FREE-TRANSVERSE VIBRATIONS OF AN AXIALLY MOVING MASS

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ABSTRACT

In this paper the non-linear analysis of an axially moving mass has been presented. It is shown that upto a certain critical value of the transport velocity, the characteristic is similar to that of the stationary state. At the critical velocity, it departs from its non-linear behaviour and the vibration mode is the same as that of the linear case with zero transport velocity. For transport velocities greater than the critical, the non-linear period T_N decreases as the initial tension and the flexural parameter μ increases. These conclusions are not borne out in a linear analysis.

NOMENCLATURE

The following symbols are used in this paper:

ρ = Mass density.
A = Cross-sectional area.
C = Const. axial transport velocity.
P = Initial tension.
I = Moment of inertia.
E = Elastic modulus.
ψ = Curvature.
S = Distance measured along the elastic arc.
l = Free length of strip between supports.
u = Axial displacement with respect to co-ordinates translating at velocity C.
v = Transverse displacement.
t = time.
x = Fixed axial co-ordinate.

w = Non-dimensional transverse displacement.
$r = I/A$ = Radius of gyration.
Ω = Frequency.
α = Non-dimensional initial tension.
η = Non-dimensional time.
ξ = Non-dimensional axial co-ordinate.
δ = Non-dimensional velocity number.
λ – Non-dimensional velocity constant.
μ = Non-dimensional radius of gyration.
T_N = Non-dimensional non-linear period.
T_L = Non-dimensional linear period.
λ_c = Critical velocity number.
T_N/T_L = Period ratio.
W_0 = Amplitude of vibration.
W_{0c} = Critical amplitude.
$\frac{\pi^2 W_0^2}{4(\alpha^2+\mu^2\pi^2)}$ = Amplitude effect.

1. Introduction

The problem of the free-transverse vibrations of an axially moving mass has attracted the attention of many workers recently on account of its many technological applications. The moving mass considered is either a string, a strip or a beam with finite flexural rigidity. Most of the analyses presented so far are concerned with linear problems [1-8]. The analogous problem of the transverse vibrations of a pipe containing a flowing fluid has also invited the attention of many workers [9-11]. Although these class of problems are generally non conservative, the analysis presented neglects certain coriolis type of force terms, thus reducing the system to a conservative one [8, 11], (because the coupling terms do not contribute to the energy of vibration) The conclusions arrived at from the linear analysis are, that the natural frequency of transverse vibration decreases with transport velocity and there exists a certain critical velocity at which the fundamental frequency of free-vibration ceases to exist. In fact for a string, all the mode frequencies vanish at the critical velocity.

Some papers have appeared recently which take into account the non-linear terms but such analysis is mostly restricted to strings [12, 13, 15]. The problem of a moving strip was considered recently by a perturbation technique but the analysis was limited to strips defined for $\mu^2 < 0.001$ and restricted to small motions.

In this paper an analysis is presented to cover all classes of moving masses namely strings, strips or beams. The solution presented is not restricted to small motions and it is valid for a larger range of strip geometry. The analysis not only provides better physical insight into the problem but also established the range of validity and the limitations of the linear analysis.

The following assumptions have been made :

(i) Damping is negligible

(ii) The axially moving mass is simply supported

(iii) The transverse displacements are measured from the equilibrium configuration defined by material moving at a constant axial transport velocity. (with no variation in transverse deflection.)

(iv) Effects of rotary inertia and transverse shear are negligible

(v) Only free-transverse vibrations are considered.

2. EQUATION OF MOTION

Fig. (1) shows the deflected position of the moving strip with its velocity components in transverse and longitudinal directions. The expressions for kinetic energy T and potential energy V are :

$$T=\frac{1}{2}\rho A\int_0^l \{(v_t+Cv_x)^2+[u_t+C(1+u_x)]^2\}\ dx \qquad [1]$$

$$V=\int_0^l P\epsilon\,dx+\frac{1}{2}AE\int_0^l \epsilon^2 dx+\frac{1}{2}EI\int_0^l \psi^2\, ds \qquad [2]$$

where $\epsilon=[(1+u_x)^2+v_x^2]^{1/2}-1$

and $\psi^2\, ds=\left[\frac{\{(1+u_x)\,v_{xx}-v_x\,u_{xx}\}^2}{\{(1+u_x^2)+v_x^2\}^{5/2}}\right]dx$

Taking $u_x<<1$, $v_x^2<1$ and $v_x^4<<v_x^2$, the kinetic and potential energies become :

$$T=\frac{1}{2}\rho A\int_0^l [(v_t+Cv_x)^2+(u_t+C)^2]\,dx \qquad [3]$$

and $$V=\frac{1}{2}\int_0^l [Pv_x^2+\frac{1}{4}AE\,v_x^4+EIv_{xx}^2\,(1-\frac{5}{2}v_x^2)]\,dx \qquad [4]$$

and the Lagrangian L is given by : $L=(T-V)$ [5]

Applying the Hamilton's Principle,

$$\delta\int_{t_1}^{t_2} L\,dt=0 \qquad [6]$$

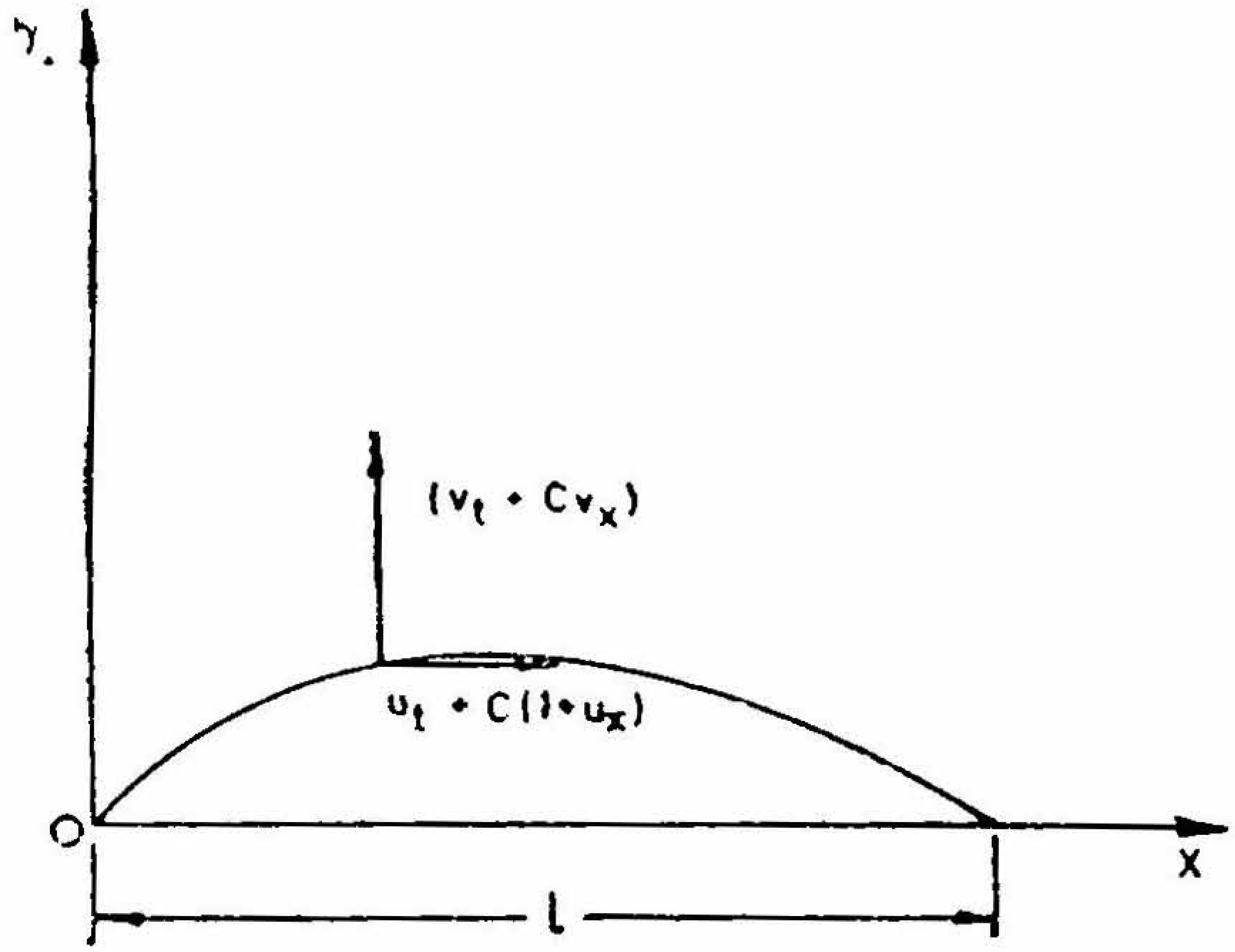

FIG. 1
Co-ordinate System and Velocity Components

and performing the variation :

$$-\int_{t_1}^{t_2}\int_0^l [\rho A v_{tt} + 2\rho AC v_{xt} + (\rho AC^2 - P) v_{xx} - (3/2)\, AE\, v_x^2 v_{xx}$$

$$+ EI\, v_{xxxx} - (5/2)\, EI\, v_x^2 v_{xxxx} - 10\, EI\, v_x v_{xx} v_{xxx} - (5/2)\, EI\, v_{xx}^3](\delta v)\, dx\, dt$$

$$-\int_{t_1}^{t_2}\int_0^l \rho A u_{tt} (\delta u)\, dx\, dt$$

$$+\int_{t_1}^{t_2} [\{AC(v_t + C v_x) - (AE/2)\, v_x^3 - P v_x\}\, \delta(v) \Big|_0^l$$

$$- EI\{(1 - [\tfrac{5}{2} v_x^2)] v_{xx}\, \delta(v_x) \Big|_0^l + \tfrac{5}{2} v_x (v_{xx})^2 - (1 - \tfrac{5}{2} v_x^2) v_{xxx}\, \delta(v) \Big|_0^l\}]\, dt \qquad [7]$$

the equation of motion for free transverse vibration is :

$$\rho A v_{tt} + 2\rho AC v_{xt} + (\rho AC^2 - P)\, v_{xx} - (3/2)\, AE\, v_x^2 v_{xx}$$

$$+ EI\, v_{xxxx} - (5/2)\, EI\, v_x^2 v_{xxxx} - 10\, EI\, v_x v_{xx} v_{xxx} - 5/2\, EI\, v_{xx}^3 = 0 \qquad [8]$$

The expression $\rho A u_{tt} = 0$ in equation [7] implies that the rate of displacement in the longitudinal direction is constant and the remaining terms in equation [7] specify the boundary conditions, *viz*,

$$v = v_{xx} = 0 \text{ at } x = 0$$

$$v = v_{xx} = 0 \text{ at } x = l \qquad [9]$$

Introducing the non-dimensional parameters :

$$w = v/l; \; \xi = x/l; \; \eta = (1/l)\left(\frac{P}{\rho A} + \frac{EI\pi^2}{\rho A l^2}\right)^{1/2} t;$$

$$\delta = \frac{C}{[P/\rho A + EI\pi^2/(\rho A l^2)]^{1/2}}; \quad \beta = \frac{C}{\sqrt{(P/\rho A)}}; \quad \lambda = \frac{C}{\sqrt{(E/\rho)}};$$

$$\alpha = (P/AE)^{1/2}; \quad \mu = r/l; \qquad [10]$$

the equation [8] in terms of non-dimensional parameter becomes:

$$w_{\eta\eta} + 2\,\delta\, w_{\xi\eta} + \frac{\delta^2(\beta^2 - 1)}{\beta^2} w_{\xi\xi} - \frac{3}{2}\,\frac{\delta^2}{\lambda^2} w_\xi^2\, w_{\xi\xi}$$

$$+ \frac{\mu^2\delta^2}{\lambda^2}[w_{\xi\xi\xi\xi} - \tfrac{5}{2} w_\xi^2\, w_{\xi\xi\xi\xi} - 10\, w_\xi\, w_{\xi\xi}\, w_{\xi\xi\xi} - \tfrac{5}{2} w_{\xi\xi}^3] = 0 \qquad [11]$$

and the boundary conditions are :

$$w(0, \eta) = w''(0, \eta) = 0$$

and $$w(1, \eta) = w''(1, \eta) = 0 \qquad [12]$$

3. SOLUTION BY THE METHOD OF HARMONIC BALANCE

The equation [11] is a highly nonlinear partial differential equation with constant co-efficients and only an approximate analysis is possible. In this paper a solution for equation [11] is given by the method of harmonic balance. A solution of equation [11] is assumed in the form

$$w(\xi, \eta) = X(\xi)\, T(\eta) \qquad [13]$$

and X and T are determined so as to approximately satisfy equation [11]. The constants in the resulting solution are chosen to satisfy the boundary conditions (12).

Substituting equation [13] in equation [11],

$$X\ddot{T}+2\,\delta\,X'\dot{T}-\frac{\delta^2(1-\beta^2)}{\beta^2}X''\,T-(3/2)\frac{\delta^2}{\lambda^2}X'^2\,X''\,T^3$$

$$+\frac{\mu^2\delta^2}{\lambda^2}[X''''\,T-(5/2)\,X'^2\,X''''\,T^3-10\,X'\,X''X'''\,T^3-(5/2)\,X''^3\,T^3]=0 \quad [14]$$

By this substitution the nonlinear partial differential eqation [11] reduces to an ordinary nonlinear differential equation [14]. Assuming $X=X_0\sin\pi\xi$ (which satisfies the boundary conditions as given by equation 12) equation [14] becomes:

$$(X_0\sin\pi\xi)\,\ddot{T}+2\delta\,(\pi X_0\cos\pi\xi)\,\dot{T}+\frac{\delta^2(1-\beta^2)}{\beta^2}\,(X_0\pi^2\sin\pi\xi)\,T$$

$$+(3/2)\frac{\delta^2}{\lambda^2}[X_0^3\pi^4\,(\cos^2\pi\xi)\,(\sin\pi\xi)\,T^3+\frac{\mu^2\delta^2}{\lambda^2}\,(X_0\pi^4\sin\pi\xi)\,T$$

$$-(5/2)\frac{\mu^2\delta^2}{\lambda^2}X_0^3\pi^6\,(\cos^2\pi\xi)\,(\sin\pi\xi)\,T^3-10\frac{\mu^2\delta^2}{\lambda^2}X_0^3\pi^6(\cos^2\pi\xi)\,(\sin\pi\xi)T^3$$

$$+(5/2)\frac{\mu^2\delta^2}{\lambda^2}X_0^3\pi^6\,(\sin^3\pi\xi)\,T^3=0.$$

Simplifying further

$$(X_0\sin\pi\xi)\,\ddot{T}+(2\delta\,X_0\pi\cos\pi\xi)\,\dot{T}+\left[\left(\frac{\delta^2(1-\beta^2)}{\beta^2}\pi^2+\frac{\mu^2\delta^2}{\lambda^2}\pi^4\right)T\right.$$

$$\left.+\left((3/8)\frac{\delta^2}{\lambda^2}X_0^2\pi^4-(5/4)\frac{\mu^2\delta^2}{\lambda^2}X_0^2\pi^6\right)T^3\right]\sin\pi\xi$$

$$+\left((3/8)\frac{\delta^2}{\lambda^2}\pi^4-(15/4)\pi^6\frac{\mu^2\delta^2}{\lambda^2}X_0^2\,T^3\right)X_0\sin 3\pi\xi=0 \quad [15]$$

Neglecting $\sin 3\pi\xi$ and setting the co-efficient of $\sin\pi\xi$ equal to zero, the equation [15] becomes,

$$\ddot{T}+\left(\frac{\delta^2(1-\beta^2)}{\beta^2}\pi^2+\frac{\mu^2\delta^2}{\lambda^2}\pi^4\right)T+\left(\frac{3}{8}\frac{\delta^2}{\lambda^2}-\frac{5}{4}\frac{\mu^2\delta^2}{\lambda^2}\pi^2\right)\pi^4X_0^2\,T^3=0 \quad [16]$$

A simple solution to equation [16] can be obtained by the method of harmonic balance. A solution of the form $T=T_0\sin\Omega\eta$ is assumed. Substituting it into eqation [16] and setting the co-efficient of sin $\Omega\eta$ equal to zero yields the frequency equation, viz.,

$$\Omega^2 = \left(\frac{\delta^2(1-\beta^2)}{\beta^2} + \frac{\mu^2\delta^2}{\lambda^2}\pi^2\right)\pi^2 + \left(\frac{9}{32}\ \frac{\delta^2}{\lambda^2} - \frac{15}{16}\ \frac{\mu^2\delta^2}{\lambda^2}\pi^2\right)\pi^4(X_0^2\ T_0^2) \qquad [17]$$

Letting $w_0 = X_0\ T_0 =$ amplitude of transverse vibration and in view of equation [10] the non-linear period after some simplification becomes :

$$T_N = 2\sqrt{\left(\frac{1}{1-\delta^2+[9/32-(15/16)\ \mu^2\pi^2]\ \pi^2 w_0^2/(\alpha^2+\mu^2\ \pi^2)}\right)} \qquad [18\text{-A}]^*$$

and consequently the linear period from equation [18] becomes :

$$T_L = 2\sqrt{[(1/1-\delta^2)]} \qquad [19]$$

and the period ratio,

$$T_N/T_L = \sqrt{\left(\frac{1-\delta^2}{1-\delta^2+[9/32-(15/16)\mu^2\pi^2]\ \pi^8\ w_0^2/(\alpha^2+\mu^2\ \pi^2)}\right)} \qquad [20]$$

Though the method of analysis is approximate, the accuracy of the results can be seen from a comparison of results of some of the recent workers [8, 13 and 16].

For the case of the travelling string the non-linear period given by equation [18] corresponds exactly to that found in[13]. As there exists no expression for the frequency of vibration in the paper by Thurman and Mote[16] for strips with transport velocity, a quantitative comparison with their rusults is not possible. But for a stationary string the fundamental frequency of vibration (by first approximation) is in excellent agreement with that given in this analysis. The results for a purely travelling beam cannot be compared as no such non-linear analysis is available. But one important result can be compared, *viz.*, the fundamental frequency of vibration of the linear system ceases to exist if the non-dimentional velocity number δ equals unity as given by equation [19], *i.e.*, when $C\ l/(EI/\rho\ A)^{\frac{1}{2}} = \pi$, the linear vibration ceases to exist. This conclusion is in close agreement with $C\ l/(EI/\rho\ A)^{\frac{1}{2}} = 3.18$ as obtained by Barokal[8].

4. RESULTS AND DISCUSSION

The results of the analysis have been presented in the forms of graphs in Figs. 2 to 11.

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*This can also be written as

$$T_N = 2\sqrt{\left(\frac{\alpha^2+\mu^2\pi^2}{(\alpha^2+\mu^2\pi^2-\lambda^2)+[9/32-(15/16)\mu^2\pi^2]\ \pi^2\omega_0^2}\right)} \qquad [18\text{-B}]$$

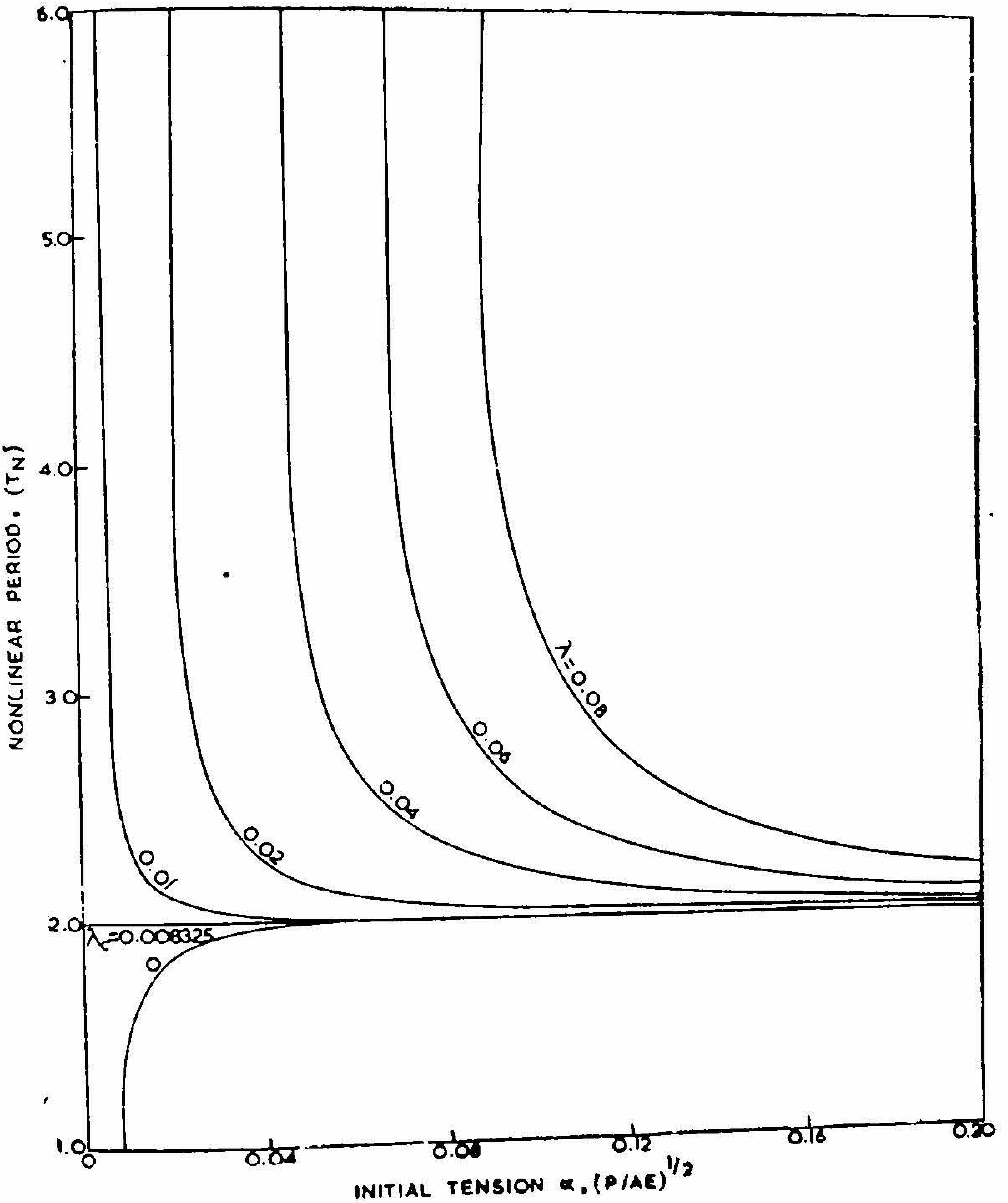

FIG. 2

Nonlinear period versus initial tension for various values of velocity constant λ.

$\mu\pi = 0{\cdot}001$ and $\omega_0 = 0{\cdot}005$

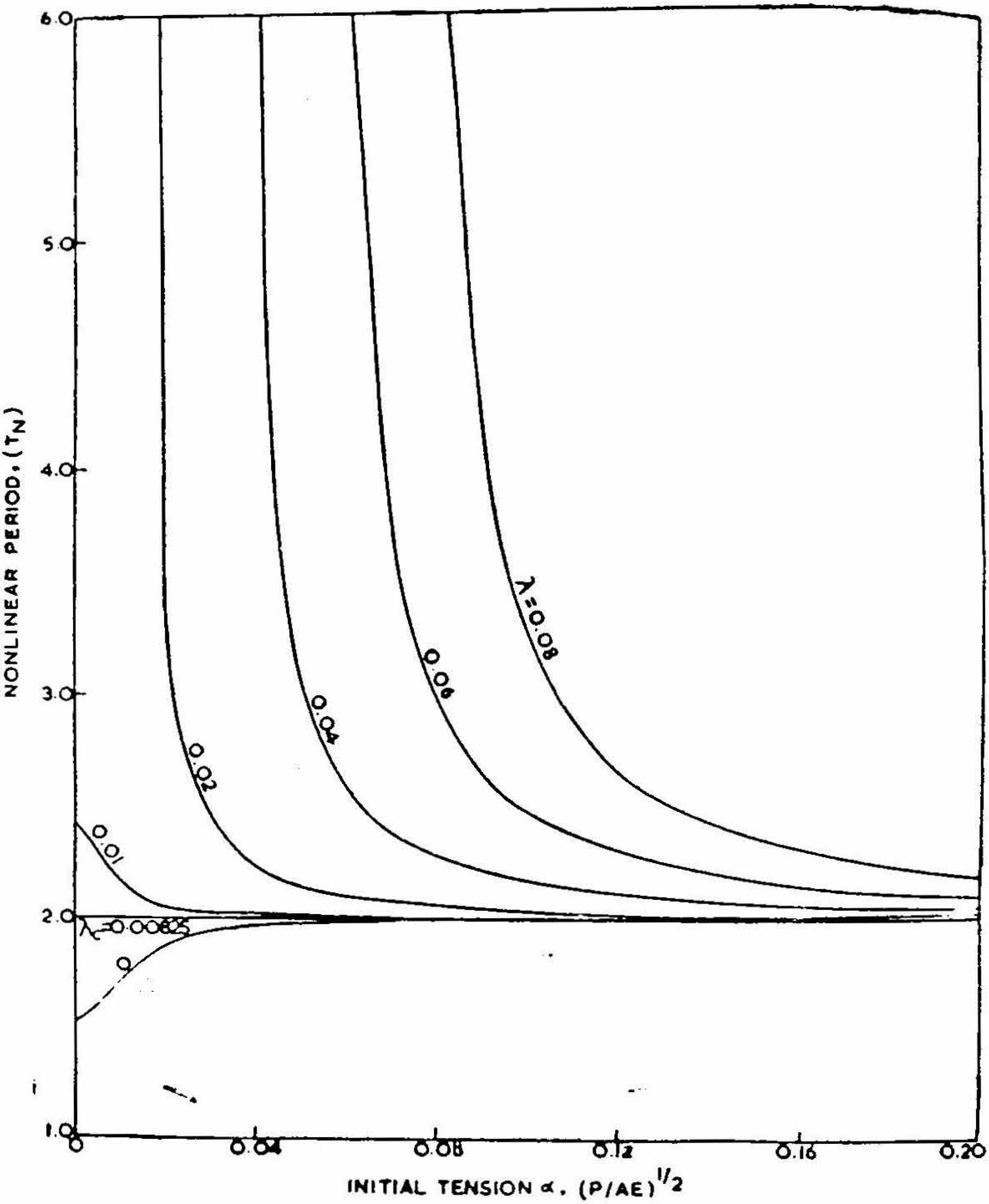

FIG. 3

Nonlinear period versus initial tension for various values of velocity constant λ· $\mu\pi = 0{\cdot}01$ and $\omega_0 = 0{\cdot}005$

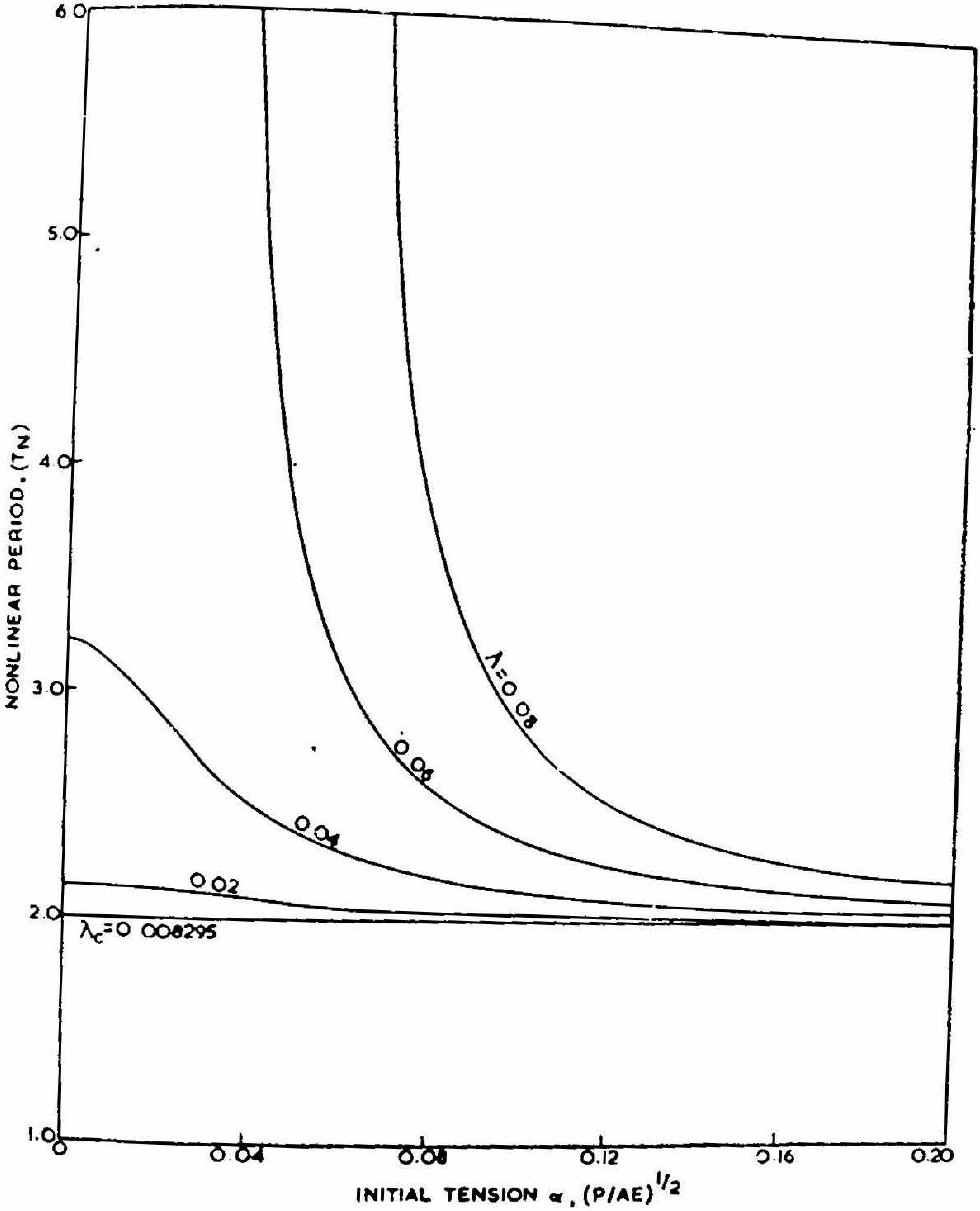

FIG. 4

Nonlinear period versus initial tension for various values of velocity constant λ.

$\mu\pi = 0{\cdot}05$ and $\omega_0 = 0{\cdot}005$

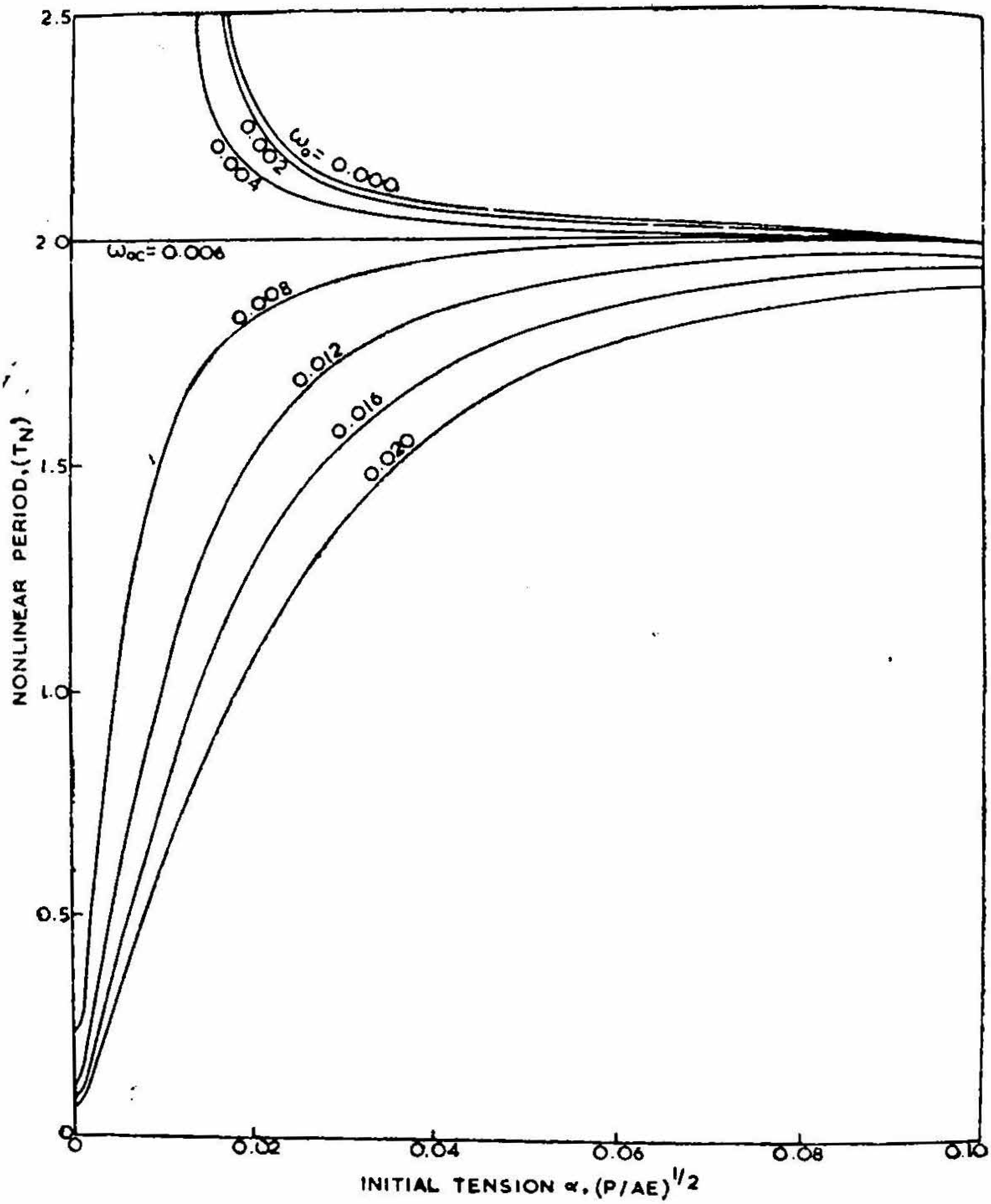

FIG. 5

Nonlinear period versus initial tension for various values of amplitude ω_0. $\mu\pi = 0{\cdot}001$ and $\lambda = 0{\cdot}01$

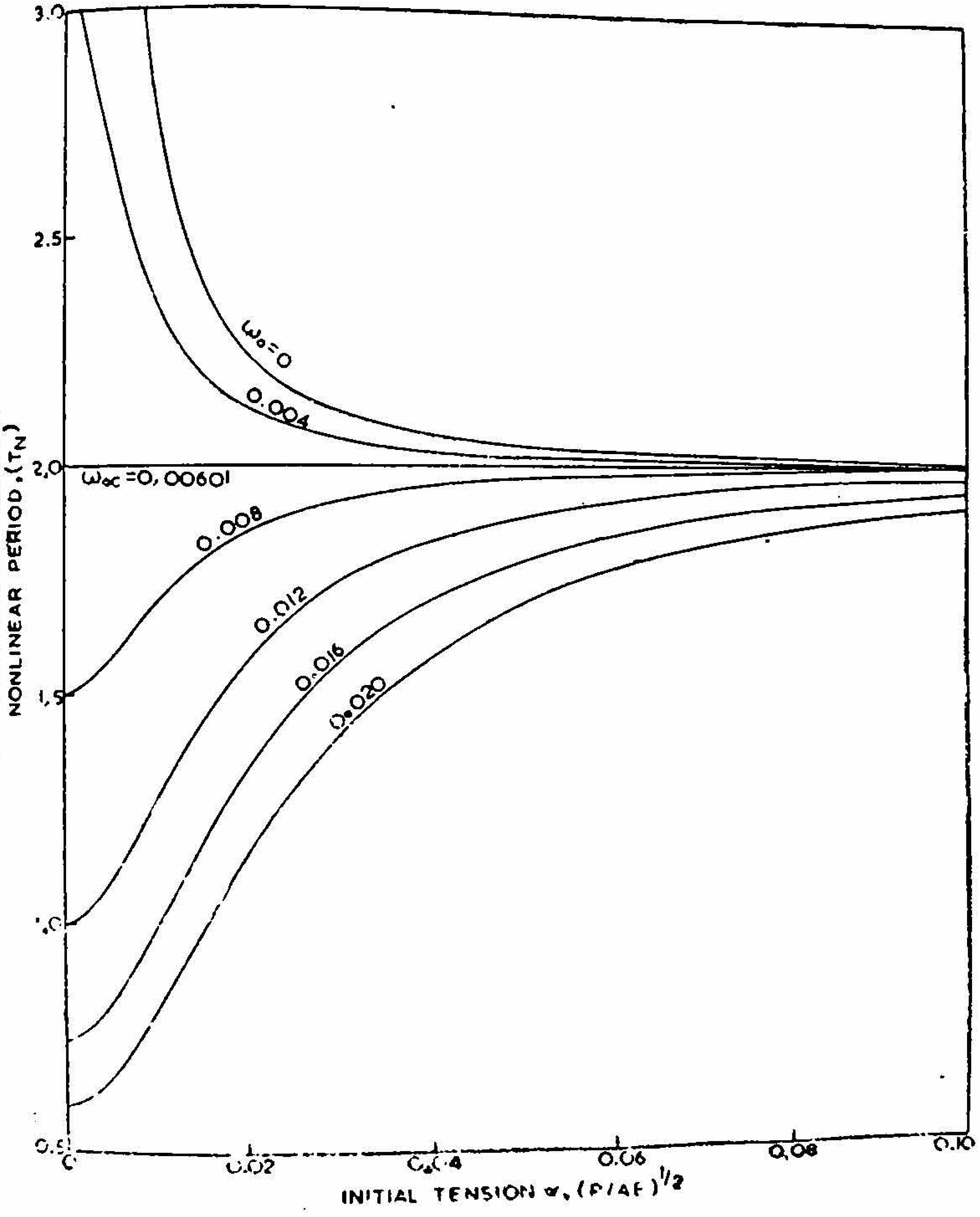

FIG. 6

Nonlinear period versus initial tension for various values of amplitude ω_0

$\mu\pi = 0{\cdot}01$ and $\lambda = 0{\cdot}01$

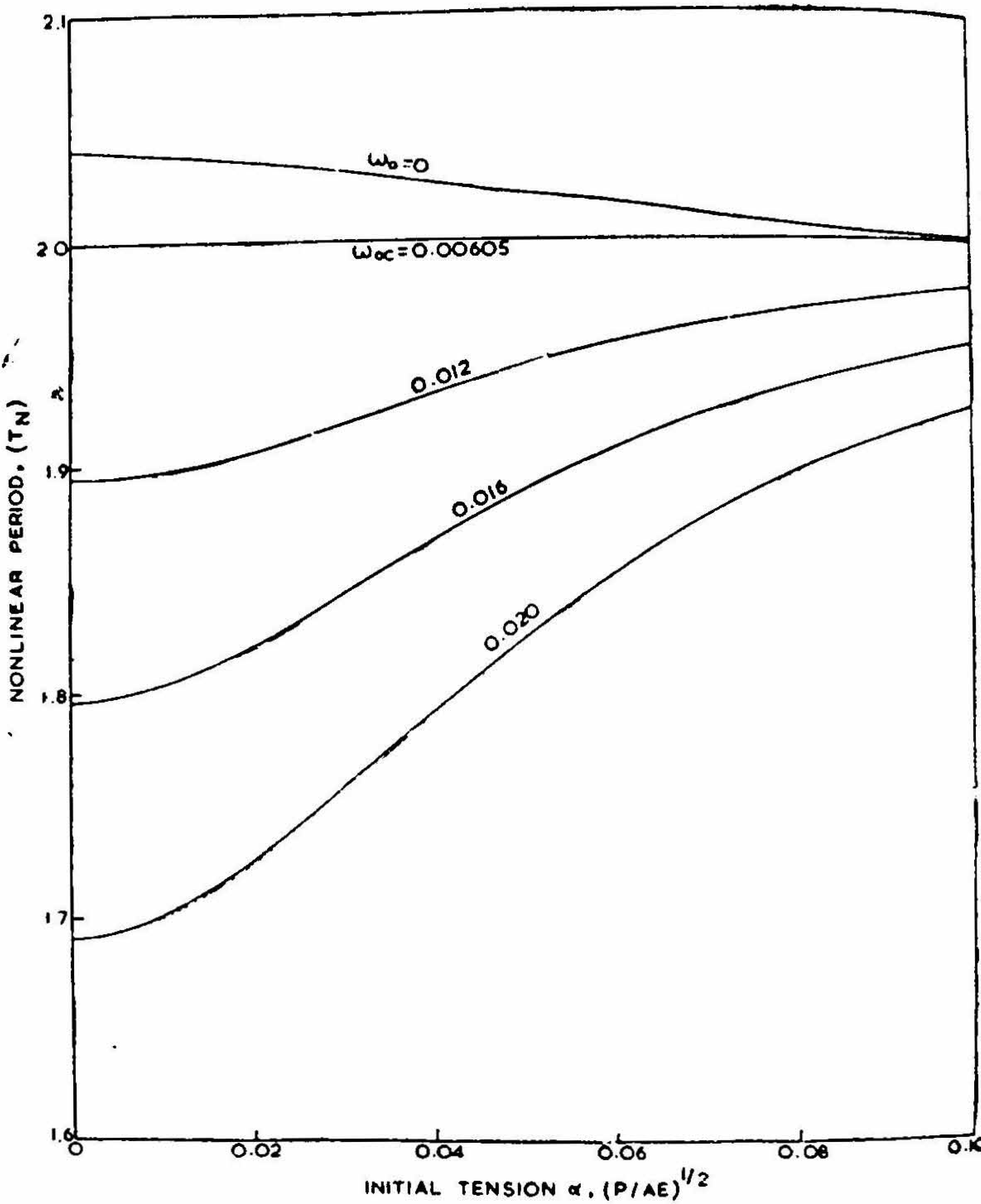

FIG. 7

Nonlinear period versus initial tension for various values of amplitude ω_0. $\mu\pi = 0{\cdot}05$ and $\lambda = 0{\cdot}01$

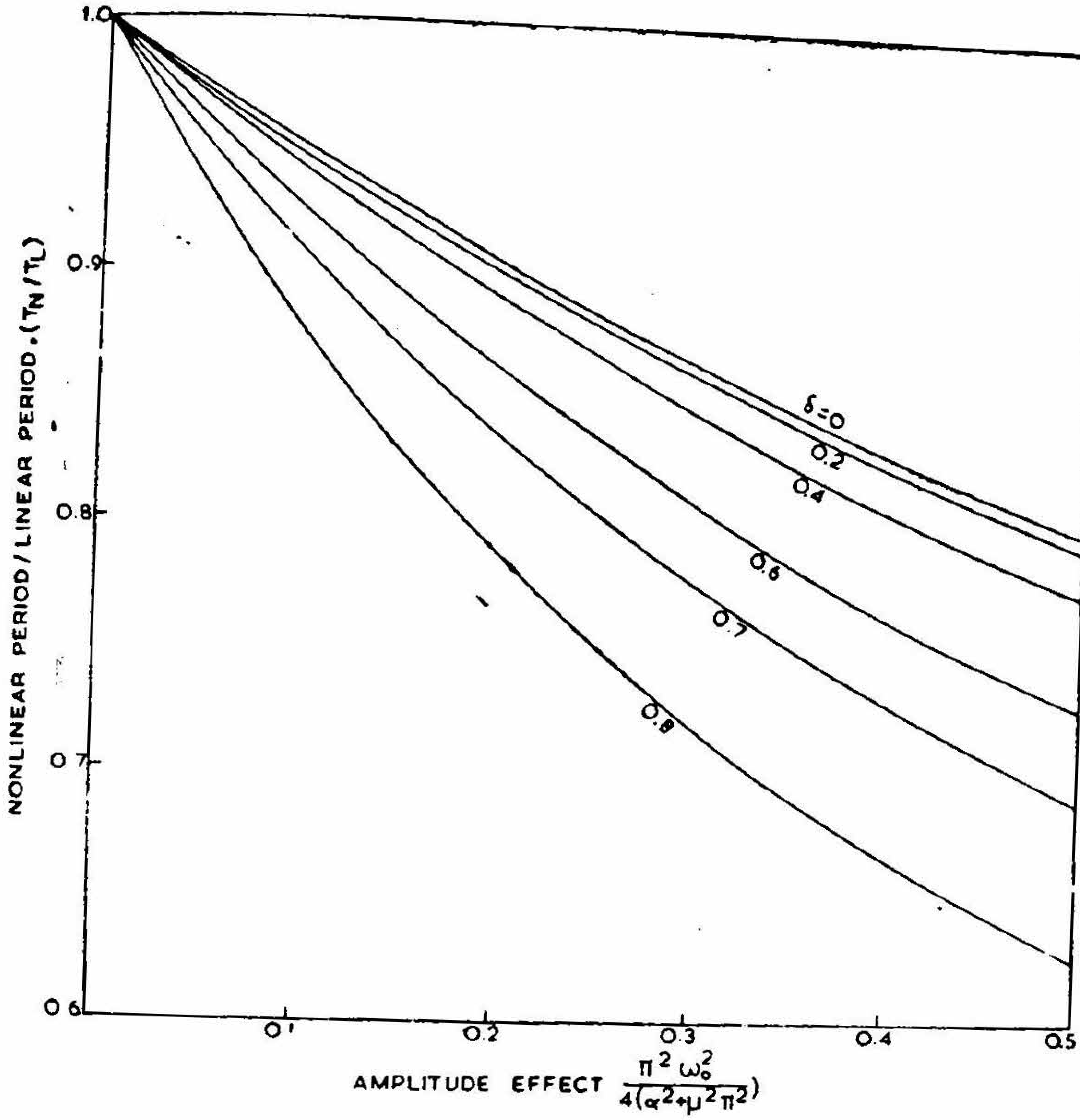

FIG. 8

Period ratio versus amplitude effect for various values of velocity number $\delta \cdot (\mu^2\pi^2 = 0{\cdot}00)$

It can be seen from Figs. 2 to 4 that for a given amplitude w_0 there exists a critical velocity constant λc. For values of $\lambda < \lambda c$ the non-linear period T_N increases as the initial tension α and flexural parameter μ increase. For values of $\lambda > \lambda c$, the non-linear period T_N decreases as α and μ increase. In other words, for transport velocities corresponding to $\lambda < \lambda c$, the characteristics are similar to that of the stationary case. The existance of critical velocity constant λc is clearly a non-linear phenomenon and it depends not only on the amplitude of vibration but also on the flexural parameter μ.

This can be seen from equation [18] and figs. 2 to 4. At the critical velocity the non-linear period T_N reduces to a constant as seen from equation [18].

For constant values of tension α and parameter μ the non-linear period T_N is found to decrease as the transport velocity constant λ increases.

In a similar manner the effect of amplitude w_0 on non-linear period T_N is shown in Figs. 5 to 7 for a constant transport velocity constant λ. For

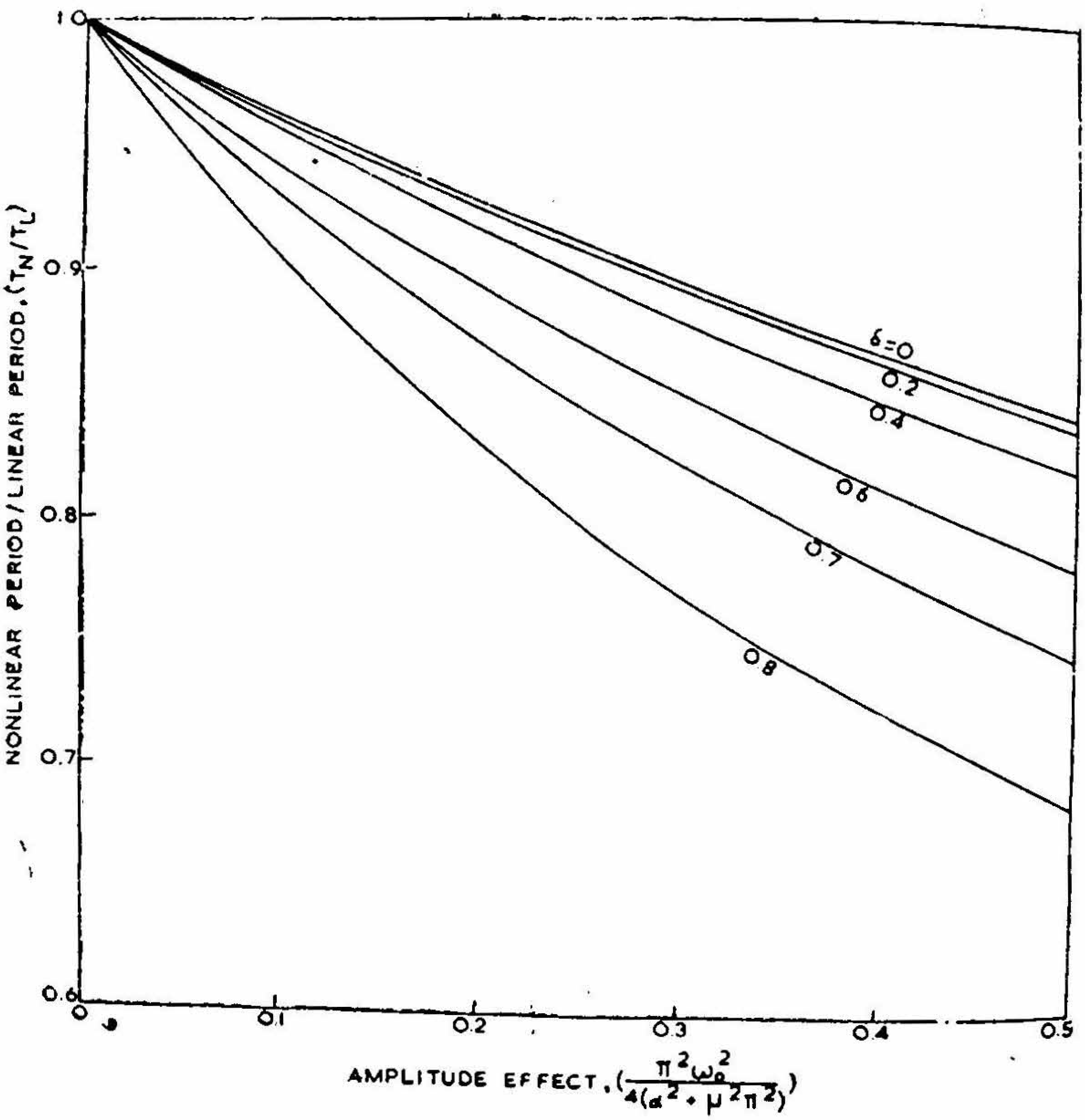

FIG. 9

Period ratio versus amplitude effect for various values of velocity number δ ($\mu^2\pi^2 = 0.08$)

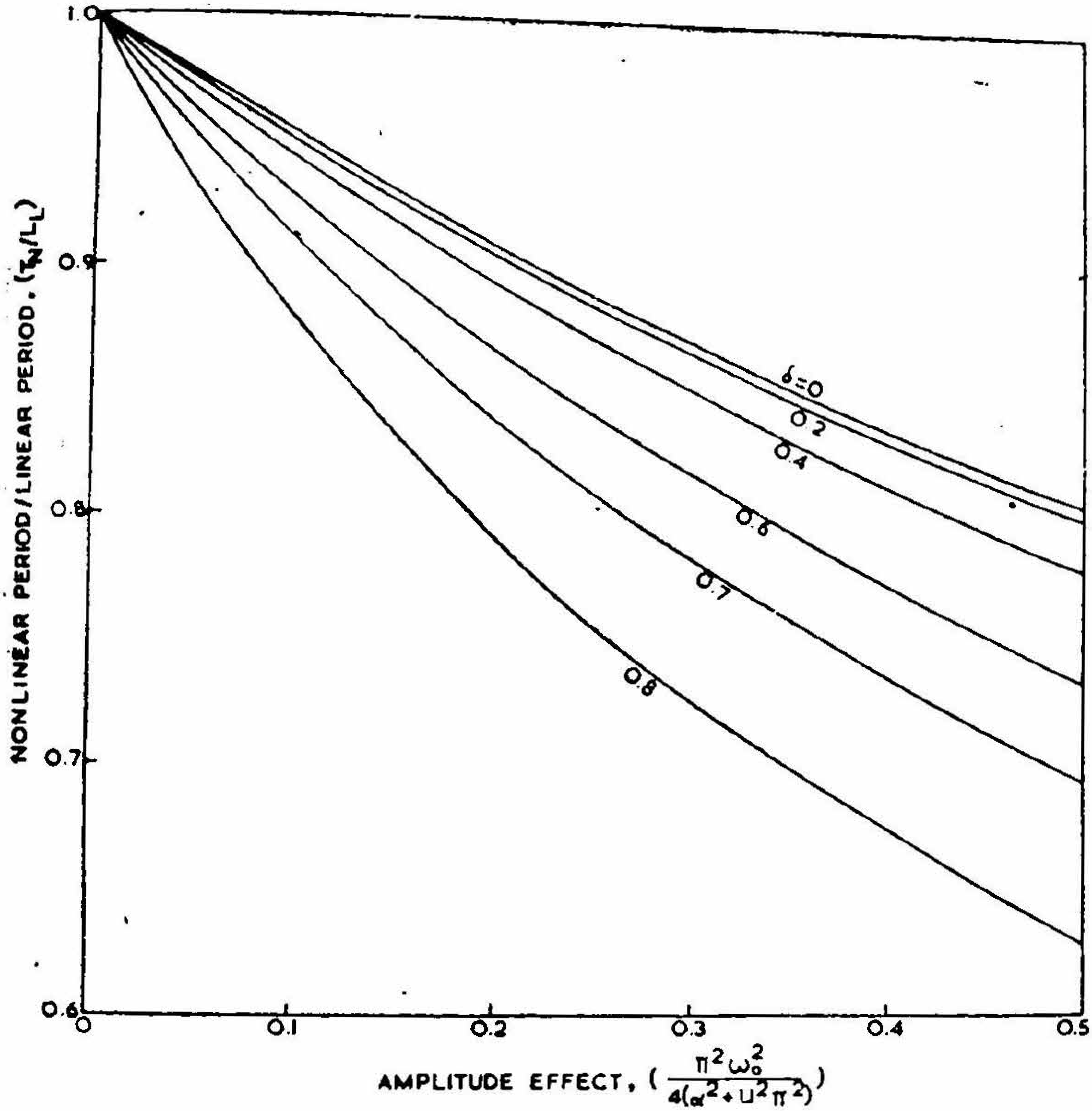

FIG. 10

Period ratio versus amplitude effect for varions values of velocity number δ·($\mu^2\pi^2$ 0·01)

$w_0 > w_{0c}$, the period T_N increases as the intial tension α and flexural parameter μ increase and for $w_0 < w_{0c}$, the period T_N is seen to decrease with an increase in α and μ. This clearly indicates a reverse effect as compared to the effect of velocity constant, λ.

Figs. 8 to 11 show the relationship between the period ratio T_N/T_L and the amplitude effect $\pi^2 w_0^2/4\ (\alpha^2+\mu^2\pi^2)$ for various values of velocity number δ and different values of μ. In all the cases the period T_N/T_L continuously decreases as the velocity number δ increases indicating the limitations of linear analysis at higher transport velocities. It is also to be noted that as the contribution due to flexural parameter μ increases the period ratio T_N/T_L increases for a particular value of amplitude effect as shown in Figs. 8 to 11. This effect is completely due to the non-linear terms associated within the expression for the period T_N in equation [20]. For $\delta = \delta c = 1$, the period ratio is zero, indicating that no vibration in fundamental mode is possible by linear analysis at this value of transport velocity. The value of transport velocity corresponding to δc has been described as critical velocity in the literature on linear analyses.

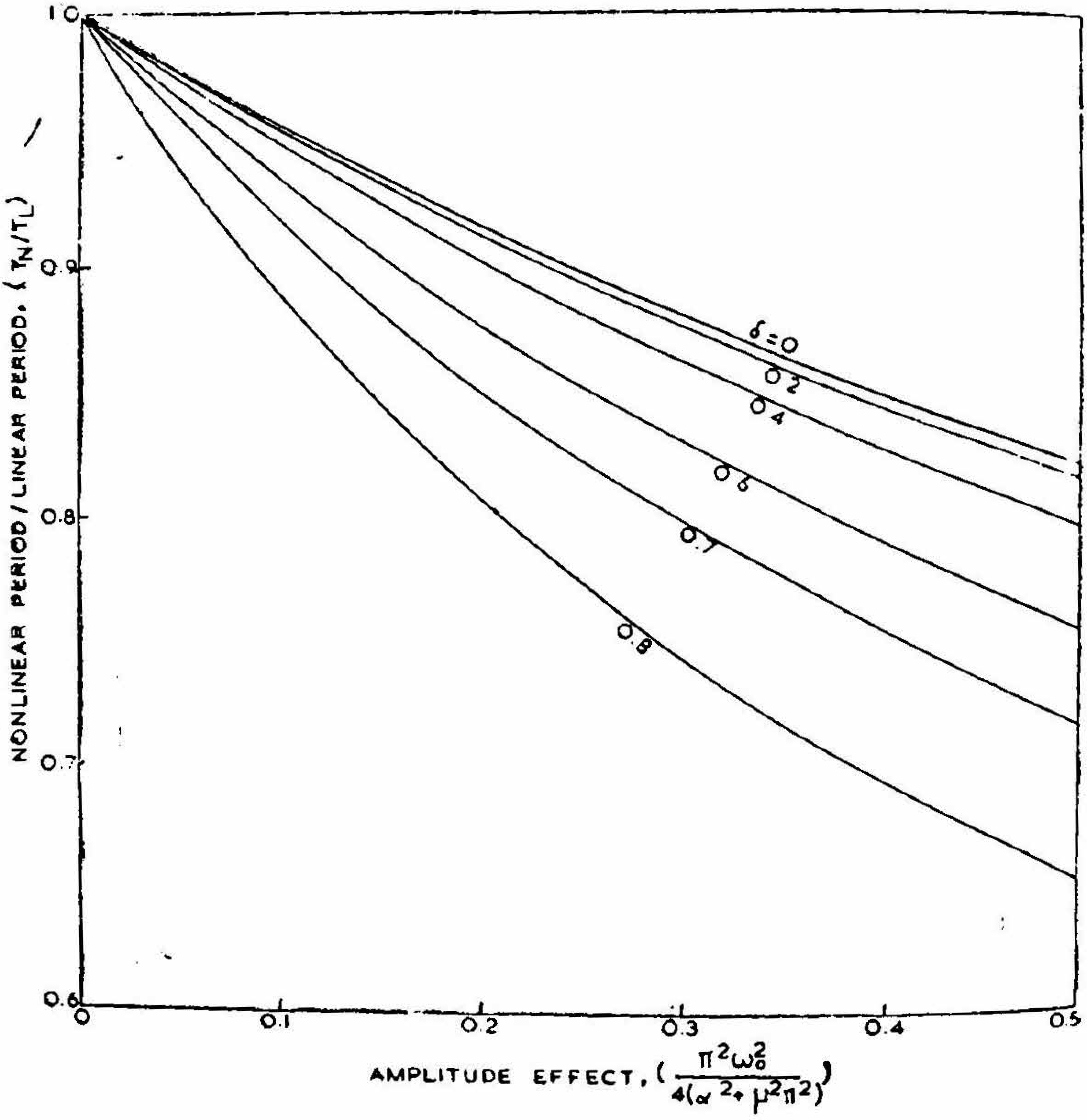

FIG. 11

Period ratio versus amplitude effect for various values of velocity number $\delta \cdot (\mu^2\pi^2 = 0.04)$

CONCLUSION

The non-linear characteristics of an axially moving mass (a string, strip or beam) is similar to that of a stationary state case upto a critical value of transport velocity. The non-linear period increases with an increase in initial tension α and flexural paramcter μ. At the critical velocity it departs from its non-linear behaviour and the vibration mode is the same as given by the linear equation with zero transport velocity. This is because of the fact that the effect of transport velocity on non-linear period is completely neutralised by the non-linear terms and the system reduces to a linear stationary case. For transport velocities greater than the critical value the non-linear period T_N decreases as the intial tension α and flexural parameter μ increase.

The contribution of non-linear terms significantly increases as the transport velocity increases and at higher velocities the linear analysis is meaningless. For the flexural parameters $\mu^2 > 0.001$ or at higher amplitudes the non-linear period is significantly modified and the effect of non-linear terms decreases for a particular transport velocity. The analysis is limited to fundamental frequency only.

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