2} \right\} - \log_e \left\{ \frac{x(1 - e^{-kx})}{(1 - e^{-k})} \right\}. \quad (7) \end{aligned}$$

For different values of k the expression (7) can be calculated. For instance when $k=5$, the values of disc thickness are plotted against radius in Fig. 1 (b). The material of the disc is steel with density 0.28 lb./cubic inch. Poissons ratio = 0.3, $\sigma_{r \text{ max}} = 20,000$ lb./square inch, and speed = 10,000 r.p.m.

By making k very large, we get

$$\log_e \frac{y}{y_0} = \frac{-\mu\omega^2}{2\sigma_{r \text{ max}}} x^2 + \frac{\mu\omega^2}{2\sigma_{r \text{ max}}}$$

i.e.,

$$y = y_0 e^{\frac{-\mu\omega^2 x^2}{2\sigma_{r \text{ max}}} + \frac{\mu\omega^2}{2\sigma_{r \text{ max}}}} \quad (\text{at } x=1, y=y_0)$$

or

$$y = y_0 e^{\frac{-\mu\omega^2 x^2}{2\sigma_{r \text{ max}}}} \left(e^{\frac{\mu\omega^2}{2\sigma_{r \text{ max}}}} \right),$$

putting

$$e^{\frac{-\mu\omega^2}{2\sigma_{r \text{ max}}}} \cong 1,$$

we have

$$y = y_0 e^{\frac{-\mu\omega^2 x^2}{2\sigma_{r \text{ max}}}}.$$

This is the shape of the well-known De-Laval disc.

An interesting conclusion obtained from this is that by assuming a unit function for the radial stress variation and making it attain its maximum value in the smallest interval by choosing k very large and computing the tangential stress from the compatibility equation, we are raising the stresses at every point in the disc to as nearly as possible the maximum permissible value. According to Holzer,¹ raising the stresses at every point to as high a value as possible is equivalent to having the work of deformation sustained at every point in the disc a maximum. A disc in which the work of deformation sustained is a maximum is a disc of minimum material.

So we have the result that De-Laval disc is also a disc of minimum material. This result is also obvious from Fig. 1 (a). As σ_r attains its maximum σ_t also becomes maximum with $k=5$, σ_r and σ_t both attain maximum value even when $x \cong 1$. In other words it is a disc of constant stress equal to the maximum stress. When $k=5$, the points lie close to a line and the shape of the disc is shown in Fig. 1 (b).

For values of $k < 1$, the disc diverges, *i.e.*, the thickness increases with the radius over a large interval and then converges. Such a shape has obviously no practical use.

When a stress is prescribed as varying as a unit function, we get different shapes for different values of k and when k is very large we get the De-Laval shape which is shown to be a disc of minimum material.

Finally, the author's thanks are due to Dr. A. Ramachandran for his constant encouragement and comments.

REFERENCE

1. Stodola and Lowenstein . . . *Steam and Gas Turbines*, Peter Smith, New York, 1945, 1, 395.

APPENDIX I

The following simplifying assumptions are made:

$$\int e^{-kx} \cdot x^{(1+\nu)} \cdot dx = -\frac{e^{-kx}}{k} \left[x^{(1+\nu)} + \frac{(1+\nu)}{k} x^\nu + \frac{(1+\nu)}{k} \cdot \frac{\nu}{k} x^{\nu-1} + \frac{(1+\nu) \cdot \nu \cdot (\nu-1)}{k \cdot k \cdot k} + \dots \right] \quad (1)$$

neglect terms containing k^2 and higher powers of k .

$$\int \frac{e^{-kx} dx}{x(1-e^{-kx})} = \int \frac{e^{-kx}}{x} (1-e^{-kx})^{-1} \cdot dx. \quad (2)$$

Expanding by binomial theorem

$$\int \frac{e^{-kx}}{x} \cdot \{1 + e^{-kx} + e^{-2kx} + e^{-3kx} + \text{etc.}\dots\}.$$

Taking only the first term

$$\int \frac{e^{-kx}}{x} = \frac{e^{-kx}}{k} \cdot \frac{1}{x} + \frac{1}{k} \cdot \left[\frac{e^{-kx}}{k} \cdot \frac{1}{x^2} + \frac{2}{k} \int \frac{e^{-kx}}{x^3} + \dots \right]$$

neglecting terms containing k^2 and higher powers of k

$$\int \frac{e^{-kx} dx}{x(1-e^{-kx})} = -\frac{e^{-kx}}{k} \cdot \frac{1}{x}.$$