Publication dates:

January, April, July and October.

Subscription rates:

Rs. 24.00 for India

$ 6.50 for U.S.A.

£ 2-25 for U.K.

Equivalent of Rs. 30.00 for other countries.

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Associate Editor
Journal of the
Indian Institute of Science
Bangalore-12, (India).

(Publishes original research carried out in the various Departments of the Institute.)

JOURNAL OF
THE INDIAN INSTITUTE OF SCIENCE

Volume 54] **Number 2** **[April 1972**

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Journal of the Indian Institute of Science
Vol. 54, No. 2, April 1972

THE USE OF BAR BUCKLING EIGENFUNCTIONS IN THE STABILITY ANALYSIS OF CLAMPED SKEW PLATES

By S. DURVASULA AND M. S. S. PRABHU
(*Department of Aeronautical Engineering, Indian Institute of Science, Bangalore-12*)

[Received: February 3, 1972]

ABSTRACT

In this paper the use of bar buckling eigenfunctions in the stability analysis of clamped skew plates is examined. These functions which are obtained from the solution of the linear, homogeneous differential equation corresponding to the buckling problem of uniform bar are used in the series expansion for deflection. The problem is formulated in oblique co-ordinates and using oblique components of stress. Galerkin method is used and the resulting set of algebraic equations is solved for the eigenvalues. Numerical calculations have been made for a few combinations of side ratio, skew angle and in-plane loading. The critical values obtained are in fair agreement with the results obtained by using beam characteristic functions indicating that these functions can well be used as an alternate set of functions in approximate solution of plate buckling problems.

1. INTRODUCTION

In many structural mechanical problems like bending, vibration and stability of bars and plates, especially when using energy methods, the solution needs to be expanded in a series of admissible functions. It is desirable, though not essential, that these functions, apart from satisfying the geometric boundary conditions, also possess some of the properties pertinent to the problem under consideration. The beam characteristic functions, that is, the functions representing the normal modes of a vibrating beam, have been used extensively in the literature for this purpose (see, for example, Refs. 1–8). They appear to be the normal choice in vibration problems, although they have been used with advantage in other problems like stability, thermal stress analysis of beams and plates.

Similarly the functions which are obtained from the solution of the linear, homogeneous differential equation corresponding to the buckling of bars may be called the " Bar Buckling Eigenfunctions ". The orthogonality relation relevant to the buckling problem is between the first derivatives of the deflection mode shapes and not between the mode shapes themselves. It is felt desirable to investigate the use of bar buckling eigenfunctions for approximate solution of plate buckling problems.

Unlike the beam characteristic functions, the bar buckling functions have not been used in formal analyses of plates except to obtain some rough estimates of the buckling load with a one-term approximation [9, 10]. In this paper, buckling of clamped skew plates under different loading conditions is considered for studying in detail the use of bar buckling eigenfunctions. The problem is formulated in terms of oblique co-ordinates with in-plane stresses represented in terms of oblique components. Deflection surface is, therefore, expressed in the form of a double series of bar buckling eigenfunctions in oblique co-ordinates. Galerkin method is used and the resulting set of simultaneous, homogeneous, linear algebraic equations is solved for eigenvalues and eigenvectors.

2. Mathematical Formulation

A sketch of the skew plate is shown in Fig 1, along with the in-plane stresses represented in terms of oblique components. The skew plate is assumed to be thin, uniform and isotropic. Using the classical,

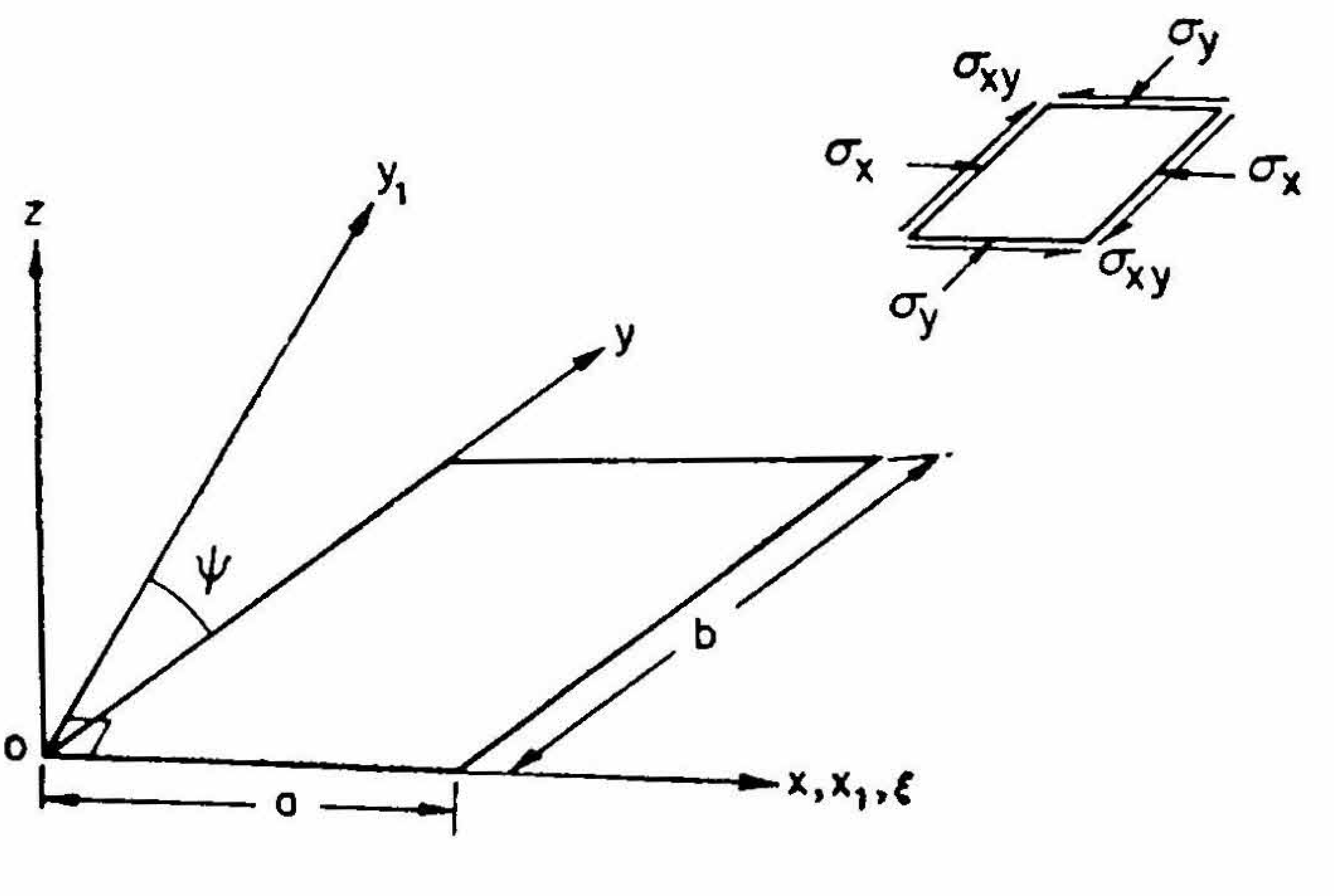

FIG. 1

Sketch of the skew plate showing the co-ordinate system and the oblique stress system small deflection thin plate theory, the differential equation for the deflection of the plate of constant thickness under the action of middle surface forces is given by,

$$D \cos \psi \nabla^4 W = -\left(N_x \frac{\partial^2 W}{\partial x^2} + 2 N_{xy} \frac{\partial^2 W}{\partial x \, \partial y} + N_y \frac{\partial^2 W}{\partial y^2} \right) \qquad [1]$$

The boundary conditions for the clamped edge are

$$W = \frac{\partial W}{\partial n} = 0 \quad \text{where } n \text{ is the outward normal to the edge} \qquad [2]$$

For example, for the edge $x = a$, the boundary conditions are

$$W = \frac{\partial W}{\partial x} = 0 \qquad [3]$$

In terms of non-dimensional co-ordinates ξ $(= x/a)$ and η $(= y/b)$, the differential equation, Eq. (1), becomes

$$W,_{\xi\xi\xi\xi} + \lambda^4 W,_{\eta\eta\eta\eta} + 2\lambda^2 (1 + 2 \operatorname{Sin}^2 \psi)\, W,_{\xi\xi\eta\eta} - 4\lambda \operatorname{Sin} \psi \,(W,_{\xi\xi\xi\eta} + \lambda^2 W,_{\xi\eta\eta\eta})$$

$$+ \bar{R}_x^* W,_{\xi\xi} + \lambda^2 \bar{R}_y^* W,_{\eta\eta} + 2\lambda \bar{R}_{xy}^* W,_{\xi\eta} = 0 \qquad [4]$$

where subscripts after a comma denote differentiation.

The deflection $W(\xi, \eta)$ is expressed as a series of bar buckling eigenfunctions satisfying the boundary conditions, Eq. (2). The deflection is expressed as,

$$W(\xi, \eta) = \sum_{m=1,2..}^{M} \sum_{n=1,2..}^{N} C_{mn} X_m(\xi)\, Y_n(\eta) \qquad [5]$$

where X_m and Y_n are the m^{th} and n^{th} bar buckling eigenfunctions (for details see Ref. 11). Substituting the expression for $W(\xi, \eta)$ from Eq. (5) in Eq. (4), we get the error function $\epsilon(\xi, \eta)$ as

$$\epsilon(\xi, \eta) = \sum_{m=1}^{M} \sum_{n=1}^{N} C_{mn} \{ X_m^{iv} Y_n + \lambda^4 X_m Y_n^{iv} + 2\lambda^2 (1 + 2 \sin^2 \psi) X_m'' Y_n''$$

$$- 4\lambda \sin \psi \,(X_m''' Y_n' + \lambda^2 X_m' Y_n''') + \bar{R}_x^* X_m'' Y_n$$

$$+ 2\lambda \bar{R}_{xy}^* X_m' Y_n' + \bar{R}_y^* X_m Y_n'' \lambda^2 \} \qquad [6]$$

This error in the interior is now orthogonalised with respect to each of the functions used in Eq. (5), i.e.,

$$\int_0^1 \int_0^1 \epsilon\,(\xi, \eta)\; X_r\,(\xi)\; Y_s(\eta)\,\cos\psi\; d\,\xi\; d\,\eta \equiv 0 \quad \text{for } r = 1, 2 \ldots M \quad s = 1, 2 \ldots N \qquad [7]$$

Substituting the expressions for the error $\epsilon\,(\xi, \eta)$ in Eq. (7), we get a set of linear, simultaneous, algebraic equations in the unknown $C'_{mn}s$ which can be expressed in the matrix form as,

$$[\,H\,]\;\{C_{mn}\} = \bar{R}_x^*[\,E\,]\{C_{mn}\} + \bar{R}_y^*[\,F\,]\{C_{mn}\} + \bar{R}_{xy}^*[\,G\,]\{C_{mn}\} \qquad [8]$$

where,

$$H_{mnrs} = I_{mr}^{40} J_{ns}^{00} + \lambda^4\, I_{mr}^{00} J_{ns}^{40} + 2\lambda^2\,(1 + 2\sin^2\psi)\; I_{mr}^{20} J_{ns}^{20}$$
$$- 4\lambda\,\sin\psi\,(I_{mr}^{30} J_{ns}^{10} + \lambda^2\, I_{mr}^{10} J_{ns}^{30})$$

$$E_{mnrs} = I_{mr}^{20} J_{ns}^{00}\;;\quad F_{mnrs} = \lambda^2\, I_{mr}^{00} J_{ns}^{20}\;;\quad G_{mnrs} = 2\lambda\, I_{mr}^{10} J_{ns}^{10} \qquad [9]$$

The I – and J – integrals are defined as follows :

$$I_{mr}^{pq} = \int_0^1 X_m^p\,(\xi)\; X_r^q\,(\xi)\; d\,\xi\;;\quad J_{ns}^{pq} = \int_0^1 Y_n^p\,(\eta)\; Y_s^q\,(\eta)\; d\,\eta \qquad [10]$$

where p, q represent the order of derivatives. The formulae and numerical values of their integrals are given in Ref. 11.

The eigenvalue problem as stated by Eq. (8) can be solved by using standard methods, by giving numerical values to any two of the three parameters $\bar{R}_x^*$, $\bar{R}_y^*$, $\bar{R}_{xy}^*$ and treating the third as the eigenvalue. For example, if the buckling parameter $\bar{R}_x^*$ is to be obtained when N_y and N_{xy} are also acting, we assign appropriate numerical values to $\bar{R}_y^*$ and $\bar{R}_{xy}^*$ and write

$$[G_1]\;\{C_{mn}\} \quad = \bar{R}_x^*\{C_{mn}\} \qquad [11]$$

where $[G_1] = [E]^{-1} \quad ([H\,] - \bar{R}_y^*\,[F] \quad - \bar{R}_{xy}^*\,[G]) \qquad [12]$

and $\bar{R}_x^*$ is the eigenvalue to be determined.

Due to the symmetry in the boundary condition, Eq. [8] can be split for convenience and computational advantage into two cases : (*a*) skew symmetric case consisting of modes with $(\,m+n\,)$ and $(\,r+s\,)$ even *i.e.*, modes which are symmetric (antisymmetric) in x-direction and symmetric (antisymmetric) in y-direction. (*b*) skew antisymmetric case consisting of modes with $(m+n)$ and $(r+s)$ odd *i.e.*, modes which are symmetric (antisymmetric) in x-direction and antisymmetric (symmetric) in y-direction. This splitting reduces the order of the matrix to be considered. The lower of the least eigenvalue from the two cases corresponds to the critical buckling load.

3, Results and Discussion

Numerical calculations have been made for a few combinations of a/b and skew angle ψ under direct and shear loadings and also for a few combined loadings. The results of the convergence study for an example case of 30° rhombic plate under the action of N_x alone is given in Table 1. It can be seen from the table that convergence of 18 terms is quite satisfactory for $\psi \leq 30°$. For higher skew angles, naturally, more terms are required to get equivalent accuracy and hence 32 terms are used. In Table 2, the buckling coefficients under the action of direct and shear loadings are given for different combinations of a/b and skew angle. Also, the results for a few combined loadings are given in Table 3. The results from Ref. 12, obtained by using beam characteristic functions, for the same order of matrices are given for comparison. From these comparisons, it is seen that the results from the use of products of bar buckling eigenfunctions are nearly the same as the results using beam characteristic functions.

TABLE 1

Convergence Study

a/b=1 ; $\psi=30°$; N_x alone acting

Skew Symmetric case

M	N	Order of Matrix	Eigen value $\bar{K}_x$
1	1	1	18.48
2	2	2	14.23
3	3	5	13.03
4	4	8	12.69
5	5	13	12.10
6	6	18	11.95
7	7	25	11.80
8	8	32	11.78

TABLE 2

Buckling coefficients under direct and shear loading

a/b	ψ	Loading	Order of Matrix	Buckling Coefficient $\bar{R}_x$ or $\bar{R}_{xy}$ Present Paper (a)	Ref. 12 (b)
0.5	0°		9	19.4	19.4
0.5	15°		18	20.9	20.8
0.5	30°		18	26.5	26.3
0.5	45°		32	39.2	38.8
1	0°	N_x alone	9	10.1	10.1
1	15°		18	10.5	10.5
1	30°		18	12.0	11.8
1	45°		32	14.5	14.3
0.5	15°		18	31.8 −62.8	31.9 −62.2
0.5	30°		18	25.5 −112	27.2 −111
0.5	45°	N_{xy} alone	32	27.6 −246	27.1 −246
1	15°		18	11.4 −22.5	11.1 −22.3
1	30°		18	9.85 −40.2	9.50 −40.0
1	45°		32	9.53 −89.6	9.35 −89.3

(a) Bar Buckling Functions (b) Beam Characteristic Functions

TABLE 3

Buckling Coefficient $\bar{R}_x$ Under Combined Loading

$a/b=0.5$; $\psi=30^\circ$; $\alpha=N_y/N_x$; $\beta=N_{xy}/N_x$

Order of Matrix $=18\times18$

α	β	$\bar{R}_x$	
		Present Paper(a)	Ref. 12(b)
1	0 5	14.8	14.4
1	1	11.8	11.5
0.5	0	23.3	23.1
0.5	0.5	17.7	17.2
0.5	1	13.6	13.2
0	0.5	20.9	20.7
0	0	26.5	26.3

(a) Bar Buckling Eigenfunctions (b) Beam Characteristic Functions

4. CONCLUSIONS

In this paper, the use of bar buckling eigenfunctions in the approximate solutions of stability problems of clamped skew plate is examined. Numerical results for the buckling coefficients are given for different skew angles and side ratios mainly under individual loading ; the results for a few cases of combined loading are also given. The results obtained from the use of bar buckling eigenfunctions are quite close to those obtained earlier by the use of beam characteristic functions. Therefore, these functions can also be equally well used in solving plate buckling problems.

5. ACKNOWLEDGEMENTS

The authors are thankful to Prof. C. V. Joga Rao for helpful criticism.

6. NOTATION

a, b	dimensions of the plate
C_{rs}	coefficient in the series expansion for deflection
D	flexural rigidity of the plate, $Eh^3/12(1-\nu^2)$
E, F, G, H, G_1	matrices defined in Eqs. [8] and [11]
E	Young's modulus of the material of the plate
h	plate thickness
I_{mr}^{pq}, J_{ns}^{pq}	integrals occurring in Eq. [10]
k_q	q^{th} root of the transcendental equation $\tan k = k$
M, N	maximum number of terms in $x-$ and $y-$directions respectively.
m, n, r, s	integers defined in Eqs. [5] and [7]
N_x, N_y, N_{xy}	in-plane forces $\sigma_x h$, $\sigma_y h$ and $\sigma_{xy} h$ respectively
$\bar{R}_x^*$, $\bar{R}_y^*$, $\bar{R}_{xx}^*$	non-dimensional mid-plane force parameters $\sigma_x a^2 h \cos^3\psi/D$, $\sigma_y a^2 h \cos^3\psi/D$, $\sigma_{xy} a^2 h \cos^3\psi/D$ respectively
$\bar{R}_x^*$, $\bar{R}_y^*$, $\bar{R}_{xy}^*$	non-dimensional mid-plane force parameters $\sigma_x b^2 h/\pi^2 D$, $\sigma_y b^2 h/\pi^2 D$, $\sigma_{xy} b^2 h/\pi^2 D$ respectively
$X_m(\xi)$, $Y_n(\eta)$	bar buckling eigenfunctions
x, y, z	oblique co-ordinate system as in Fig. 1
$W(\xi, \eta)$	deflection of the plate
ξ, η	non-dimensional co-ordinates, x/a and y/b respectively
σ_x, σ_y, σ_{xy}	in-plane stresses (oblique components)
ν	Poisson's ratio
$\epsilon(\xi, \eta)$	error in the interior
λ	side ratio, a/b
ψ	skew angle, as defined in Fig. 1
∇^2	skew differential operator $-\sec^3\psi\left(\frac{\partial^2}{\partial x^2} - 2\sin\psi\frac{\partial^2}{\partial x\,\partial y} + \frac{\partial^2}{\partial y^2}\right)$

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