ELECTRO-MAGNETIC BOUNDARY VALUE PROBLEM OF THE DIELECTRIC SPHERE EXCITED BY DELTA-FUNCTION ELECTRIC AND MAGNETIC SOURCES

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(Received: November 7, 1972)

ABSTRACT

The electromagnetic boundury value problem of the dielectric sphere excited by delta-function electric and magnetic sources applied normally across an arbitrary plane, has been solved. The possibility of symmetric as well as unsymmetric TE, TM and hybrid modes have been investigated.

1. INTRODUCTION

There has been some work done on dielectric resonators of different shapes, including spheres, during previous years[1-6]. These resonators made of materials of high dielectric constant and low loss factor lend themselves to a number of different applications. In 1941, Startton and Chu[7] considered the problem of forced oscillations of a conducting sphere which is excited in an infinite number of *TM* modes by a delta-function electric source field normally across the equatorial plane. But, as far as the author is aware of, there has been very little work done on the forced oscillations of a dielectric sphere.

In this paper, the electromagnetic boundary value problem of the dielectric sphere excited by delta function electric and magnetic sources applied normally across an arbitrary plane has been solved. The possibility of exciting both symmetric and unsymmetric *TE*, *TM* and hybrid modes have been investigated.

2. STATEMENT OF THE PROBLEM

The geometry of the structure is given in Fig. 1. Spherical eoordinates (r, θ, ϕ) are used, A dielectric sphere of radius 'a' and constants ϵ_1, μ_1, σ_1, is embedded in another dielectric medium of contants ϵ_0, μ_0, σ_0. The dielectric sphere is excited by delta-function electric and magnetic field sources in a direction normal to the plane $z=z_1=a \cos \theta_1$.

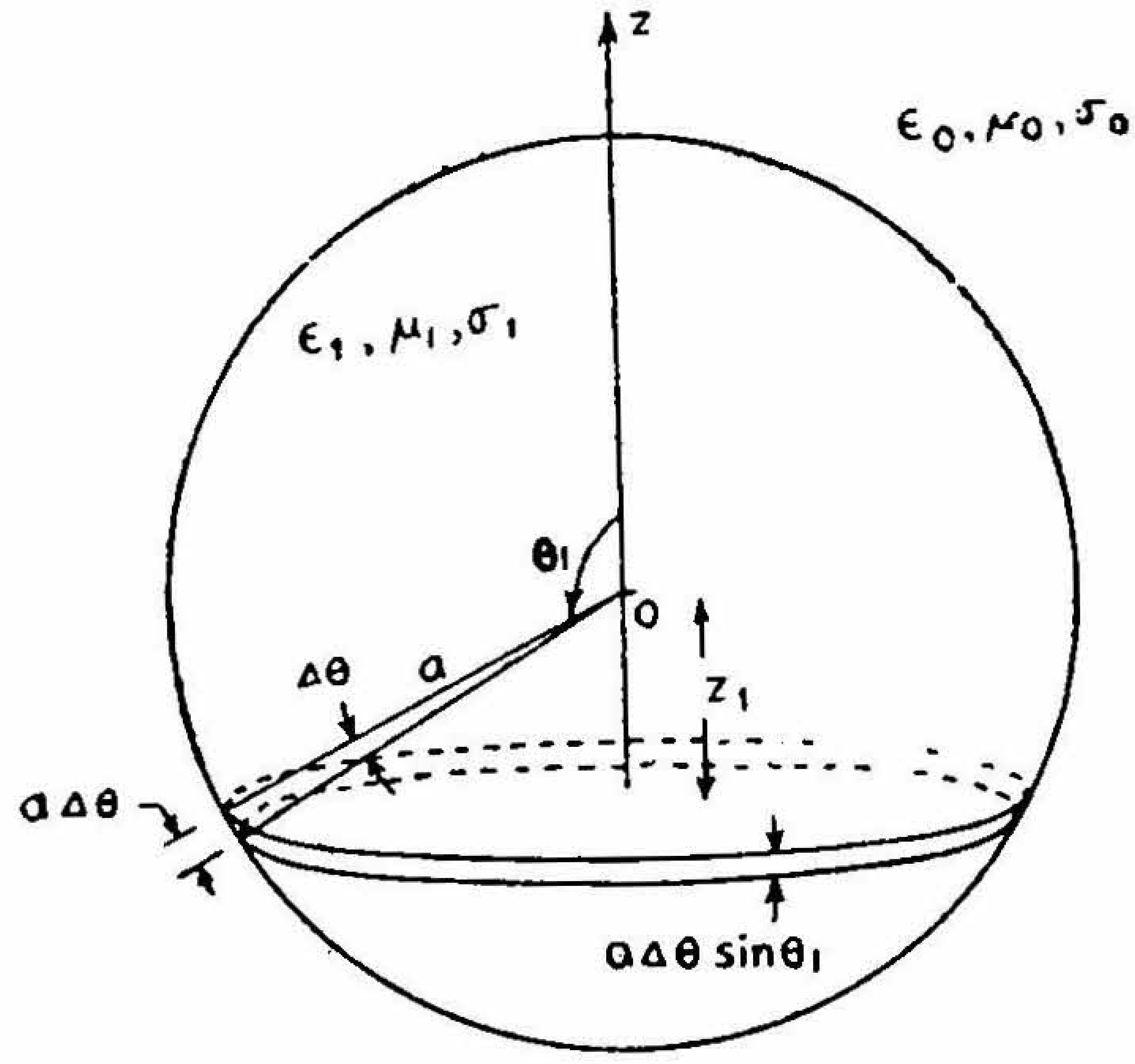

FIG. 1
Geometry of the structure

The object of this paper is to solve the electromagnetic boundary-value problem and discuss the possibility of the existence of hybrid, *TE* and *TM* modes.

3. HYBRID MODES

Let the excitation of the dielectric sphere be a combination of an electric field $E'e^{-jwt}$ and a magnetic field $H' e^{-jwt}$ applied uniformly over the plane $z = z_1 = a \cos \theta_1$, and in a direction normal to this plane.

Let

$$E' = E_0 \cos m\phi \qquad [1]$$

$$H' = H_0 \sin m\phi \qquad [2]$$

Both E' and H' have components E_r' E_θ' and H_r' H_θ', in the r and θ directions respectively. These components of the applied electric and magnetic fields can be expanded in series of spherical harmonics as follows :

$$E_r^1(r, \theta, \phi) = \frac{-n(n+1)}{k_1} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} D_{mn}(r) \cos(m\phi)\, P_n^m(\cos\theta)_e{}^{-jwt} \qquad [3]$$

$$\sin \theta E_\theta', k_1(r, \theta, \phi) = -\frac{1}{k_1} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_{mn}(r) \cos(m\phi)\, P_n^m(\cos\theta)\, e^{-jws} \qquad [4]$$

$$H'_\gamma(r,\theta,\phi) = \frac{-n(n+1)}{j\omega\mu_1} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} D'_{mn}(r) \sin(m\phi) P_n^m(\cos\theta) e^{-j\omega t} \quad [5]$$

$$\sin\theta\, H'_\theta(r,\theta,\phi) = -\frac{1}{j\omega\mu_1} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C'_{mn}(r) \sin(m\phi) P_n^m(\cos\theta) e^{-j\omega t} \quad [6]$$

where

$$D_{mn}(r) = \frac{-k_1(2n+1)(n-m)!}{n(n+1)\,2\pi\,(n+m)!} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} E'_r P_n^m(\cos\theta) \cos(m\phi) \sin\theta\, d\theta d\phi \quad [7]$$

$$D'_{mn}(r) = \frac{-j\omega\mu_1(2n+1)(n-m)!}{n(n+1)\,2\pi\,(n+m)!} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} H'_r P_n^m(\cos\theta) \sin(m\phi) \times \sin\theta\, d\theta\, d\phi \quad [8]$$

$$C_{mn}(r) = \frac{-k_1(2n+1)(n-m)!}{2\pi\,(n+m)!} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} E'_\theta P_n^m(\cos\theta) \sin\theta \cos(m\phi) \times \sin\theta\, d\theta\, d\phi \quad [9]$$

$$C'_{mn}(r) = \frac{-j\omega\mu_1(2n+1)(n-m)!}{2\pi\,(n+m!} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \sin\theta\, H'_\theta P_n^m(\cos\theta) \times \sin(m\phi)\, d\theta\, d\phi \quad [10]$$

Let E_0 and H_0 be δ-function sources at the plane $z = a_1 = a\cos\theta_1$ for all r and for all θ, *i.e.*,

$$E_0 = \frac{-V}{a\Delta\theta \sin\theta'} \text{ for } \theta_1 - \frac{\Delta\theta}{2} < \theta < \theta_1 + \frac{\Delta\theta}{2} \quad [11]$$

$$H_0 = \frac{-U}{a\Delta\theta\sin\theta_1} \qquad \text{,,} \qquad \text{,,}$$

$$\left.\begin{matrix} E_0 = 0 \\ H_0 = 0 \end{matrix}\right\} \begin{matrix} \text{for } \theta < \theta_1 - (\Delta\theta/2) \text{ and for} \\ \theta > \theta_1 + (\Delta\theta/2) \end{matrix} \quad [12]$$

and

$$E'_r = E' \cos\theta_1, \quad E'_\theta = -E' \sin\theta_1$$

$$H'_r = H' \cos\theta_1, \quad H'_\theta = -H' \sin\theta_1$$

$$D_{mn}(r) = D_{mn}(a) = \frac{-k_1(2n+1)(n-m)!}{n(n+1)\,2\pi\,(n+m)!}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} E_r' P_n^m(\cos\theta)\cos(m\phi)\sin\theta\,d\theta\,d\phi$$

$$= \frac{k_1(2n+1)(n-m)!}{n(n+1)\,2\pi\,(m+n)!}\,\frac{2\pi V}{a\sin\theta_1}\cos\theta_1\,P_n^m(\cos\theta_1)\sin\theta_1$$

$$= \frac{V}{a}\,\frac{k_1(2n+1)(n-m)!}{n(n+1)(n+m)!}\cos\theta_1\,P_n^m(\cos\theta_1) \qquad [13]$$

$$D'_{mn} = D'_{mn}(a)$$

$$= \frac{-j\omega\mu_1(2n+1)(n-m)!}{n(n+1)\,2\pi\,(n+m)!}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} H_r' P_n^m(\cos\theta)\cos(m\phi)\sin\theta d\,d\phi$$

$$= \frac{j\omega\mu_1(2n+1)(n-m)!}{n(n+1)\,2\pi\,(n+m)!}\,\frac{2\pi U\cos\theta}{a\sin\theta_1}\,P_n^m(\cos\theta_1)\sin\theta_1$$

$$= \frac{U}{a}\,\frac{j\omega\mu_1(2n+1)(n-m)!}{n(n+1)(n+m)!}\cos\theta_1\,P_n^m(\cos\theta_1) \qquad [14]$$

$$C_{mn}(r) = C_{mn}(a)$$

$$= \frac{-k_1(2n+1)(n-m)!}{2\pi\,(n+m)!}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\sin\theta\,E_\theta' P_n^m(\cos\theta)\cos(m\phi)\sin\theta d\,\theta d\,d\phi$$

$$= \frac{2\pi V}{a}\,\frac{\sin^3\theta_1}{\sin\theta_1}\,\frac{P_n^m(\cos\theta_1)\,k_1(2n+1)(n-m)!}{2\pi\,(n+m)!}$$

$$= \frac{V}{a}\sin^2\theta_1\,P_n^m(\cos\theta_1)\,k_1\frac{(2n+1)(n-m)!}{(n+m)!} \qquad [15]$$

$$C'_{mn}(r) = C_{mn}(a)$$

$$= \frac{-j\omega\mu_1(2n+1)(n-m)!}{2\pi\,(n+m)!}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\sin\theta\,H_\theta' P_n^m(\cos\theta)\cos(m\phi)\sin\theta\,d\theta\,d\phi$$

$$= \frac{2\pi U\sin^3\theta_1}{a\sin\theta_1}\,P_n^m(\cos\theta_1)\,\frac{j\omega\mu_1(2n+1)(n-m)!}{2\pi\,(n+m)!}$$

$$= \frac{U}{a}\sin^2\theta_1\,P_n^m(\cos\theta_1)\,\frac{j\omega\mu_1(2n+1)(n-m)!}{(n+m)!} \qquad [16]$$

The field components inside the dielectric sphere are [8]

$$E_r^i = -\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} n(n+1)\, A_{mn} \cos(m\phi)\, P_n^m(\cos\theta) \frac{j_n(k_1 r)}{k_1 r} e^{-j\omega t} + E_r' \quad [17]$$

$$E_\theta^i = -\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{d}{dt}\left\{P_n^m(\cos\theta)\right\} A_{mn} \cos(m\phi) \frac{1}{k_1 r}\left[k_1 r\, j_n(k_1 r)\right]' e^{-j\omega t} + E_\theta'$$

$$-\frac{1}{\sin\theta}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A'_{mn}\, m \cos(m\phi)\, P_{mn}(\cos\theta)\, j_n(k_1 r)\, e^{-\omega t} \quad [18]$$

$$E_\phi^i = \frac{1}{\sin\theta}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{mn} P_n^m(\cos\theta)\, m \sin(m\phi) \frac{1}{k_1 r}[k_1 r\, j_n(k_1 r)]' e^{j\omega t}$$

$$+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A'_{mn} \frac{d}{d\theta}\left\{P_n^m(\cos\theta)\right\} \sin(m\phi)\, j_n(k_1 r) e^{-j\omega t} \quad [19]$$

$$H_r^i = -\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A'_{mn} \sin(m\phi) \frac{n(n+1)}{j\omega\mu_1} P_n^m(\cos\theta) \frac{1}{r} j_n(k_1 r)\, e^{-j\omega t} + Hr' \quad [20]$$

$$H_\theta^i = \frac{k_1}{j\omega\mu_1} \frac{1}{\sin\theta} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{mn} P_n^m(\cos\theta)\, m \sin(m\phi)\, j_n(k_1 r) e^{-j\omega t}$$

$$-\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A'_{nm} \sin(m\phi) \frac{1}{j\omega\mu_1} \frac{d}{d\theta}\{P_n^m(\cos\theta)\} \frac{1}{r}[k_1 r\, j_n(k_1 r)]' e^{-j\omega t} + H_\theta'$$

[21]

$$H\phi^i = \frac{k_1}{j\omega\mu_1} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{mn} \frac{d}{d\theta}\left\{P_n^m(\cos\theta)\right\} \cos(m\phi)\, j_n(k_1 r) e^{-j\omega t}$$

$$-\frac{1}{\sin\theta}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A'_{mn}\, m \cos(m\phi)\, P_n^m(\cos\theta) \frac{1}{j\omega\mu_1} \frac{1}{r}\left[k_1 r\, j_n(k_1 r)\right]' e^{-j\omega t}$$

[22]

The field components outside the dielectric sphere are :

$$E_r^e = -\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} n(n+1)\, B_{mn} \cos(m\phi)\, P_n^m(\cos\theta) \frac{h_n^{(1)}(k_0 r)}{k_0 r} e^{-j\omega t} \quad [23]$$

$$E_\theta^e = -\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{d}{d\theta}\{P_n^m(\cos\theta)\}\, B_{mn} \cos(m\phi) \frac{1}{k_0 r}[k_0 r\, h_n^{(1)}(k_0 r)]' e^{-j\omega t}$$

$$-\frac{1}{\sin\theta}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} B'_{mn} \cos(m\phi)\, m\, P_n^m(\cos\theta)\, h_n^{(1)}(k_0 r)\, e^{-j\omega t} \quad [24]$$

$$E_\phi^e = \frac{1}{\sin\theta} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} B_{mn} P_n^m (\cos\theta)\, m \sin(m\phi) \frac{1}{k_0 r} [k_0 r\, h_r^{(1)} (k_0 r)]' e^{-j\omega t}$$

$$+ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} B'_{mn} \sin(m\phi) \frac{d}{d\theta} \left\{ P_n^m (\cos\theta) \right\} h_n^{(1)}(k_0 r) e^{-j\omega t} \qquad [25]$$

$$H_r^e = -\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} B'_{mn} \sin(m\phi) \frac{n(n+1)}{j\omega\mu_0} P_n^m (\cos\theta) \frac{1}{r} h_n^{(1)}(k_0 r) e^{-j\omega t} \qquad [26]$$

$$H_\theta^e = \frac{k_0}{j\omega\mu_0 \sin\theta} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} B_{mn} P_n^m (\cos\theta)\, m \sin(m\phi)\, h_n^{(1)}(k_0 r) e^{-j\omega t}$$

$$- \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} B'_{mn} \sin(m\phi) \frac{1}{j\omega\mu_0} \frac{d}{d\theta} \left\{ P_n^m (\cos\theta) \right\} \frac{1}{r} \left[k_0 r\, h_n^{(1)} (k_0 r) \right]' e^{-j\omega t} \qquad [27]$$

$$H_\phi^e = \frac{k_0}{j\omega\mu_0} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} B_{mn} \frac{d}{d\theta} \{ P_n^m (\cos\theta) \} \cos(m\phi)\, h_r^{(1)} (k_0 r)\, e^{-j\omega t}$$

$$- \frac{1}{\sin\theta} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} B'_{mn}\, m \cos(m\phi)\, P_{mn} (\cos\theta) \frac{1}{j\omega\mu_0} \frac{1}{r} [k_0 r\, h_n^{(1)} (k_0 r)]'\, e^{-j\omega t} \qquad [28]$$

where

$k_1 = \omega\sqrt{[\mu_1 (\epsilon_1 + j\sigma_1/\omega)]}$

$k_0 = \omega\sqrt{[\mu_0 (\epsilon_0 + j\sigma_0/\omega)]}$

ω = angular frequency

A_{mn}, A'_{mn}, B_{mn}, B'_{mn} are amplitude constants.

Applying the boundary conditions that

$E_\theta^i = E_\theta^e$, $E_\phi^i = E_\phi^e$, $H_\theta^i = H_\theta^e$, $H_\phi^i = H_\phi^e$

$\epsilon_1 E_r^i = \epsilon_0 E_r^e$, $\mu_1 H_r^i = \mu_0 H_r^e$

at $r = a$, and making use of equations [3], [4], [5], [6] and the following relation [29]:

$$\frac{d P_n^m (\cos\theta)}{d\theta} = \frac{1}{\sin\theta} \{ (n-m+1)\, n\, P_{n+1}^m (\cos\theta) - (n+m)(n+1)\, P_{n-1}^m (\cos\theta \}$$

$$\frac{1}{(2n-1)} \qquad [29]$$

and remembering that $P_n^m(\cos\theta)=0$ for $n=m$, the following six equations [30] to [35] are obtained for each value of m.

$$-\sum_{n=m}^{\infty}\frac{1}{\sin\theta}\left\{(n-m+1)\,n\,P_{n+1}^m-(n+m)\,(n+1)\,P_{n-1}^m\right\}\frac{A_{mn}}{(2n+1)}\,\frac{1}{k_1 a}\,[k_1 a j_n(k_1 a)]'$$

$$-\frac{1}{k_1}\sum_{n=m}^{\infty}C_{mn}(a)\frac{P_n^m}{\sin\theta}-\frac{1}{\sin\theta}\sum_{n=m}^{\infty}A'_{mn}P_n^m\,m\,j_n(k_1 a)$$

$$=-\sum_{n=m}^{\infty}\frac{1}{\sin\theta}\left\{(n-m+1)\,n\,P_{n+1}^m-(n+m)\,(n+1)\,P_{n-1}^m\right\}\frac{1}{(2n+1)}\times$$

$$\times B_{mn}\frac{1}{k_0 a}\left[k_0 a\,h_n^{(1)}(k_{0a})\right]'-\frac{1}{\sin\theta}\sum_{n=m}^{\infty}B_{mn}P_n^m h_n^{(1)}(k_0 a) \qquad [30]$$

$$\sum_{u=m}^{\infty}\frac{m}{\mathrm{sim}\theta}A_{mn}P_n^m\frac{1}{k_1 a}\,[k_1 a\,j_n(k_1 a)]'$$

$$+\sum_{n=m}^{\infty}A'_{mn}\frac{1}{\sin\theta}\left\{(n-m+1)\,n\,P_{n+1}^m-(n+m)\,(n+1)\,P_{n-1}^m\right\}\frac{1}{2n+1}j_n(k_1 a)\times$$

$$=\sum_{n=m}^{\infty}B_{mn}\frac{mP_n^m}{\sin\theta}\frac{1}{k_0 a}\left[k_0 a\,h_n^{(1)}(k_0 a)\right]'$$

$$+\sum_{n=m}^{\infty}B'_{mn}\frac{1}{\sin\theta}\left\{(n-m+1)n\,P_{n+1}^m-(n+m)\,(n+1)\,P_{n-1}^m\right\}\frac{1}{2n+1}h_n^{(1)}(k_0 a) \qquad [31]$$

$$\frac{k_1}{\mu_1}\sum_{n=m}^{\infty}mA_{mn}\frac{P_n^m}{\sin\theta}j_n(k_1 a)$$

$$-\frac{1}{\mu_1}\sum_{n=m}^{\infty}A'_{mn}\frac{1}{\sin\theta}\left\{(n-m+1)n\,P_{n+1}^m-(n+m)\,(n+1)\,P_{n-1}^m\right\}\frac{1}{(2n+1)}\times$$

$$\times\frac{1}{a}[k_1 a\,j_n(k_1 a)]'-\frac{1}{\mu_1}\sum_{n=0}^{\infty}C'_{mn}(a)\frac{P_n^m}{\sin\theta}$$

$$=\frac{k_0}{\mu_0}\sum_{n=m}^{\infty}B_{mn}\frac{P_n^m}{\sin\theta}h_n^{(1)}(k_0 a)$$

$$-\frac{1}{\mu_0}\sum_{n=m}^{\infty}B'_{mn}\frac{1}{\sin\theta}\left\{(n-m+1)n\,P_{n+1}^m-(n+m)(n+1)\,P_{n-1}^m\right\}\frac{1}{(2n+1)}\,\frac{1}{a}\times$$

$$\times[k_0 a\,h_n^{(1)}(k_0 a)]' \qquad [32]$$

$$\frac{k_1}{\mu_1}\sum_{n=m}^{\infty} A_{mn} \frac{1}{\sin\theta}\{(n-m+1)nP_{n+1}^m - (n+m)(n+1)P_{n-1}^m\} \frac{1}{2n+1} j_n(k_1 a)$$

$$-\frac{1}{\mu_1}\sum_{n=m}^{\infty} A'_{mn}\, m \frac{1}{a}[k_1 a\, j_n(k_1 a)]' \frac{P_n^m}{\sin\theta}$$

$$= \frac{k_0}{\mu_0}\sum_{n=m}^{\infty} B_{mn} \frac{1}{\sin\theta}\{(n-m+1)n\, P_{n+1}^m - (n+m)(n+1)P_{n-1}^m\} \times$$

$$\times \frac{1}{(2n+1)} h_n^{(1)}(k_0 a) - \frac{1}{\mu_0}\sum_{n=m}^{\infty} B'_{mn} \frac{m}{a}[k_0 a\, h_n^{(1)}(k_0 a)]' \frac{P_n^m}{\sin\theta} \qquad [33]$$

$$\sum_{n=m}^{\infty} \frac{A_{mn} P_n^m(\cos\theta)\, j_n(k_1 a)}{k_1 a} + \sum_{n=m}^{\infty} \frac{1}{k_1} D_{mn}(a)\, P_n^m(\cos\theta)$$

$$= \sum_{n=m}^{\infty} B_{mn} \frac{P_n^m(\cos\theta)\, h_n^{(1)}(k_0 a)}{k_0 a} \frac{\epsilon_0}{\epsilon_1} \qquad [34]$$

$$\sum_{n=m}^{\infty} A'_{mn} \frac{1}{\mu_1} P_n^m(\cos\theta)\, j_n(k_1 a) + \sum_{n=m}^{\infty} \frac{1}{\mu_1} D'_{mn}(a)\, P_n^m(\cos\theta)$$

$$= \sum_{m=n}^{\infty} B'_{mn} \frac{1}{\mu_0} P_n^m(\cos\theta)\, h_n^{(1)}(k_0 a) \frac{\mu_0}{\mu_1} \qquad [35]$$

[For brevity, $P_n^m(\cos\theta)$ is put $= P_n^m$]

Equating the coefficients of $[P_n^m(\cos\theta)/\sin\theta]$ on either side of equations [30] to [33] and those of $P_n^m(\cos\theta)$ on either side of cquations [34] and [35] the following six equations [36] to [41] are obtained

$$\frac{(n-m)(n-1)}{(2n-1)} A_{m,n-1} \frac{1}{k_1 a}[k_1 a\, j_{n-1}(k_1 a)]'$$

$$-\frac{(n+2)(n+m+1)}{(2n+3)} A_{m,n+1} \frac{1}{k_1 a}[k_1 a\, j_{n+1}(k_1 a)]'$$

$$+\frac{1}{k_1} C_{mn}(a) + A'_{mn}\, m\, j_n(k_1 a)$$

$$= \frac{(n-m)(n-1)}{(2n-1)} B_{m,n-1} \frac{1}{k_0 a}[k_0 a h_{n-1}^{(1)}(k_0 a)]'$$

$$-\frac{(n+2)(n+m+1)}{(2n+3)} B_{m,n+1} \frac{1}{k_0 a}[k_0 a\, h_{n+1}^{(1)}(k_0 a)]'$$

$$+ B'_{mn}\, m\, h_n^{(1)}(k_0 a) \qquad [36]$$

$$m A_{mn} \frac{1}{k_1 a} \left[k_1 a j_n (k_1 a) \right]' + A'_{m,n-1} \frac{(n-1)(n-m)}{(2n-1)} j_{n-1}(k_1 a)$$

$$- A'_{m,n+1} \frac{(n+2)(n+m+1)}{(2n-3)} j_{n+1}(k_1 a)$$

$$= B_{mn} m \frac{1}{k_0 a} \left[k_0 a h_n^{(1)}(k_0 a) \right]' + B'_{m,n-1} \frac{(n-1)(n-m)}{(2n-1)} h_{n-1}^{(1)}(k_0 a)$$

$$- B'_{m,n+1} \frac{(n+2)(n+m+1)}{(2n+3)} h_{n+1}^{(1)}(k_0 a) \qquad (37)$$

$$\frac{k_1}{\mu_1} m A_{mn} j_n (ka) - \frac{1}{\mu_1} \frac{A'_{m,n-1}}{(2n-1} (n-m)(n-1) \left[k_1 a j_{n-1}(k_1 a) \right]'$$

$$- \frac{1}{\mu_1} \frac{A'_{m1} n+1}{(2n+3)} (n+m+1)(n+2) \frac{1}{a} \left[k_1 a j_{n-1}(k_1 a) \right]' - \frac{1}{\mu_1} C'_{mn}(a)$$

$$= \frac{k_0}{\mu_0} m B_{mn} h_n^{(1)}(k_0 a) - \frac{1}{\mu_0} \frac{B'_{mn}}{(2n-1)} (n-m)(n-1) \frac{1}{a} \left[k_0 a h_{n-1}^{(1)}(k_0 a) \right]'$$

$$+ \frac{1}{\mu_0} \frac{B'_{m,n+1}}{(2n+3)} (n+m+1)(n+2) \frac{1}{a} \left[k_0 a h_{n+1}^{(1)}(k_0 a) \right]' \qquad (38)$$

$$\frac{k_1}{\mu_1} A_{m,n-1} \frac{(n-m)(n-1)}{(2n-1)} j_{n-1}(k_1 a) - \frac{k_1}{\mu_1} A_{m,n+1} \frac{(n+m+1)(n+2)}{(2n+3)} j_{n+1}(k_1 a)$$

$$+ \frac{1}{\mu_1} A'_{mn} m \frac{1}{a} [k_1 a j_n (k_1 a)]'$$

$$= \frac{k_0}{\mu_0} B_{m,n-1} \frac{(n-m)(n-1)}{(2n-1)} h_{n-1}^{(1)}(k_0 a)$$

$$- \frac{k_0}{\mu_0} B_{m,n+1} \frac{(n+m+1)(n+2)}{(2n+3)} h_{n+1}^{(1)}(k_0 a) - \frac{1}{\mu_0} B'_{mn} m \frac{1}{a} [k_0 a h_n^{(1)}(k_0 a)]' \qquad [39]$$

$$\frac{A_{mn} j_n (k_1 a)}{k_1} + \frac{D_{mn}(a)}{k_1} = \frac{\epsilon_0}{\epsilon_1} \frac{B_{mn} h_n^{(1)}(k_0 a)}{k_0} \qquad [40]$$

$$\frac{A'_{mn}}{\mu_1} j_n (k_1 a) + \frac{1}{\mu_1} D'_{mn}(a) = \frac{\mu_0}{\mu_1} \frac{B'_{mn}}{\mu_0} h_n^{(1)}(k_0 a) \qquad [41]$$

For the lowest value of n, *i.e.*, for $n=m$, equations [36] to [41] become

$$-\frac{(m+2)(2m+1)}{(2m+3)} A_{m,m+1} \frac{1}{k_1 a} [k_1 a j_{m+1}(k_1 a)]' + \frac{1}{k_1} C_{mm}(a) + A'_{mm} m j_m(k_1 a)$$

$$= -\frac{(m+2)(m+1)}{(2m+3)} B_{m,m+1} \frac{1}{k_0 a} [k_0 a h^{(1)}_{m+1}(k_0 a)]' + B'_{mm} m h^{(1)}_m(k_0 a) \qquad [42]$$

$$m A_{mm} \frac{1}{k_1 a} [k_1 a j_m(k_1 a)]' - A'_{m,m+1} \frac{(m+2)(2m+1)}{(2m+3)} j_{m+1}(k_1 a)$$

$$= m B_{mm} \frac{1}{k_0 a} [k_0 a h^{(1)}_m(k_0 a)]' - B'_{m\ m+1} \frac{(m+2)(2m+1)}{(2m+3)} h^{(1)}_{m+1}(k_0 a) \qquad [43]$$

$$\frac{k_1}{\mu_1} m A_{mm} j_m(k_1 a) + \frac{1}{\mu_1} \frac{A'_{m,m+1}}{(2m+3} (2m+1)(m+2) \frac{1}{a} [k_1 a j_{m+1}(k_1 a)]' - \frac{1}{\mu_1} C'_{mm}(a)$$

$$= \frac{k_0}{\mu_0} m B_{mm} h^{(1)}_m(k_0 a) + \frac{B'_{m,m+1}}{(2m+3)} (2m+1)(m+2) \frac{1}{a} [k_0 a h^{(1)}_{m+1}(k_0 a)]' \qquad [44]$$

$$-\frac{k_1}{\mu_1} A_{m,m+1} \frac{(2m+1)(m+2)}{(2m+3)} j_{m+1}(k_1 a) - \frac{1}{\mu_1 a} A'_{mm} \frac{m}{a} [k_1 a j_m(k_1 a)]'$$

$$= -\frac{k_0}{\mu_0} B_{m,m+1} \frac{(2m+1)(m+2)}{(2m+3)} h^{(1)}_{m+1}(k_0 a) - \frac{1}{\mu_0} B'_{mm} \frac{1}{a} [k_0 a h^{(1)}_m(k_0 a)]' \qquad [45]$$

$$A_{mn} \frac{j_m(k_1 a)}{k_1} + \frac{D_{mm}(a)}{k_1} = \frac{\epsilon_0}{\epsilon_1} \frac{B_{mm} h_m^{(1)}(k_0 a)}{k_0} \qquad [46]$$

$$A'_{mm}\, j_m(k_1 a) + \frac{1}{\mu_1} D'_{mm}(a) = \frac{\mu_0}{\mu_1} \frac{B_{m'm}}{\mu_0} h_m^{(1)}(k_0 a) \qquad [47]$$

For $n = m+1$, equations (40) and (41) become

$$A_{m,m+1} \frac{j_{m+1}(k_1 a)}{k_1} + \frac{D'_{m,m+1}(a)}{k_1} = \frac{\epsilon_0}{\epsilon_1} B_{m,m+1} \frac{h_{m+1}^{(1)}(k_0 a)}{k_0} \qquad [48]$$

$$\frac{A'_{m,m+1}}{\mu_1} \frac{j_{m+1}(k_1 a)}{k_1} + \frac{1}{\mu_1} D'_{m,m+1}(a) = \frac{\mu_0}{\mu_1} \frac{B'_{m,m+1}}{\mu_0} h_{m+1}^{(1)}(k_0 a) \qquad [49]$$

Equations (42)-(49) are eight equations in the eight unknown amplitude coefficients A_{mm}, B_{mm}, $A_{m,m+1}$, $B_{m,m+1}$, A'_{mm}, B'_{mm}, $A'_{m,m+1}$, $B'_{m,m+1}$, and hence these unknown coefficients can be solved for.

Putting $n = m+1$, $m+2$, $m+3$, $\cdots$ etc. in equations (36)-(41), the other higher order coefficients $A_{m,m-2}$, $B_{m,m+2}$, $A'_{m,m+2}$, $B'_{m,m+2}$, $\cdots$ etc. can be solved for.

Since equations (36)-(41) contain the amplitude coefficients A_{mn}, B_{mn}, A'_{mn}, B'_{mn}, $A_{m,n-1}$, $B_{m,n-1}$, $A'_{m,n-1}$, $B'_{m,n-1}$, $A_{m,n+1}$, $B_{m,n+1}$, $A'_{m,n+1}$, $B'_{m,n+1}$, it is not possible to separate out only the coefficients A_{mn}, B_{mn}, A'_{mn}, B'_{mn}, for the same value of n.

Hence the boundary conditions are satisfied not for a single value of n but for $(n-1)$, n, and $(n+1)$ combined together. This shows that for a hybrid mode, for any particular value of m, the electric and magnetic field components consists of an infinite number of terms for values of n varying from m to ∞.

Since for each value of m, the infinite number of amplitude coefficients A_{mn}, A'_{mn}, B_{mn}, B'_{mn} can be solved for $n = m$ to ∞, it can be concluded that for each value of m, there exists a corresponding hybrid mode.

4. UNSYMMETRIC TM AND TE MODES

For unsymmetric TM mode $(m \neq 0)$, let the excitation be $E'e^{-j\omega t}$ applied uniformly over the plane $z = z_1 = a \cos\theta_1$, and in a direction normal to this plane.

$$\text{Let } E' = E_0 \cos m\phi \qquad [50]$$

E' has components E'_r and E'_θ in the r and θ directions. These components E'_r and E'_θ can be expanded in series of spherical harmonics as given by equation [3], [4], [7], [9], [13] and [15.]

The field components inside the dielectric sphere are

$$E_\gamma^i = -\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} n(n+1)\ A_{mn}\cos(m\phi)\ P_n^m(\cos\theta)\frac{j_n(k_1 r)}{k_1 r}e^{-jwt} + E_r' \qquad [51]$$

$$E_\theta^i = -\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{d}{d\theta}\{P_n{}^m(\cos\theta)\}A_{mn}\cos(m\phi)\frac{1}{k_1 r}[k_1 r\ j_n(k_1 r)]' e^{-jwt} + E_\theta' \qquad [52]$$

$$E_\phi{}^i = \frac{1}{\sin\theta}\sum_{n-0}^{\infty}\sum_{m=0}^{\infty} A_{mn}^{-}\ P_n^m(\cos\theta)\ m\sin(m\phi)\frac{1}{k_1 r}\left[k_1 r j_n(k_1 r)\right]' e^{-j\omega t} \qquad [53]$$

$$H_r^i = 0 \qquad [54]$$

$$H_\theta^i = \frac{k_1}{j\omega\mu_1}\frac{1}{\sin\theta}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{mn}\ P_n^m(\cos\theta)\ m\sin(m\phi)\ j_n(kr)\ e^{-j\omega t} \qquad [55]$$

$$H_\phi^i = \frac{k_1}{j\omega\mu_1}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{mn}\frac{d}{d\theta}\left\{P_n^m(\cos\theta)\right\}\cos(m\phi)\ j_n(k_1 r)\ e^{-j\omega t} \qquad [56]$$

The field components outside the sphere are

$$E_r^e = -\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} n(n+)\ B_{mn}\cos(m\phi)\ P_n^m(\cos\theta)\frac{h_n^{(1)}(k_0 r)}{k_0 r}e^{-j\omega t} \qquad [57]$$

$$E_\theta^e = -\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{d}{d\theta}\{P_n^m(\cos\theta)\}B_{mn}\cos(m\phi)\frac{1}{k_0 r}[k_6 r\ h_n^{(1)}(k_0 r)]' e^{-j\omega t} \qquad [58]$$

$$E_\phi^e = \frac{1}{\sin\theta}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} B_{mn}\ P^{mn}(\cos\theta)\ m\sin(m\phi)\frac{1}{k_0 r}[k_0 r\ h_n^{(1)}(k_0 r)]' e^{-j\omega t} \qquad [59]$$

$$H_r{}^e = 0 \qquad [60]$$

$$H_\theta^e = \frac{k_0}{j\omega\mu_0\sin\theta}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} B_{mn}\ P_n^m(\cos\theta)\ m\sin(m\phi)\ h_n^{(1)}(k_0 r)\ e^{-j\omega t} \qquad [61]$$

$$H_\phi^e = \frac{k_0}{j\omega\mu_0}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} B_{mn}\frac{d}{d\theta}\{P_n^m(\cos\theta)\}\cos(m\phi)\ h_n^{(1)}(k_0 r)\ e^{-j\omega t} \qquad [62]$$

Applying the boundary conditions that at $r=a$,

$E_\theta^i = E_\theta^e$, $E_\phi^i = E_\phi^e$, $H_\theta^i = H_\theta^e$

and

$H_\phi^i = H_\phi^e$,

we obtain

$$\frac{A_{mn}}{k_1 a}\,[k_1 r\, j_n(k_1 r)]'_{r=a} + \frac{C_{mn}(a)}{k_1} = \frac{B_{mn}}{k_0 a}\,[k_0 r\, h_n^{(1)}(k_0 r)]'_{r=a} \qquad [63]$$

$$\frac{A_{mn}}{k_1}\,[k_1 r\, j_n(k_1 r)]'_{\gamma=a} = \frac{B_{mn}}{k_0}\,[k_0 r\, h_n^{(1)}(k_0 r)]'_{r=a} \qquad [64]$$

$$\frac{k_1}{\mu_1} A_{mn}\, j_n(k_1 a) = \frac{k_0}{\mu_0} B_{mn}\, h_n^{(1)}(k_0 a) \qquad [65]$$

$$\frac{k_1}{\mu_1} A_{mn} j_n(k_1 a) = \frac{k_0}{\mu_0} B_{mn}\, h_n^{(1)}(k_0 a) \qquad [66]$$

Equations (65) and (66) are the same. Equations (63), (64) and (65) are three independent equations in two unknowns A_{mn} and B_{mn}. Hence there is no unique solution for A_{mn} and B_{mn}.

This shows that unsymmetric *TM* modes are not possible on the dielectric sphere.

It can be shown in a similar manner that unsymetric *TE* modes are also not possible on the dielectric sphere.

5. SYMMETRIC *TM* AND *TE* MODES

For a symmetric *TM* mode ($m=0$), let the excitation be $E' e^{-jwt}$ applied uniformly over the plane $z=z_1=a\cos\theta_1$, and in a direction normal to this plane. Let $E'=E_0$ and E' has components $E'_r=E_0\cos\theta_1=E_{r0}$ and $E'_\theta=E_{\theta_0}=-E_0\sin\theta_1$ in the r and θ directions. These compenents E'_r and E'_θ can be expanded in series of spherical harmonics as given below :

$$E_{\theta_0} = -\frac{1}{k_1}\sum_{n=0}^{\infty} C_{0n}(r)\, P_n^1(\cos\theta)\, e^{-jwt} \qquad [67]$$

$$E_{r_0} = -\frac{1}{k_1}\sum_{n=0}^{\infty} n(n+1) D_{0n}(r)\, P_n(\cos\theta) e^{-jwt} \qquad [68]$$

where

$$C_{0n}(r) = \frac{-k_1(2n+1)}{2\pi}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} E_{\theta_0} P_n^1(\cos\theta)\sin\theta\, d\theta\, d\phi \qquad [69]$$

$$D_{0n}(r) = \frac{-k_1(2n+1)}{n(n+1)\,2\pi}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} Er_0\, P_n(\cos\theta)\sin\theta\, d\theta\, d\phi \qquad [70]$$

If E_0 is a δ-function source given by equations [11] and [12], then

$$C_{0n}(r) = C_{0n}(a) = -\frac{V}{a}\frac{k_1(2n+1)}{2n(n+1)} \sin\theta_1 \, P_n^1(\cos\theta_1) \qquad [71]$$

and

$$D_{0n}(r) = D_{0n}(a) = \frac{V}{a} \cos\theta_1 \, P_n(\cos\theta_1) \, \frac{k_1(2n+1)}{2n(n+1)} \qquad [72]$$

The field components inside the sphere are

$$E_\gamma^i = -\sum_{n=0}^{\infty} n(n+1) A_{0n} P_n(\cos\theta) \frac{j_n(k_1 r)}{k_1 r} e^{-j\omega t} + E_{r0} \qquad [73]$$

$$E_\theta^i = -\sum_{n=0}^{\infty} \frac{d}{d\theta}\{P_n(\cos\theta)\} A_{0n} \frac{1}{k_1 r}[k_1 r j_n(k_1 r)]' e^{-j\omega t}$$

$$= -\sum_{n=0}^{\infty} P_n^1(\cos\theta) A_{0n} \frac{1}{k_1 r}[k_1 r j_n(k_1 r)]' e^{-j\omega t} \qquad [74]$$

$$H_\phi^i = \frac{k_1}{j\omega\mu_1} \sum_{n=0}^{\infty} A_{0n} P_n^1(\cos\theta) j_n(k_1 r) e^{-j\omega t} \qquad [75]$$

The field components outside the sphere are

$$E_r^e = -\sum_{n=0}^{\infty} n(n+1) B_{0n} P_n(\cos\theta) \frac{h_n^{(1)}(k_0 r)}{k_0 r} e^{-j\omega t} \qquad [76]$$

$$E_\theta^e = -\sum_{n=0}^{\infty} B_{0n} P_n^1(\cos\theta) \frac{1}{k_0 r}[k_0 r h_n^{(1)}(k_0 r)]' e^{-j\omega t} \qquad [77]$$

$$H_\phi^e = \frac{k_0}{j\omega\mu_0} \sum_{n=0}^{\infty} B_{0n} P_n^1(\cos\theta) h_n^{(1)}(k_0 r) e^{-j\omega t} \qquad [78]$$

Applying the boundary conditions that at $r=a$, $E_\theta^e = E_\theta^i$ and $H_\phi^e = H_\phi^i$, we obtain

$$\frac{A_{0n}}{k_1 a}[k_1 a j_n(k_1 a)]' + \frac{C_{0n}(a)}{k_1} = \frac{B_{0n}}{k_0 a}[k_0 a h_n^{(1)}(k_0 a)]' \qquad [79]$$

and

$$\frac{k_1}{\mu_1} A_{0n} j_n(k_1 a) = \frac{k_0}{\mu_0} B_{0n} h_n^{(1)}(k_0 a) \qquad [80]$$

Using equation [7.] in equation [79] and [80], the amplitude coefficient A_{0n} and B_{0n} can be uniquely determined for each value of n.

$$B_{0n} = \frac{C_{0n}(a)\, j\, \omega\, \mu_0}{k_1 k_0 h_n^{(1)}(k_0 a)} \frac{i}{Z_n} \tag{81}$$

$$A_{0n} = \frac{k_0\, h_n^{(1)}(k_0 a)}{k_1\, j_n(k_1 a)} B_{0n} \tag{82}$$

where

$$Z_n = \left\{ \frac{j\,\omega\,\mu_1}{k_1^2\, a} \frac{[k_1\, a\, j_n(k_1\, a)]'}{j_n(k_1\, a)} - \frac{j\,\omega\,\mu_0}{k_0^2\, a} \frac{[k_0\, a\, h_n^{(1)}(k_0\, a)]'}{h_n^{(1)}(k_0\, a)} \right\} \tag{83}$$

Z_n is the difference between the radial wave impedances E_θ / H_ϕ inside and outside the sphere. Free oscillation of the sphere results when these impedances are matched, that is, when $Z_n = 0$, an equation whose roots determine the characteristic or resonant frequencies of the natural modes of oscillation.

The field is thus determined unequely both inside and outside the dielectric sphere for each value of n. This shows that TM_{0n} modes exist for $n = 0, 1, 2, \cdots$ for the dielectric sphere.

It can similarly be shown that symmetric TE_{0n} modes also exist for $n = 0, 1, 2, \cdots$ for the dielectric sphere.

6. CONNLUSION

The following conclusions can be drawn from the above investigation on the dielectric sphere :

(i) It is not possible to excite unsymmetric *TM* and *TE* modes on the dielectric sphere.

(ii) It is possible to excite symmetric *TM* and *TE* modes, as well as symmetric and unsymmetric hybrid modes on the structrure.

Numerical calculations and experimental verification of the results obtained will be reported in subsequent papers.

7. ACKNOLEDGMENT

The author acknowledges with thanks the facilities offered by Dr. S. Dhawan, Director of the Indian Institute of Science.

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