A ROTATING DISC IN CONSTANT PURE SHEAR

By S. Kumar and C. V. Joga Rao

(*Department of Aeronautical Engineering, Indian Institute of Science, Bangalore-3*)

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Summary

The disc of constant pure shear can be obtained by taking $\sigma_\theta = -\sigma_r = \sigma$, a positive constant, throughout the disc. Substituting this in the equation of equilibrium (in polar co-ordinates) for a symmetrical stress distribution we get a homogenous differential equation of the first order. On integrating this equation we get the equation of the thickness distribution for the required disc. Taking $\frac{r}{a}, \frac{h}{h_0}$ as abscissa and ordinate, and $\frac{\sigma_a}{\sigma}$ as a parameter, we plot the forms of different profiles.

Introduction

Investigations have been made by several workers to find the stress distribution in rotating discs of different profiles and to find profiles corresponding to given distribution of stresses. De Laval[1] has designed a rotating disc of uniform strength. Biezeno and Grammel[2] have given a number of profiles of the rotating discs and their corresponding stress distributions. Stresses in a rotating disc of exponential thickness profile have recently been given by Lee.[3] The authors[4] have recently investigated stresses around a circular hole in rotating discs of hyperbolic and parabolic profiles. This paper presents a new type of disc in which the resulting pure shear stress is constant nearly everywhere in the disc.

Nomenclature

r, θ = polar co-ordinates

ω = angular velocity of the disc

ρ = density of the material of the disc

σ_r = radial stress

σ_θ = tangential stress

$\tau_{r\theta}$ = shearing stress in the r, θ plane

R = body force

h = thickness of the disc

h_0 = thickness of the disc at the periphery

a = the extreme radius of the disc

$\frac{\sigma_a}{\sigma}$ = S, a parameter

σ = a constant stress

$\sigma_a = \frac{\rho\omega^2 a^2}{2}$

$\mathrm{H} = \frac{h}{h_0}$

$x = \frac{r}{a}$

The equations of equilibrium in polar co-ordinates are

$$\left.\begin{aligned} &\frac{\partial\sigma_r}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\sigma_r - \sigma_\theta}{r} + \mathrm{R} = 0 \\ &\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta} + \frac{\partial\tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} = 0 \end{aligned}\right\} \quad (1)$$

If the thickness of the disc is small in comparison to its radial dimensions, we can neglect the variation of the tangential and radial stress over the thickness of the disc. Also, the stress distribution being symmetrical about the axis of rotation, $\tau_{r\theta}$ is zero and the stresses are not dependent on θ but on r alone. Since $\sigma_\theta = -\sigma_r$, the resulting stress is pure shear. If we consider an infinitesimally small radial element, this resulting pure shear will bisect the directions of σ_θ and σ_r.

In case of a disc of variable thickness the first of equations (1) becomes

$$\frac{d}{dr}(hr\sigma_r) - h\sigma_\theta + \rho\omega^2 r^2 h = 0 \quad (2)$$

The second of equations (1) vanishes altogether. Now for pure shear there should be two perpendicularly acting uniform normal stresses, one compressive and the other tensile. So here we take

$$-\sigma_r = \sigma_\theta = \sigma, \text{ a positive constant} \quad (3)$$

Therefore, equation (2) becomes

$$-\sigma\frac{d}{dr}(hr) - h\sigma + \rho\omega^2 r^2 h = 0$$

or

$$\frac{dh}{h} = \left(\frac{\rho}{\sigma}\omega^2 r - \frac{2}{r}\right)dr$$

Integrating this we obtain

$$\log h = \frac{\rho}{\sigma}\,\frac{\omega^2 r^2}{2} - 2\log r + \log k \tag{4}$$

where $\log k$ is a constant of integration

or

$$h = \frac{k\, e^{\frac{\rho}{\sigma}\frac{\omega^2 r^2}{2}}}{r^2} \tag{5}$$

Now k being the constant of integration it can be determined by the boundary conditions of the disc. Let us assume that at $r = a$, $h = h_0$

$$\therefore \quad h_0 = \frac{k\, e^{\frac{\rho}{\sigma}\frac{\omega^2 a^2}{2}}}{a^2} \tag{6}$$

Substituting the value of k from (6) in (5) we get

$$h = \frac{e^{\frac{\rho}{\sigma}\frac{\omega^2 r^2}{2}}}{r^2}\;\frac{a^2 h_0}{e^{\frac{\rho}{\sigma}\frac{\omega^2 a^2}{2}}} \tag{7}$$

This equation gives the thickness profile of the required rotating disc. To study the variation of h with respect to r, we introduce convenient parameters.

Equation (7) can be written as

$$\frac{h}{h_0} = \frac{a^2}{r^2}\, e^{-\frac{\rho\omega^2 a^2}{2}\cdot\frac{1}{\sigma}\left(1-\frac{r^2}{a^2}\right)}$$

Now putting

$$\frac{h}{h_0} = \mathrm{H},\quad \frac{\rho\omega^2 a^2}{2} = \sigma_a \text{ and } \frac{r}{a} = x$$

we have

$$\mathrm{H} = \frac{1}{x^2\, e^{\frac{\sigma_a}{\sigma}(1-x^2)}} \tag{8}$$

This is the general equation of the profile of the disc. To make the disc finite in the radial dimensions and also to facilitate the process we choose

$$0 < x < 1$$

i.e., 'a' has been taken as the peripheral radius of the disc.

At the outer boundary of the disc σ_r should be zero, but this condition violates our hypothesis of $-\sigma_r = \sigma_\theta = \sigma$, a constant. However, this condition may be approximately satisfied by placing a rim at the periphery of the disc.

Corresponding to different values of the parameter $\frac{\sigma_a}{\sigma}$ we get different shapes of the disc. Table I shows seven sets of values of $\frac{h}{h_0}, \frac{r}{a}$ correspond-

TABLE I

x	H for S =						
	1	2	3	4	5	7·5	10
1·0	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000
0·9	1·0209	0·8442	0·6982	0·5774	0·4775	0·2975	0·1846
0·7	1·2255	0·7359	0·4419	0·2654	0·1594	0·0445	0·0124
0·5	1·8893	0·8925	0·4216	0·1991	0·0941	0·0144	0·0022
0·3	4·4723	1·8002	0·7246	0·2888	0·1174	0·0121	0·0012
0·1	37·1747	13·8122	5·1308	1·9253	0·7084	0·0496	0·0050

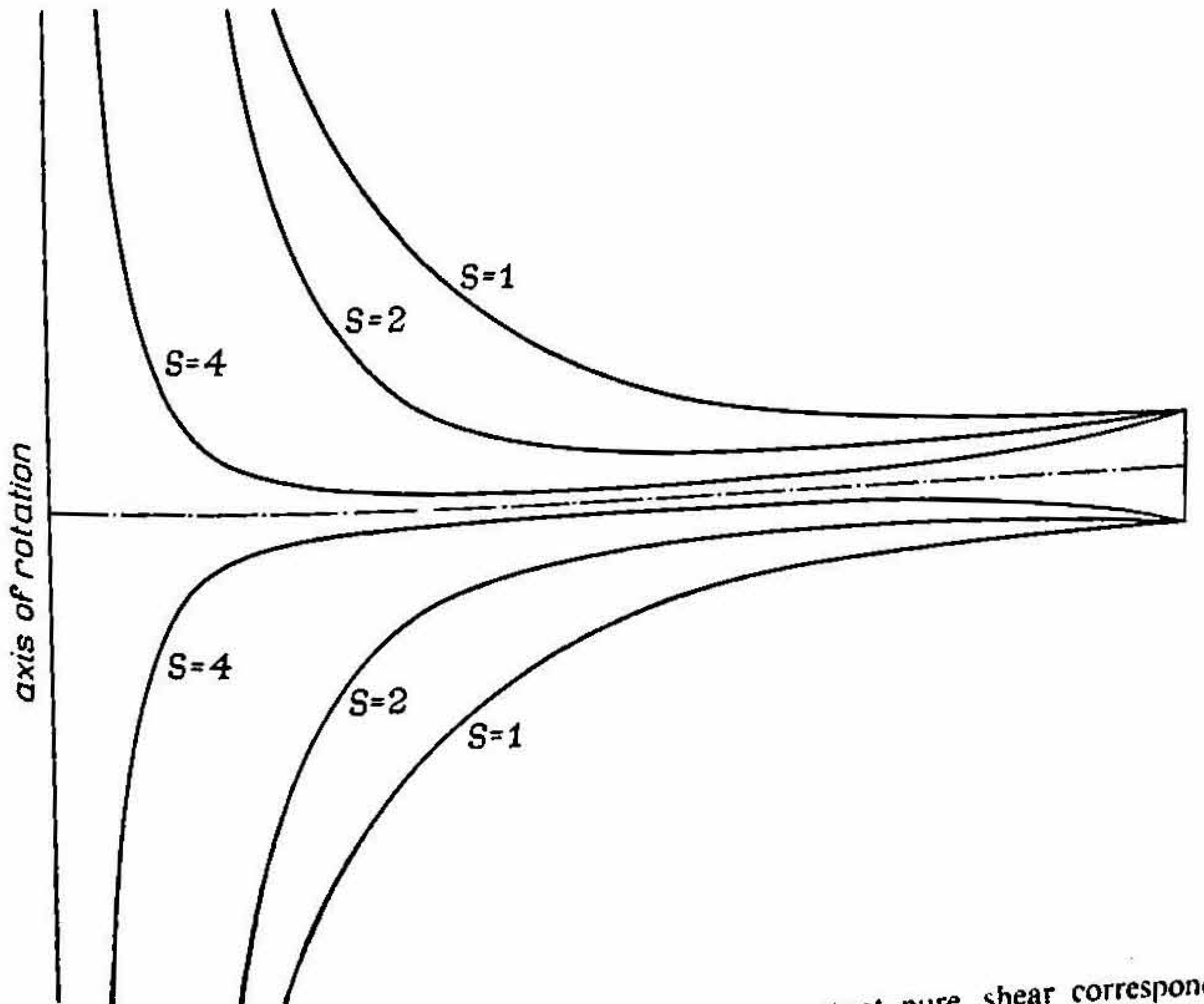

FIG. 1. Non-dimensional profiles of the disks of constant pure shear corresponding to $S = \frac{\sigma_a}{\sigma} = 1, 2$ and 4.

ing to seven values of $\frac{\sigma_a}{\sigma}$ varying from 1 to 10. The corresponding profiles are plotted in Fig. 1. We find from the curves in Fig. 1, by giving some suitable values to h_0 and a, that the shapes of the discs obtained from the values of $\frac{\sigma_a}{\sigma}$ in the vicinity of 4 may be suitable for practical use. Again by varying a in (7) we find that this equation holds good for any value of a, though very high values of a make the disc only of theoretical interest, *e.g.*, if a is made a finitely great number, h has to tend to zero to maintain the idea of finite and constant stress throughout the disc. The drawback in this disc is that any particular form of the disc is most suitable only for one angular velocity.

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