PROBLEMS CONNECTED WITH THE RHOMBUS

I. Elastic Torsion

By S. K. Lakshmana Rao and K. T. Sundara Raja Iyengar
(*Civil and Hydraulic Engineering Section, Indian Institute of Science, Bangalore-3, India*)

Received August 9, 1954

1. Introduction

Since the time of Saint-Venant the problem of elastic torsion of prismatical bars has been engaging the attention of numerous investigators. The semi-inverse method of solution for this problem initiated by Saint-Venant himself has yielded exact solutions for several important cross-sections. Quite a number of cross-sections have also been successfully dealt with by the method of conformal mapping,[1,2] where one transforms the region of the given cross-section onto the inside of a unit circle. However, in many cases it is impossible to obtain exact solutions and one has to depend on approximate techniques among which the variational, numerical and graphical ones are important.[3] Synge[4] has given a new method of approximation for problems in elasticity, based on the method of the hypercircle in function space developed by himself and Prager.[5] These and other methods provide upper and lower bounds for the solution.

The present paper deals with the problem of elastic torsion of a uniform, isotropic bar whose cross-section is a rhombus. In Part A below, we use the method of Rayleigh-Ritz to obtain an approximate solution of the problem and also notice bounds for the torsional rigidity by some other methods. We use the technique of relaxation in Part B to obtain a numerical solution of the problem taking the acute angle of the rhombus equal to 60°.

2. Statement of the Problem

The problem of torsion consists in the determination of the warping function $\phi(x, y)$ which is harmonic in the region of the cross-section and has prescribed normal derivative on the boundary. If $\psi(x, y)$ is the harmonic conjugate of $\phi(x, y)$, we have

$$\nabla^2\psi(x, y) = 0 \text{ at all interior points of the region,}$$

and

$$\psi(x, y) = \tfrac{1}{2}(x^2 + y^2) + \text{constant, on the boundary.}$$

Writing

$$\Psi(x, y) = \psi(x, y) - \tfrac{1}{2}(x^2 + y^2)$$

the problem is equivalent to the determination of (the stress function) Ψ such that

$$\nabla^2 \Psi + 2 = 0 \qquad \text{at interior points} \tag{2.1}$$

and

$$\Psi = \text{constant} \qquad \text{on the boundary} \tag{2.2}$$

The constant may be taken equal to zero for a simply connected region. The stresses are, with the usual notation of (Ref. 1).

$$\tau_{zx} = \mu a \frac{\partial \Psi}{\partial y}, \quad \tau_{zy} = -\mu a \frac{\partial \Psi}{\partial x} \tag{2.3}$$

and the torsional rigidity is

$$D = 2\mu \int\int \Psi \, dx dy \tag{2.4}$$

PART A

We are concerned with the bar whose cross-section is the rhombus (side a, acute angle 2θ) bounded by the lines

$$\frac{x}{\cos\theta} \pm \frac{y}{\sin\theta} = \pm a \qquad \text{(Fig. 1)}$$

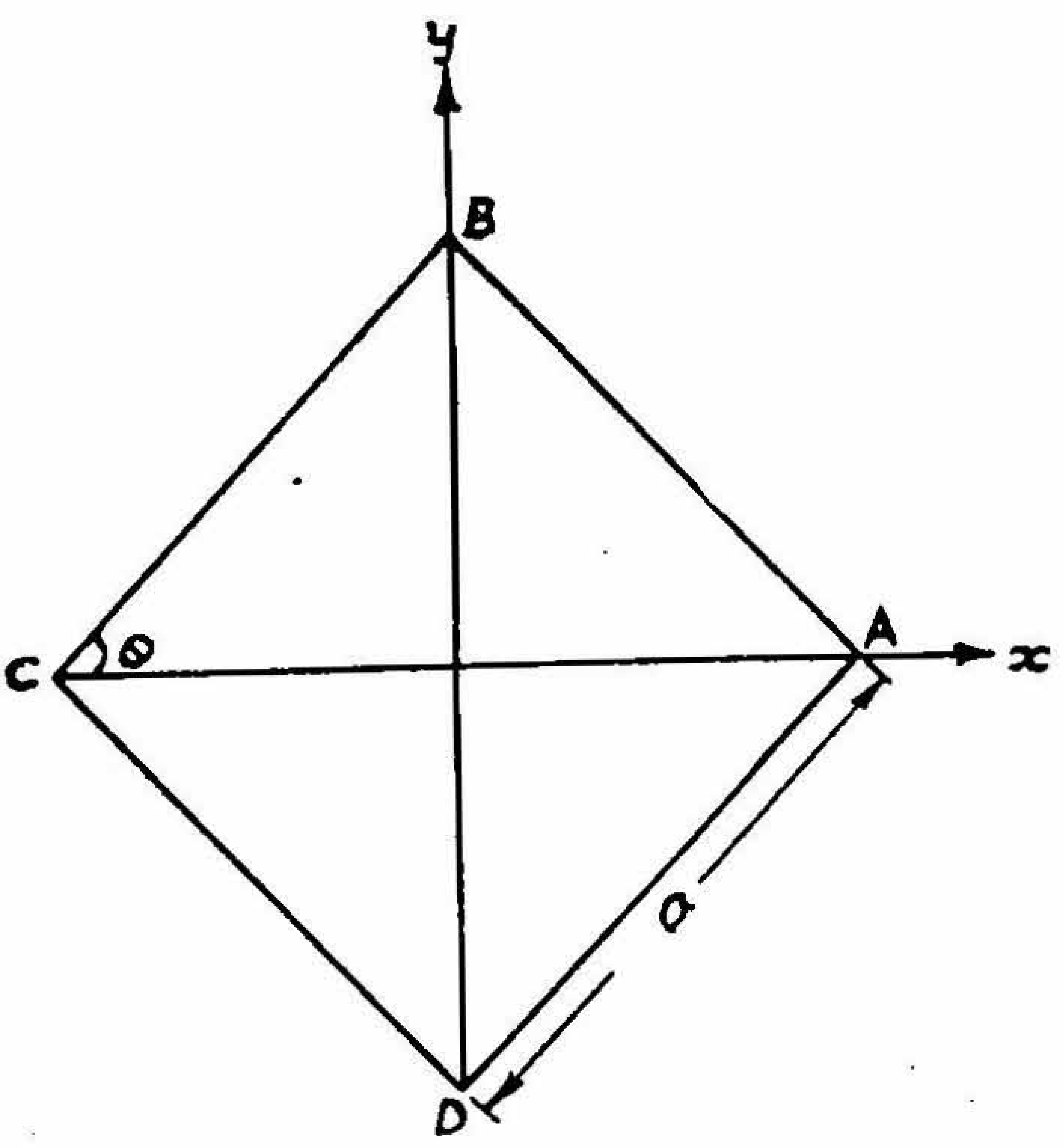

FIG. 1.

3. To obtain an approximate solution of the problem by the Rayleigh-Ritz method which consists in minimizing an "Energy Integral" of the form

$$I(f) = \iint [(\nabla f)^2 - 4f]\, dxdy \tag{3.1}$$

f assuming the same boundary values as the wanted function Ψ, we take (for a first approximation)

$$f_1(x, y) = c_1 \left[\left(\frac{x}{\cos\theta} + \frac{y}{\sin\theta}\right)^2 - a^2\right]\left[\left(\frac{x}{\cos\theta} - \frac{y}{\sin\theta}\right)^2 - a^2\right]; \tag{3.2}$$

this obviously vanishes on the boundary of the region. The condition of minimization of (3.1), *viz.*,

$$c_1 \iint f_1(x, y)\, \nabla^2 f_1(x, y)\, dxdy = -2 \iint f_1(x, y)\, dxdy \tag{3.3}$$

gives

$$c_1 = \frac{5}{32}\,\frac{(\sin 2\theta)^2}{a^2}.$$

The corresponding approximate torsional rigidity is

$$D_1 = 2\mu c_1 \iint f_1(x, y)\, dxdy = \frac{5}{36}\,\mu (\sin 2\theta)^3 a^4. \tag{3.4}$$

To have a better approximation for the solution, we take

$$f_2(x, y) = \left[\left(\frac{x}{\cos\theta} + \frac{y}{\sin\theta}\right)^2 - a^2\right]\left[\left(\frac{x}{\cos\theta} - \frac{y}{\sin\theta}\right)^2 - a^2\right]$$
$$\times \left[c_1 + c_2\left(\frac{x}{\cos\theta} + \frac{y}{\sin\theta}\right)^2 + c_3\left(\frac{x}{\cos\theta} - \frac{y}{\sin\theta}\right)^2\right] \tag{3.5}$$
$$= c_1 g_1(x, y) + c_2 g_2(x, y) + c_3 g_3(x, y).$$

c_1, c_2, c_3 which are to be obtained from the minimization conditions, *viz.*,

$$\sum_{j=1}^{3} c_j \iint g_i\, \nabla^2 g_j\, dxdy = -2 \iint g_i\, dxdy \qquad (i = 1, 2, 3) \tag{3.6}$$

are solutions of the linear equations

$$\left.\begin{aligned}
\frac{8}{15} a^2 c_1 + \frac{48}{175} a^4 c_2 + \frac{48}{175} a^4 c_3 &= \sin^2\theta \cos^2\theta, \\
\frac{16}{35} a^2 c_1 + \frac{152}{315} a^4 c_2 + \frac{8}{105} a^4 c_3 &= \frac{1}{3}\sin^2\theta \cos^2\theta, \\
\frac{17}{35} a^2 c_1 + \frac{31}{210} a^4 c_2 + \frac{152}{315} a^4 c_3 &= \frac{1}{3}\sin^2\theta \cos^2\theta.
\end{aligned}\right\} \tag{3.7}$$

Employing Kramer's rule, we have

$$c_1 = \frac{5}{32} \frac{19079}{20033} \frac{(\sin 2\theta)^2}{a^2}, \quad c_2 = \frac{15}{16} \frac{631}{20033} \frac{(\sin 2\theta)^2}{a^2},$$

$$c_3 = \frac{15}{32} \frac{593}{20033} \frac{(\sin 2\theta)^2}{a^2}. \tag{3.8}$$

The approximate stress function is then given by (3.5) and (3.8); and the. torsional rigidity is

$$D_2 = 2\mu \int\int (c_1 g_1 + c_2 g_2 + c_3 g_3)\, dxdy = (0.1400)\, \mu a^4 (\sin 2\theta)^3. \tag{3.9}$$

It is well known that the approximate torsional rigidity found by the Rayleigh-Ritz method falls short of the exact value. Denoting the latter by D, we have then $D > D_1$, D_2 of which the better result is

$$D > (0\cdot 1400)\, \mu a^4 (\sin 2\theta)^3. \tag{3.10}$$

4. An upper bound for the torsional rigidity will now be obtained by a method due to Friedrichs (Ref. 1, pp. 323–27) which consists in maximizing an integral of the form

$$J(W) = -\int\int \left[(\nabla W)^2 - 4x \frac{\partial W}{\partial y} + 4x^2 \right] dxdy; \tag{4.1}$$

W is always a harmonic function and may be chosen with advantage to be a polynomial in x and y consisting of only odd powers in x as well as y. We take

$$W_1 = k_1\, xy \tag{4.2}$$

and

$$W_2 = l_1\, xy + l_2 (x^3 y - xy^3) \tag{4.3}$$

for first and second approximations and have

$$J(W_1) = -\frac{\sin 2\theta}{6} (k_1^2 - 4k_1 \cos^2\theta + 4\cos^2\theta)\, a^4, \tag{4.4}$$

and

$$J(W_2) = -\frac{1}{6} l_1^2 a^4 (\sin 2\theta) - \frac{1}{280} l_2^2 a^8 (\sin 2\theta)(10 - 7\sin^2 2\theta)$$
$$- \frac{2}{15} l_1 l_2 a^6 (\cos 2\theta)(\sin 2\theta) + \frac{2}{3} l_1 a^4 (\sin 2\theta)(\cos^2\theta)$$
$$+ \frac{2}{15} l_2 a^6 (\sin 2\theta)(\cos^2\theta)(2\cos^2\theta - \sin^2\theta) - \frac{2}{3} a^4$$
$$\times (\sin 2\theta)(\cos^2\theta).$$

The conditions for maximum of J (W_1) and J (W_2) yield

$$k_1 = 2\cos^2\theta \tag{4.6}$$

and

$$\left.\begin{aligned} l_1 &= 2\cos^2\theta - \frac{28(\sin 2\theta)^2\cos 2\theta}{38 + 7(\sin 2\theta)^2}, \\ l_2 &= \frac{70(\sin 2\theta)^2}{38 + 7(\sin 2\theta)^2}\cdot\frac{1}{a^2} \end{aligned}\right\} \tag{4.7}$$

Writing

$$-\mu\, J(W_1) = D_1', \quad -\mu\, J(W_2) = D_2'$$

we have $D < D_1', D_2'$ of which the better result is

$$D < D_2'. \tag{4.8}$$

In the case of a rhombus with acute angle equal to 60°, we note

$$D_1 = (0\cdot 09022)\,\mu a^4, \quad D_1' = (0\cdot 1082)\,\mu a^4,$$

$$D_2 = (0\cdot 09093)\,\mu a^4, \quad D_2' = (0\cdot 0951)\,\mu a^4.$$

Combining (3.10) and (4.8) we have

$$(0\cdot 09093)\,\mu a^4 < D < (0\cdot 0951)\,\mu a^4, \tag{I}$$

5. Saint-Venant had observed (1856) that of all cross-sections with a given area, the circle has the maximum torsional rigidity though the result was proved in a mathematical sense only recently (Ref. 6, p. 121). This can be expressed by the inequality

$$2\pi\, D \leqslant \mu A^2 \tag{5.1}$$

where A is the area of the cross-section. In the present case, $A = a^2 \sin 2\theta$ so that we have

$$D \leqslant \frac{\mu a^4 (\sin 2\theta)^2}{2\pi} = 0\cdot 1592\,\mu a^4 (\sin 2\theta)^2. \tag{5.2}$$

In the case of the rhombus with $2\theta = 60°$, we have

$$D \leqslant (0\cdot 1194)\,\mu a^4. \tag{5.3}$$

6. We may obtain another upper bound for the torsional rigidity by symmetrizing the given cross-section and using the fact that the torsional rigidity thereby increases.[6] Symmetrizing the rhombus with respect to a perpendicular to one of its sides, we change it into a rectangle whose sides are a, $a \sin 2\theta$. Using the value of the torsional rigidity of this rectangle (Ref. 1, p. 148), we have for the rhombus,

$$D < \mu \frac{a^4 (\sin 2\theta)^3}{3} - \frac{64 \mu a^4 (\sin 2\theta)^4}{\pi^5} \sum_{n=0}^{\infty} \frac{\tanh (2n+1) \frac{\pi}{2} \operatorname{cosec} 2\theta}{(2n+1)^5} \tag{6·1}$$

When $2\theta = 60°$, we note that

$$D < (0.1051)\, \mu\, a^4 \tag{6.2}$$

Since by repeated symmetrization a rhombus is convertible into a square, we also note

$$D < (0.1406)\, \mu a^4 (\sin 2\theta)^2. \tag{6.3}$$

In the case of $2\theta = 60°$ this is

$$D < (0·10545)\, \mu\, a^4 \tag{6.4}$$

though (6.2) gives a better estimate.

7. Recently Weinberger[7] has proved an inequality connecting the torsional rigidities of two (or more) disjoint regions and of the union of the regions. The result can be stated in the form

$$D_{12\cdots n} \geqslant D_1 + D_2 + \ldots + D_n \tag{7.1}$$

where D_i is the torsional rigidity of the i-th region R_i and $D_{12\cdots n}$ is that of the union $R = R_1 + R_2 + \ldots + R_n$. The proof of this result uses an equation of the type

$\oint_r \frac{\partial \Psi}{\partial n}\, ds = - 2A(r)$ and can be easily shown to be applicable in our problem. As the region ABCD (Fig. 1) is the union of the triangular regions ABD, BCD we have

$$D_{(ABCD)} \geqslant D_{(ABD)} + D_{(BCD)} \tag{7.2}$$

When $2\theta = 60°$, ABD and BCD are equilateral triangles so that we have

$$D_{(ABCD)} \geqslant 2\, D_{(ABD)} = \frac{\sqrt{3}}{40} \mu a^4$$

i.e.,

$$D \geqslant (0.0433)\, \mu a^4 \tag{7.3}$$

This is not satisfactory in view of (I); however it is of some interest. It is observed in (Ref. 7) that (7.1) reduces to an equality relation when and only when the common boundary lines of the regions R_i are level lines of Ψ for the composite region. Comparison of (I) and (7.3) leaves no room for equality in the latter, thus forcing the conclusion that the shorter diagonal of a rhombus of acute angle 60° cannot be a level line of Ψ.

Comparing the several inequalities for D, we note the best possible estimates for the torsional rigidity of a rhombus of acute angle 60°.

$$(0\cdot 0909)\,\mu a^4 < D < (0\cdot 0951)\,\mu a^4.$$

PART B

8. We now take up the numerical solution of the problem of torsion of a rhombus of acute angle 60°, using the relaxation technique. The function to be determined is Ψ and is governed by the conditions stated in (2.1) and (2.2). To have a non-dimensional equation, we write

$$x = a\xi,\ y = a\eta,\ \Psi = a^2 \chi(\xi, \eta). \tag{8.1}$$

Then

$$\frac{\partial^2 \chi}{\partial \xi^2} + \frac{\partial^2 \chi}{\partial \eta^2} = -2 \tag{8.2}$$

and the stresses are given by

$$\tau_{zx} = \mu a a \frac{\partial \chi}{\partial \eta}, \quad \tau_{zy} = -\mu a a \frac{\partial \chi}{\partial \xi} \tag{8.3}$$

and the torsional rigidity is

$$D = 2\mu a^4 \iint \chi(\xi, \eta)\, d\xi\, d\eta. \tag{8.4}$$

The cross-sectional shape suggests the use of a triangular net. If d is the side of the mesh, put $d = al$ (l is a pure number). Equation (8.2) is now replaced by finite difference equations of the form[8]

$$\tfrac{1}{6}(\chi_1 + \chi_2 + \chi_3 + \chi_4 + \chi_5 + \chi_6) - \chi_0 + \tfrac{1}{2} l^2 = 0 \tag{8.5}$$

where 1, 2, 3, 4, 5, 6 are the nodal points around zero and l is the side of the net (a pure number). We start with the triangular net of side $d = \frac{a}{3}$ so that $l = 1/3$.

The values of χ at the nodal points on the boundary are zero. If a_1, b_1, c_1, d_1 are the internal nodes, by symmetry $\chi(a_1) = \chi(c_1)$ and $\chi(b_1) = \chi(d_1)$. From (8.5) we are led to the equations

$$\left.\begin{aligned} 2\chi(a_1) - 5\chi(b_1) + \tfrac{1}{3} &= 0 \\ 3\chi(a_1) - \chi(b_1) - \tfrac{1}{6} &= 0 \end{aligned}\right\} \tag{8.6}$$

giving

$$\chi(a_1) = \chi(c_1) = \frac{7}{78}, \quad \chi(b_1) = \chi(d_1) = \frac{4}{39}. \tag{8.7}$$

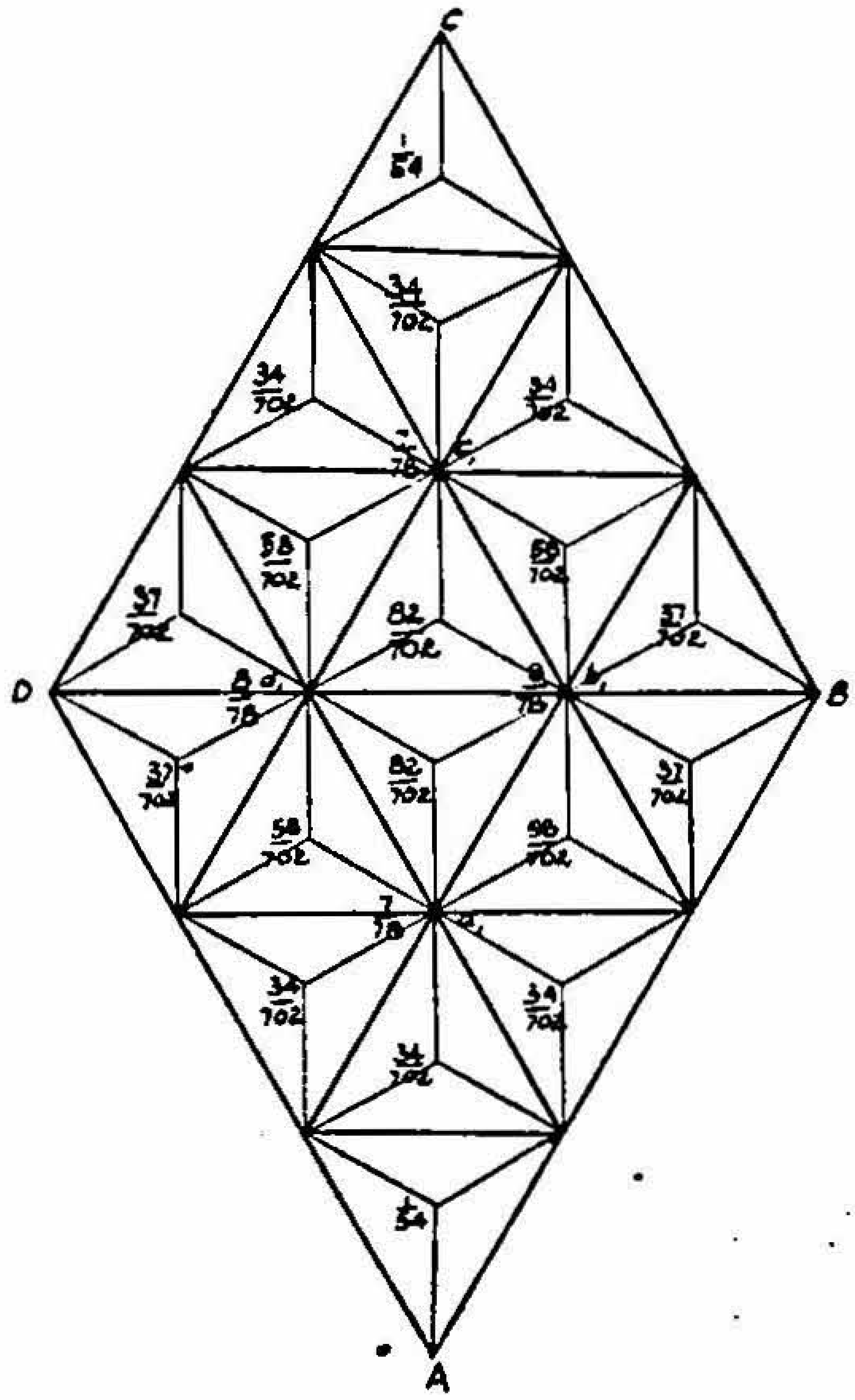

FIG. 2.

We may now proceed to a finer net (hexagonal) of mesh length $d = \frac{a}{3\sqrt{3}}$ so that $l = \frac{1}{3\sqrt{3}}$. The corresponding finite difference equations take the form

$$(X_1 + X_2 + X_3) - 3X_0 + \frac{1}{18} = 0. \tag{8.8}$$

Due to symmetry of the region, we may confine our attention to a fourth of the rhombus, which will be a 30°, 60°, 90° triangle. From (8.7) and (8.8) the X-values for the new nodes are determined (Fig. 2). Advancing now to a still finer (triangular) net of mesh length $d = \frac{a}{9}\left(l = \frac{1}{9}\right)$ we note the corresponding finite difference equation to be

$$X_1 + X_2 + X_3 + X_4 + X_5 + X_6 - 6X_0 + \frac{1}{27} = 0. \tag{8.9}$$

The X-values at these nodes are determined by using the values calculated above in (8.9). Starting with these initial values we proceed to liquidate the residuals. The final X-values after relaxation are given in Fig. 3.

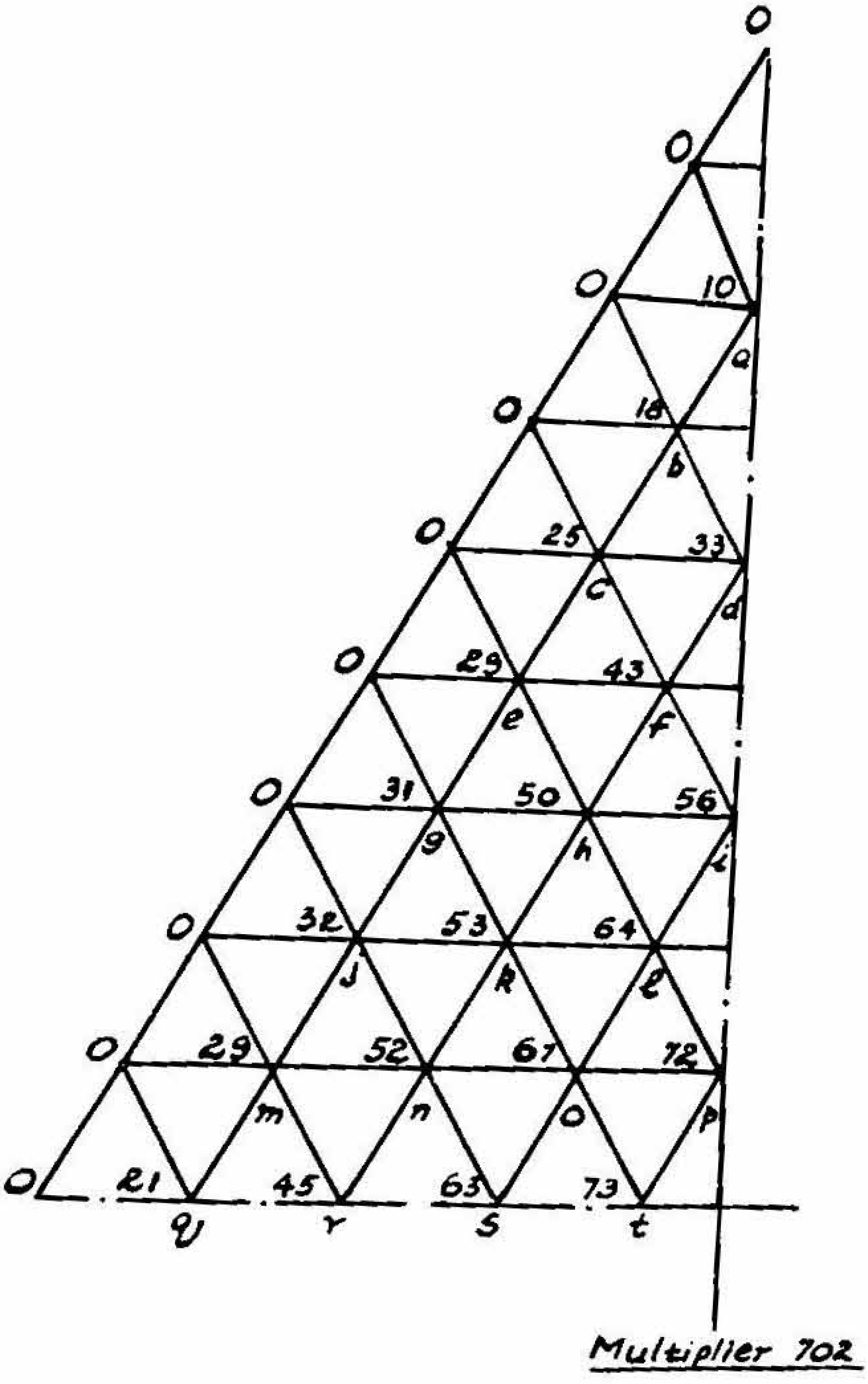

FIG. 3.

Going now to the four-stage advance to a finer (hexagonal) net of mesh length $d = \frac{a}{9\sqrt{3}}$ we use the finite difference equation

$$X_1 + X_2 + X_3 - 3X_0 + \frac{1}{162} = 0 \tag{8.10}$$

to obtain the X-values at the nodes. These values are now taken as starting assumptions for a *triangular* net of mesh side $d = \frac{a}{9\sqrt{3}}$ and the residuals (R_0) are obtained by using the relation

$$R_0 = X_1 + X_2 + X_3 + X_4 + X_5 + X_6 - 6X_0 + \frac{1}{81}. \tag{8.11}$$

The initial values (with multiplier 6318) and the resulting residuals are indicated to the left and right of the nodal points in Fig. 4.

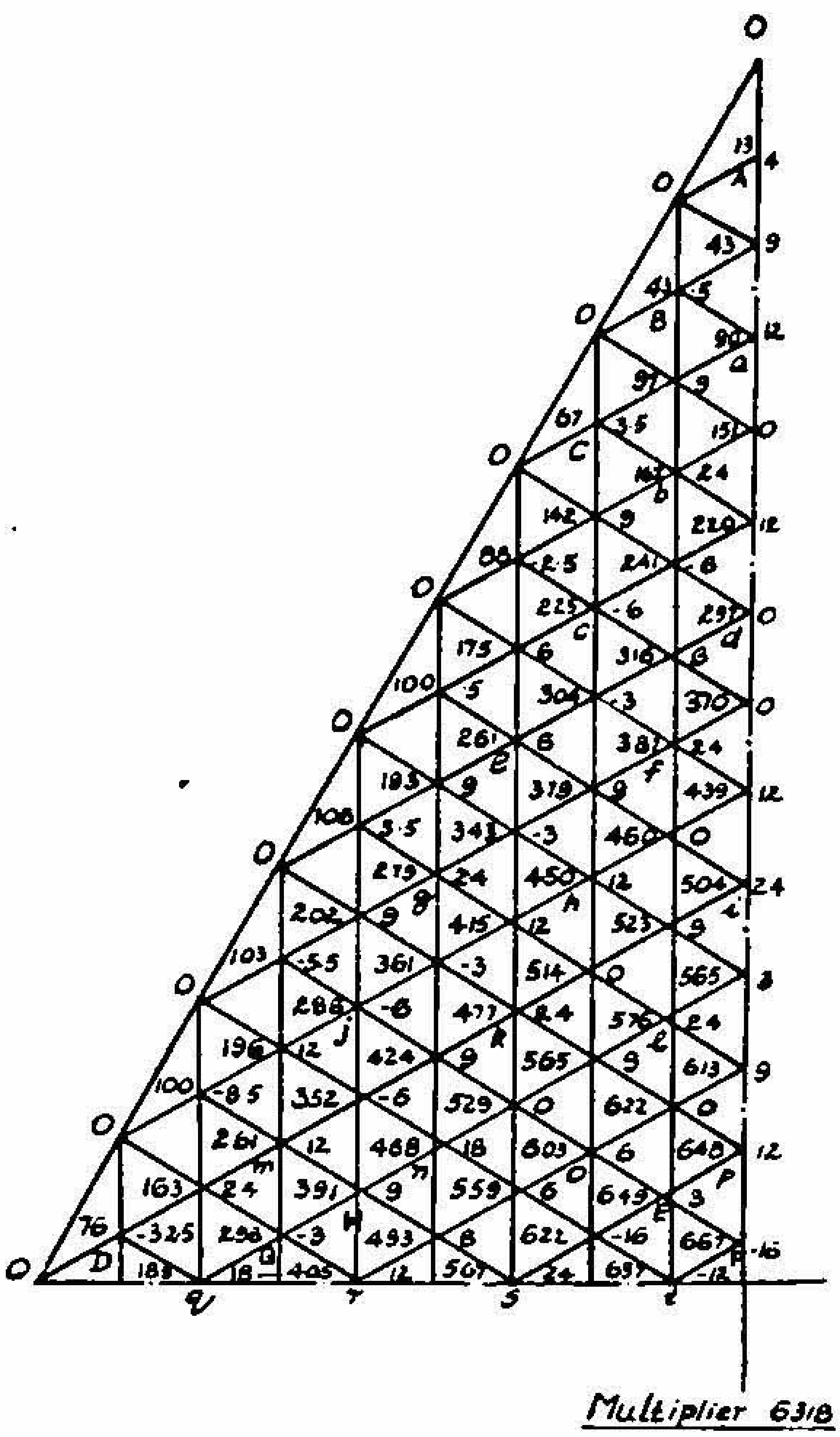

FIG. 4.

We may now proceed to liquidate the residuals. Symmetry of the region and irregular stars necessitate the use of several relaxation patterns in addition to the standard one (see Fig. 5).

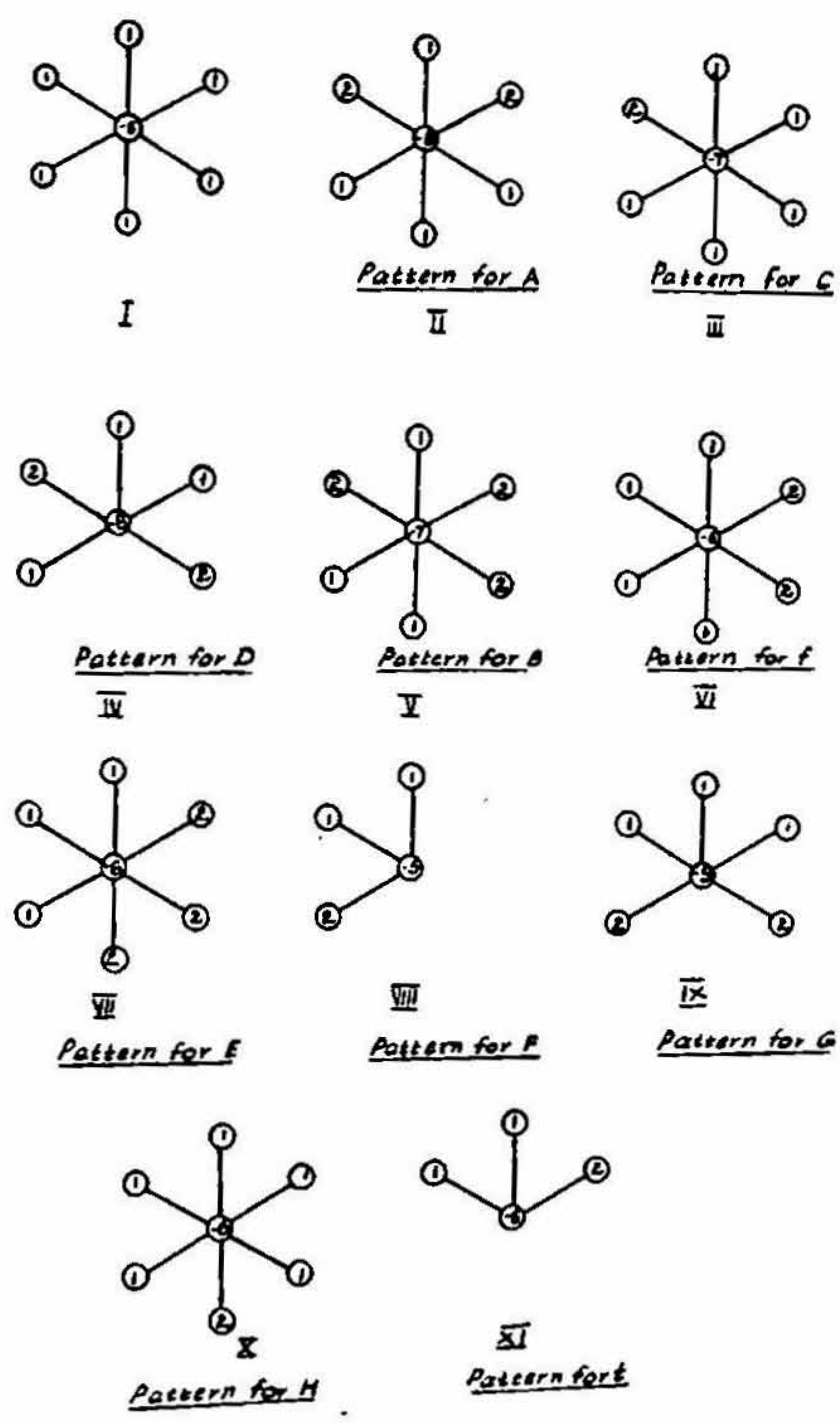

FIG. 5. Relaxation Patterns.

The accepted solution is shown in Fig. 6.

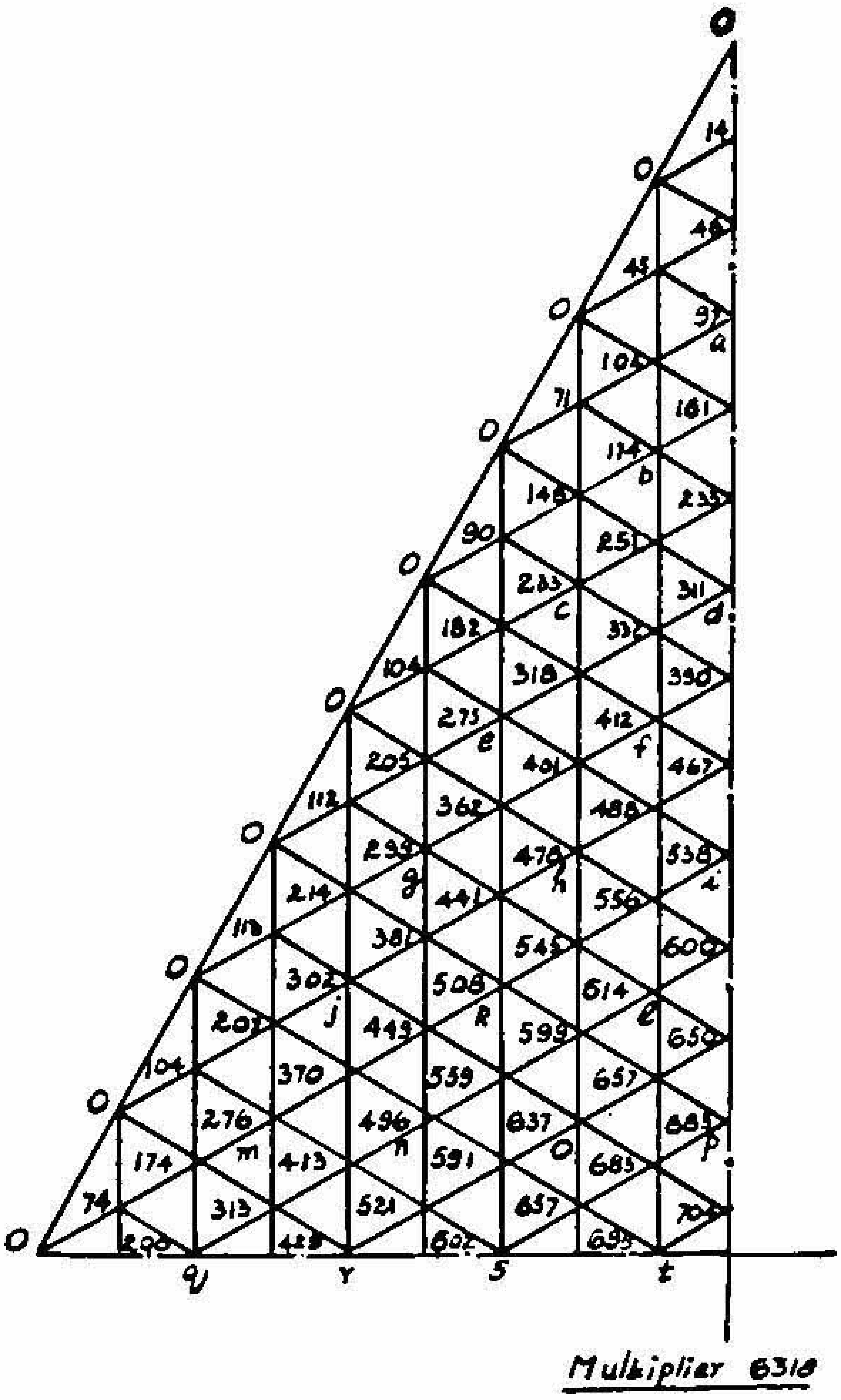

FIG. 6.

We need $\int\int \chi d\xi d\eta$ for the determination of the torsional rigidity (8.4). For the triangular mesh, the contribution of a single traingle is seen to be

$$\int\int \chi d\xi \, d\eta = \left(\frac{\chi_a + \chi_b + \chi_c}{3} + \frac{l^2}{6}\right) \frac{\sqrt{3}}{4} \, l^2. \tag{8.12}$$

Actual evaluation gives

$$D = 0 \cdot 09121 \mu a^4.$$

This compares favourably with the limits given at the end of Part A.

The best thanks of the authors are due to Professor N. S. Govinda Rao for his interest in this paper as well as encouragement.

References

1. Sokolnikoff, I. S. .. *A Mathematical Theory of Elasticity*, McGraw-Hill, 1946.
2. Higgins, T. J. .. *Amer. J. Phys.*, 1942, **10**, 248–59.
3. ———— .. *J. Appl. Phys.*, 1943, **14**, 469–80.
4. Synge, J. L. .. *Proc. Roy. Irish Acad.*, 1950, **53**, 41–64.
5. Prager, W. and Synge, J. L. .. *Quart. Appl. Math.*, 1947, **5**, 241–71.
6. Polya, G. and Szego, G. .. *Isoperimetric Inequalities in Mathematical Physics*, Princeton University Press, 1951.
7. Weinberger, H. F. .. *Jour. Math. and Phys.*, 1953, **32**, 54–62.
8. Southwell, R. V. .. *Relaxation Methods in Theoretical Physics*, Oxford, 1946.