TORSION OF MULTIPLY CONNECTED SECTIONS

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Synopsis

In the numerical solution of doubly connected sections in Torsion, only one of the boundary values, either inner or outer, can be arbitrarily assumed. Usually the value assumed. will be zero. For the second value, the method proposed by Southwell,[1] though exact, is tedious. In this article an alternative method, giving fairly accurate results is proposed.

The governing differential equation of uniform torsion on a prismatic bar is

$$\nabla^2 \phi = -2 \tag{1}$$

together with the boundary condition[2] $\dfrac{d\phi}{ds} = 0$ (ϕ is the stress function).

The numerical solution for singly connected regions is quite straightforward. But for doubly or multiply connected regions the solution becomes tedious and involves enormous amount of patient Relaxation. The aim of this paper is to seek an alternative procedure which will minimise the labour without sacrificing the accuracy.

The main problem will be the determination of the inner boundary values. In doubly connected regions the unknown boundary value will be only one since the other boundary value can be assumed to be zero without any loss of generality.

The differential equation must be satisfied at the inner boundary. The value of the stress function ϕ at the inner boundary will be hereafter designated by ϕ_i.

This is otherwise expressed as

$$\oint \frac{\partial \phi}{\partial n}\, ds = -2\,\mathrm{A}$$

where A is the area enclosed by the inner boundary curve. For our purpose, it will be convenient to have it in the former form only, *i.e.*,

$$(\nabla^2 \phi)_i = -2$$

Referring to the membrane analogy this only means that in the equation $\nabla^2\phi = K$, K must be equal to -2 at all points.

In the membrane analogy experiments, the hole in the section will correspond to plate in the membrane. From the membrane over the solid section, we remove the portion corresponding to the hole and put a plate in its place, the membrane can take various shapes, with the plate remaining horizontal. The problem is then, to adjust the height of the plate correctly such that $(\nabla^2\phi)_i = -2$.

The torque of a prismatical bar is

$$T = 2\iint_R \phi\, dx dy$$

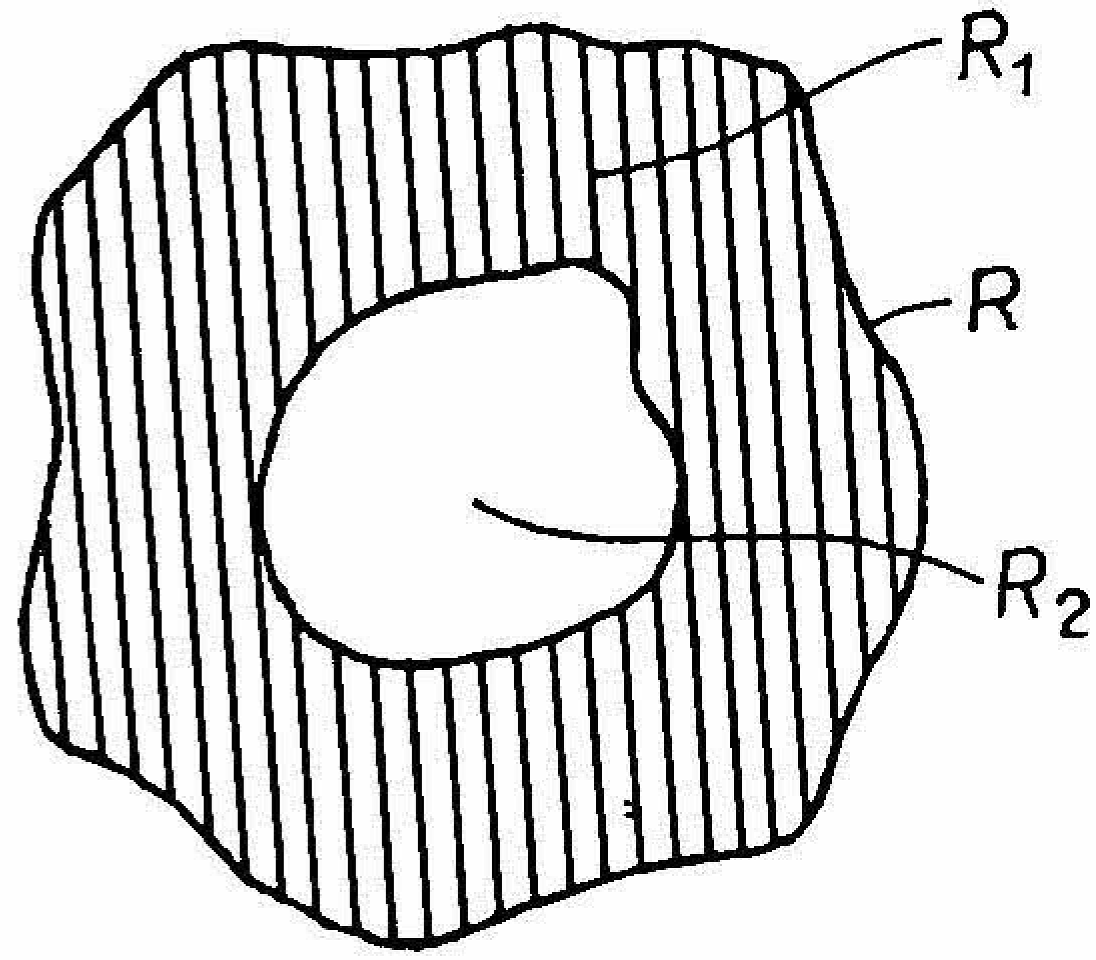

FIG. 1.

Similarly for the hollow section

$$T = 2\iint_{R_1} \phi\, dx dy + 2CA$$

The torque inequality can be written down as[3]

$$T_s \geqslant T_p + T_h \tag{2}$$

where

T_s = Torque of solid shaft

T_p = Torque of pierced shaft

T_h = Torque due to material removed in the hole

$$T_s = 2\iint_{R_1} \phi_s dx dy + 2\iint_{R_2} \phi_s dx dy$$

$$T_p = 2\iint_{R_1} \phi_p dx dy + 2\,CA$$

$$T_h = 2\iint_{R_2} \phi_h dx\, dy$$

where ϕ_h is the stress function of the shaft corresponding to the hollow portion.

Introducing the membrane height z which corresponds to stress function ϕ, we can write equation 2 as

$$2\iint_{R_1} z_s dxdy + 2\iint_{R_2} z_s dxdy \geqslant 2\iint_{R_1} z_p dxdy + 2CA + 2\iint_{R_2} z_h dxdy$$

So,

$$\iint_{R_1} z_s \, dxdy = \iint_{R_1} z_p \, dxdy. \tag{3}$$

This equation is continuously true, *i.e.*, if this is true for R_1, it will be true for $R_1 + dR_1$, the addition being done on both sides.

For the Equation (3) to be true it is not necessary that z_s and z_p must be identical. It will be a particular case when the boundary of the hole coincides with the stress line in the corresponding solid shaft.

We are concerned with the important case when $z_p \neq z_s$ at every point.

We shall consider centrally situated circular boundaries. The results hold approximately for nearly circular boundaries centrally situated. Then we can write z_p and z_s in the form of Fourier series. This is possible if the Dirichlets' conditions are satisfied.

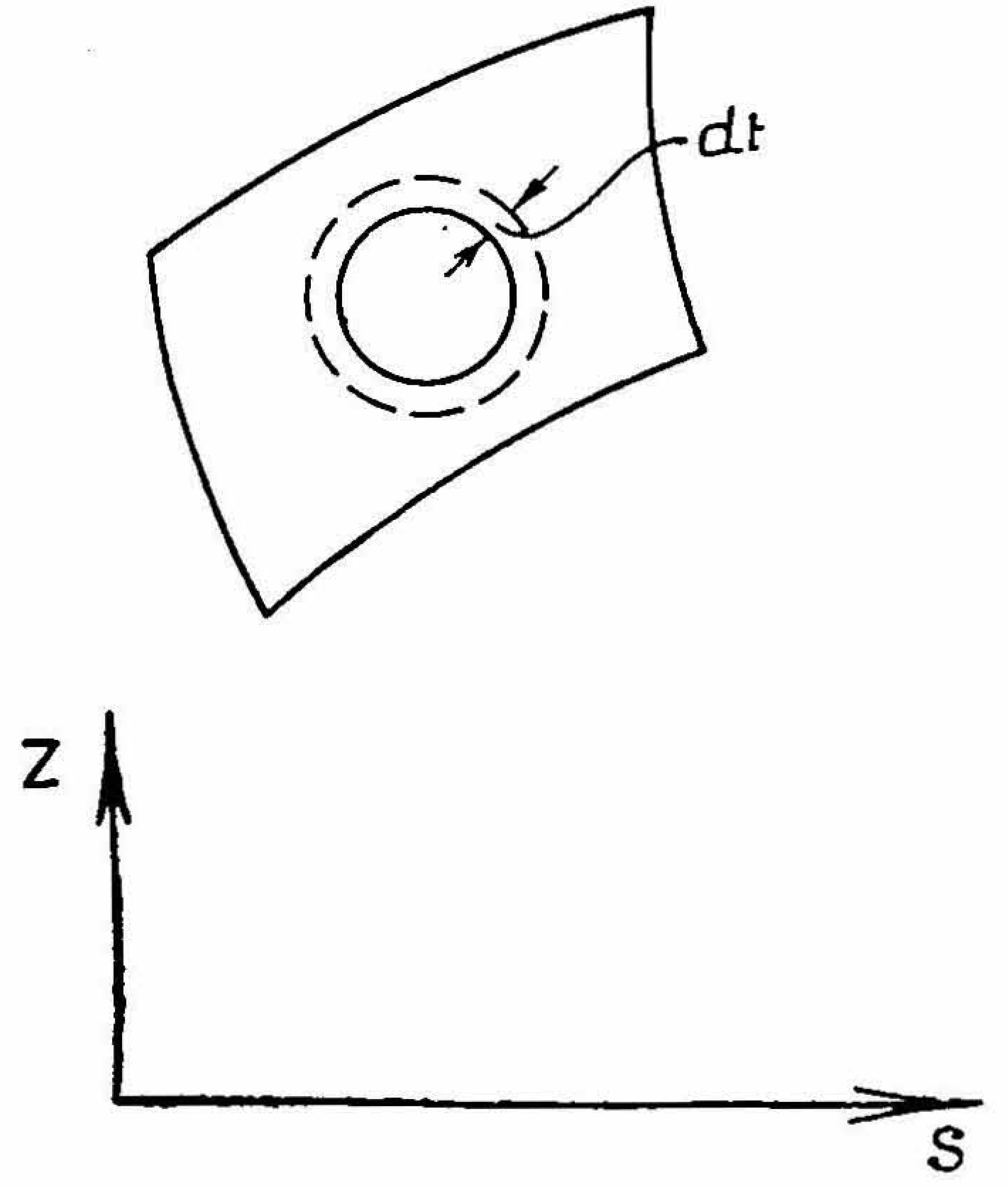

FIG. 2.

On the membrane of the solid shaft take a strip of infinitesimal width 'dt' adjacent or along the curve which corresponds to the hole in the solid shaft.

Take the corresponding curve in the membrane of the hollow shaft. z_p and z_s are functions of x and y. Open out the curves and lay them along s-axis. z_p and z_s are defined in the interval 0 to 2π.

Since we are not concerned with the behaviour of these functions beyond this interval, it is enough if all the Dirichlets' conditions are satisfied within this interval.

The two functions are continuous and single valued; $f(0)=f(2\pi)$.

So we can represent both z_p and z_s by Fourier Series.

$$z_s = b_{10} + \sum_{m_1=1}^{\infty} b_{m_1} \cos m_1 s + \sum_{m_1=1}^{\infty} a_{m_1} \sin m_1 s$$

$$z_p = b_{20} + \sum_{m_2=1}^{\infty} b_{m_2} \cos m_2 s + \sum_{m_2=1}^{\infty} a_{m_2} \sin m_2 s$$

$$b_{10} = \frac{1}{2\pi} \int_0^{2\pi} z_s \, ds \qquad b_{20} = \frac{1}{2\pi} \int_0^{2\pi} z_p \, ds$$

$$b_{m_1} = \frac{1}{\pi} \int_0^{2\pi} z_s \sin m_1 s ds \qquad b_{m_2} = \frac{1}{\pi} \int_0^{2\pi} z_p \sin m_2 s ds$$

$$a_{m_1} = \frac{1}{\pi} \int_0^{2\pi} z_s \cos m_1 s ds \qquad a_{m_2} = \frac{1}{\pi} \int_0^{2\pi} z_p \cos m_2 s ds$$

In the new system of axes

$$\iint_{dR_1} z_s \, dxdy = dt \int_0^{2\pi} z_s \, ds$$

$$\iint_{dR_1} z_p \, dxdy = dt \int_0^{2\pi} z_p \, ds$$

Equating both, according to Equation (3)

$$\int_0^{2\pi} z_s \, ds = \int_0^{2\pi} z_p \, ds \qquad (4)$$

As discussed in Equation 3, this will be true even if z_s and z_p are not identical. They can be as shown in the figure, *i.e.*, they can be sinusoidal functions oscillating about the same axis, displaced by a distance from the axis-s.

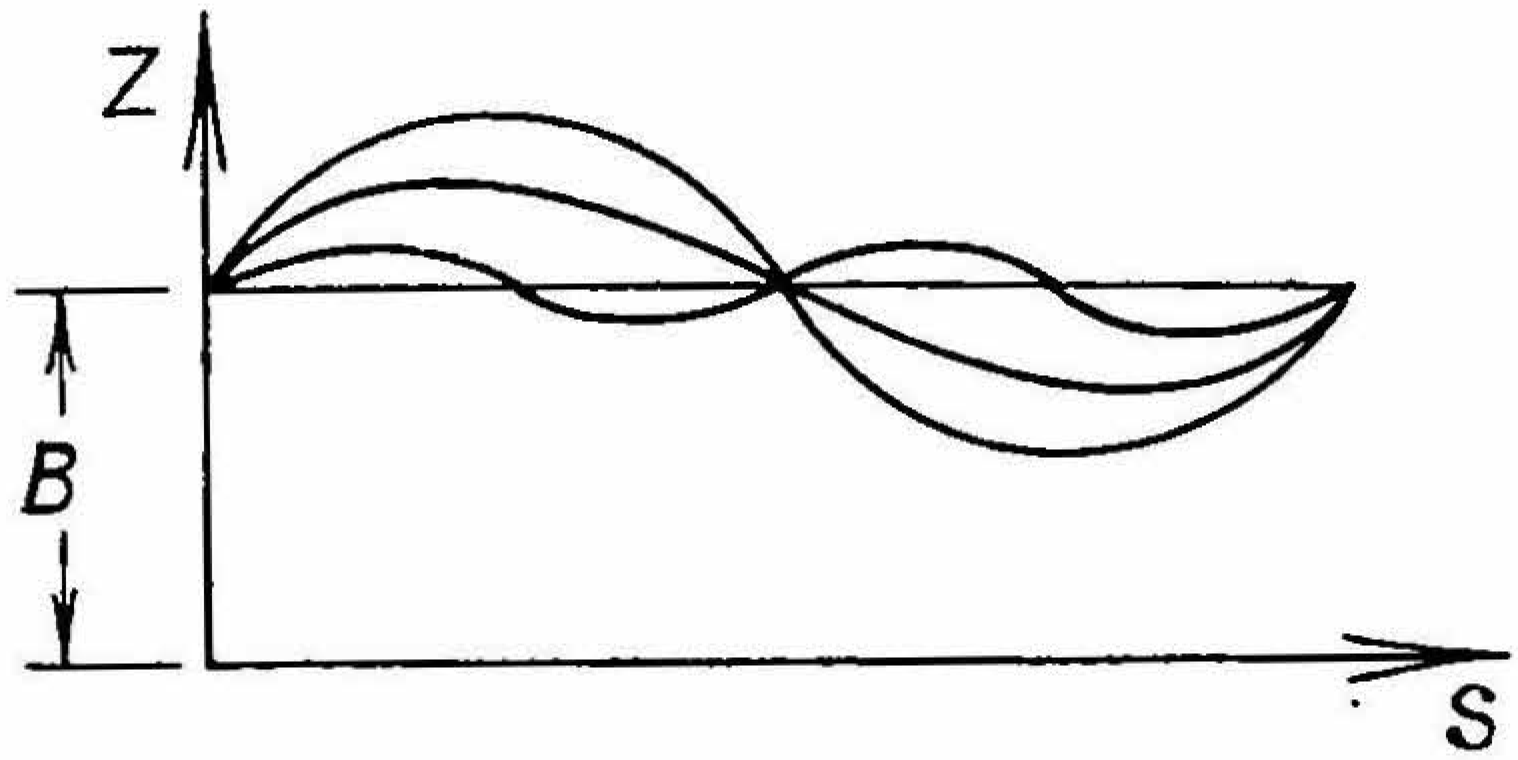

FIG. 3.

So this relation does not help us as it is to determine the function completely, *i.e.*, we cannot determine the Fourier coefficients.

We can seek for a relationship between the coefficients of the two expansions z_p and z_s means of the Parseval's Theorem.

$$\int_0^{2\pi} f(z)^2\, dz = b_0^2 + \tfrac{1}{2} \sum_{k=1}^{\infty} (a_k^2 + b_k^2)$$

where

$$f(z) = b_0 + \sum_{k=1}^{\infty} a_k \sin kx + \sum_{k=1}^{\infty} b_k \cos kx$$

$$\int_0^{2\pi} z_p^2\, ds = b_{20}^2 + \tfrac{1}{2} \sum_{m_2=1}^{\infty} (a_{m_2}^2 + b_{m_2}^2)$$

$$\int_0^{2\pi} z_s^2\, ds = b_{10}^2 + \tfrac{1}{2} \sum_{m_1=1}^{\infty} (a_{m_1}^2 + b_{m_1}^2)$$

Here we will have to make one more assumption which is approximate.

The values of z_p and z_s depend mainly on b_{20} and b_{10}. (This does not mean that the rest of the terms are negligible always, but their sum will be comparatively small. The sum of their squares, *i.e.*, $\sum_{k=1}^{\infty} (a_k^2 + b_k^2)$ can be neglected).

Then

$$\int_0^{2\pi} z_s^2\, ds = b_{10}^2$$

$$\int_0^{2\pi} z_p^2\, ds = b_{20}^2$$

From equation (4)

$$\int_0^{2\pi} z_s \, ds = \int_0^{2\pi} z_p \, ds$$

multiply both sides by $\frac{1}{2\pi}$

$$\frac{1}{2\pi} \int_0^{2\pi} z_s \, ds = \frac{1}{2\pi} \int_0^{2\pi} z_p \, ds$$

i.e.,

$$b_{10} = b_{20}$$

So

$$b_{10}^2 = b^2_{20}$$

Therefore,

$$\int_0^{2\pi} z_s^2 \, ds = \int_0^{2\pi} z_p^2 \, ds \tag{5}$$

Consider at this stage, the internal boundary curve of the hole, both in the hollow and corresponding solid shafts. On the internal boundary of the hollow shaft ϕ_i = constant, *i.e.*, $(z_p)_i$ = constant = C.

Equation (5) will, then, be

$$\int_0^{2\pi} C^2 \, ds = \int_0^{2\pi} z_s^2 \, ds$$

$$\phi_i^2 = C^2 = \frac{\int_0^{2\pi} z_s^2 \, ds}{\int_0^{2\pi} ds} \tag{6}$$

This is the required relation which helps us to evaluate ϕ_i from the values of ϕ_s on the corresponding curve in the solid shaft.

(ϕ_s = stress function of the solid shaft.)

Evaluation of the right-hand side of Equation 6 can be done in the following way.

Divide the curve into *n*-divisions, preferably equal. Then

$$\begin{aligned}\int_0^{2\pi} z_s^2 \, ds &= \int_0^{2\pi} \phi_s^2 \, ds \\ &= \epsilon \, [\phi_{s1}^2 + \phi_{s2}^2 + \ldots + \phi_{sn}^2] \\ &= \epsilon \sum_{n=1}^{n} \phi_{sn}^2\end{aligned}$$

$$C^2 = \frac{\epsilon \sum_{n=1}^{n} \phi_{sn}^2}{n\epsilon}$$

$$= \frac{\sum_{n=1}^{n} \phi_{sn}^2}{n}$$

$$C = \sqrt{\frac{\sum_{n=1}^{n} \phi_{sn}^2}{n}}$$

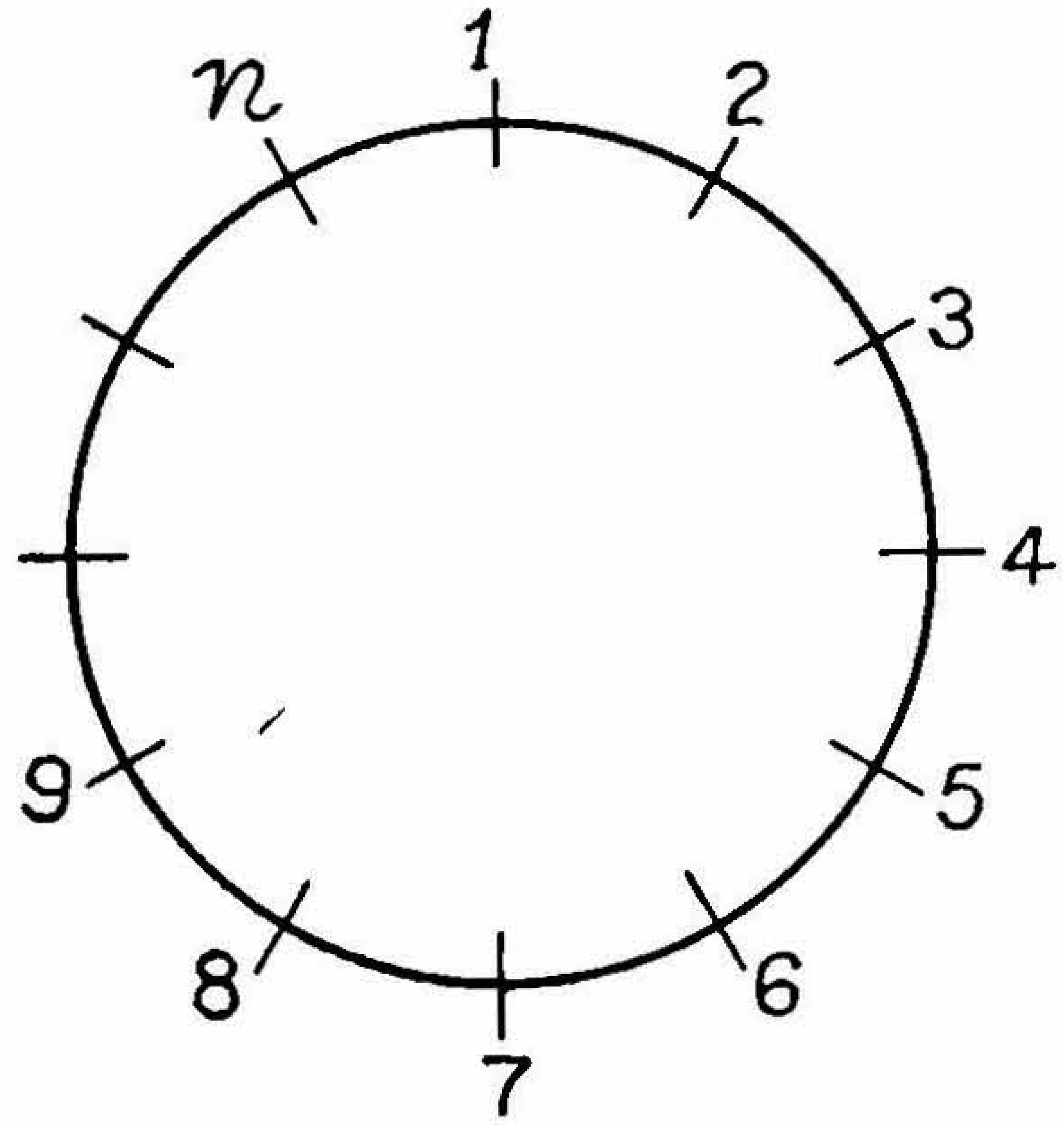

FIG. 4.

This method can be called 'R.M.S. Value' method, indicating the characteristic feature involved in it.

Examples are given in the appendix.

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APPENDIX

1. *Rectangular shaft with a circular hole at the centre.—*

$$C^2 = \frac{1}{6}\left(\frac{163^2}{2} + 160^2 + 153 \cdot 5^2 + 150^2 + 147 \cdot 25^2 + 149^2 + \frac{149^2}{2}\right)$$

$$= \frac{100^2}{6}\,(1.33 + 2.56 + 2.36 + 2 \cdot 25 + 2.17 + 2.22 + 1.11)$$

$$= \frac{13 \cdot 80 \times 100^2}{6} = 2 \cdot 30 \times 100^2$$

$$C = 1 \cdot 517 \times 100 = 151 \cdot 7$$

Exact value $= 149$.

This has been worked out by the author by the method of Southwell[1] for $6'' \times 10''$ rectangle with $2''$ diameter hole.

2. *Splined Shaft*[4]—

$$C^2 = \frac{\int_0^{2\pi} \phi_s{}^2\, ds}{\int_0^{2\pi} ds} = \left[\frac{5\,R}{54} \times 2300^2 + \frac{R}{18} \times 2360^2 + \frac{R}{18}\, 2485^2 + \frac{4\,R}{27}\, 2540^2\right] \frac{54}{19\,R}$$

$$= \frac{54}{19} \times 100^2 \left[\frac{1}{6} \times \frac{5}{9} \times 529 + \frac{1}{3} \times \frac{1}{6} \times 557 + \frac{1}{18} \times 617 + \frac{4}{27} \times 645\right]$$

$$= 100^2\,[49 \cdot 0 + 31 + 34 \cdot 3 + 95 \cdot 6]\,\frac{54}{19}$$

$$= 100^2\,[209 \cdot 9]\,\frac{54}{19} = 100^2 \times 596 \cdot 7$$

$$C = 2442$$

Exact value $= 2374$.

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