CIRCULAR INHOMOGENEITY IN AN INFINITE STRIP

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ABSTRACT

Stress field due to circular inhomogeneity in an infinite strip has been investigated in this paper. Stresses which are given in terms of infinite integrals have been calculated numerically for some particular cases. When the ratio of the radius of circular inclusion to the semi-width of the strip is .1, the stress distribution at the equilibrium boundary in the strip is approximately the same as the stress distribution due to circular inclusion in the infinite medium.

INTRODUCTION

Two-dimensional inclusion problems have been mainly investigated by using point-force technique coupled with complex variable methods[1,2], strain-energy method[3], and theory of Hilbert problem[4]. The above methods are found unsuitable for the present problem. Point-force technique requires the knowledge of complex potentials which give the elastic field due to a point force in an infinite strip containing inhomogeneity and strain energy method needs considerable guessing of the equilibrinm boundary. However theory of Hilbert problem coupled with the superposition principle could be applied to obtain the stress field everywhere.

The solution is obtained by the superposition of two stress systems. The first system corresponds to the inhomogeneity in an infinite medium. The second stress system is obtained by applying on the edges of the strip, normal and shearing tractions opposite to those obtained in the first system.

FIRST STRESS SYSTEM

The problem of circular inhomogeneity in an infinite medium may be solved as follows :

Consider a circular region ($|z| \leqslant r'$; $z = x + iy$) situated in an infinite elastic plane. This circular region will be called inhomogeneity if its elastic constants are different from those of the outer region and inclusion if the elastic constants are the same as those of outer region. The outer region will be called matrix. The elastic constants of inhomogeneity are

taken as μ_1 and k_1 and those of matrix μ_2 and k_2, where μ is the rigidity modulus and $k = 3 - 4\sigma$ for plane strain case and $k = (3 - \sigma)/(1 + \sigma)$ for generalized plane stress, σ being Poisson's ratio. We shall denote the inhomogeneity and matrix by S^+ and S^- regions respectively.

Let the inhomogeneity in the absence of matrix undergo a prescribed deformation $(\epsilon_1 x, \epsilon_2 y)$ which in the presence of the matrix will attain a different-equilibrium configuration. The following conditions must be satisfied at the boundary $|z| = r'$:

$$u^+ - u^- = -\epsilon_1 x = g_1(t)$$

$$v^+ - v^- = -\epsilon_2 y = g_2(t) \tag{1}$$

$$\phi^+(t) + t\,\overline{\phi'^+(t)} + \overline{\psi^+(t)} = \phi^-(t) + t\,\overline{\phi'^-(t)} + \overline{\psi^-(t)} \tag{2}$$

and $\quad \mu_2 k_1 \phi^+(t) - \mu_2\, t\,\overline{\phi'^+(t)} - \mu_2\,\overline{\psi^+(t)}$

$$= \mu_1 k_2 \phi^-(t) - \mu_1 t\,\overline{\phi'^-(t)} - \mu_1\,\overline{\psi^-(t)} + 2\mu_1\mu_2\, g_1(t) + i\, g_2(t) \tag{3}$$

where the $+$ and $-$ supercripts stand for the S^+ and S^- regions respectively, t is the boundary point, $\phi(t)$ and $\psi(t)$ ate analytie functions as used by Muskhelishvili[5].

If we put

$$\Omega(z) = \phi^+(z) - z\overline{\phi}'^-(r'^2/z) - \overline{\psi}^-(r'^2/z), \quad z \in s^+$$

and $\quad \Omega(z) = \phi^-(z) - z\overline{\phi}'^+(r'^2/z) - \overline{\phi}^+(r'^2/z), \quad z \in s^-$

then, $\Omega(z)$ is holomorphic in the whole plane except possibly at the origin and infinity.

Further. if

$$\omega(z) = \mu_2 k_1 \phi^+(z) + \mu_1 z\overline{\phi}'^-(r'^2/z) + \mu_1\,\overline{\psi}^-(r'^2/z), \quad z \in s^+$$

and $\quad \omega(z) = \mu_1 k_2 \phi^-(z) + \mu_2 z\overline{\phi}'^+(r'^2/z) + \mu_2\overline{\psi}^+(r'^2/z), \quad z \in s^-$

then it may be seen from (3) that $\omega(z)$ satisfies the equation

$$\omega^+(t) - \omega^-(t) = 2\mu_1\mu_2\, g_1(t) + i\, g_2(t) \text{ on } |z| = r' \tag{4}$$

The function $\omega(z)$ is sectionally holomorphic everywhere except at the point of infinity where it has pole of order one.

Using the fact that the stresses should vanish at infinity and are bounded at the origin, $\Omega(z)$ and $\omega(z)$ can be determined. The analytic functions

$\phi(z)$ and $\psi(z)$ for S^+ and S^- can be easily found from $\Omega(z)$ and $\omega(z)$ and are given by

$$\phi^+(z) = -\frac{\mu_1 \mu_2 (\epsilon_1 + \epsilon_2) z}{2\mu_2 - \mu_2 + k_1 \mu_2},$$

$$\phi^-(z) = \frac{\mu_1 \mu_2 r'^2 (\epsilon_1 - \epsilon_2)}{z(\mu_2 + \mu_1 k_2)},$$

$$\psi^+(z) = \frac{\mu_1 \mu_2 (\epsilon_1 - \epsilon_2) z}{\mu_2 + \mu_1 k_2},$$

$$\psi^-(z) = -\frac{2\mu_1 \mu_2 r'^2 (\epsilon_1 + \epsilon_2)}{z(2\mu_1 - \mu_2 + \mu_2 k_1)} + \frac{\mu_1 \mu_2 (\epsilon_1 - \epsilon_2) r'^4}{z^3 (\mu_1 + \mu_2 k_1)}. \qquad [5]$$

If the center of inhomogeneity is taken as $(0, y_0)$, the corresponding stresses for inhomogeneity and matrix can be found out and are given by

$$\{(T_{xx})_1\}^+ = -\frac{2\mu_1 \mu_2 (\epsilon_1 + \epsilon_2)}{2\mu_1 - \mu_2 + \mu_2 k_1} - \frac{\mu_1 \mu_2 (\epsilon_1 - \epsilon_2)}{\mu_2 + \mu_1 k_2},$$

$$\{(T_{yy})_1\}^+ = -\frac{2\mu_1 \mu_2 (\epsilon_1 + \epsilon_2)}{2\mu_1 - \mu_2 + \mu_2 k_1} + \frac{\mu_1 \mu_2 (\epsilon_1 - \epsilon_2)}{\mu_2 + \mu_1 k_2},$$

$$\{(T_{xy})_1\}^+ = 0,$$

$$\{(T_{xx})_1\}^- = \frac{4A x^2 \{x^2 - 3(y - y_0)^2\}}{\{x^2 + (y - y_0)^2\}^3} + \frac{B\{x^2 - (y - y_0)^2\}}{\{(x^2 + (y - y_0)^4\}^2}$$
$$- \frac{3C\{x^4 + (y - y_0)^4 - 6x^2(y - y_0)^2\}}{\{x^2 + (y - y_0)^2\}^4},$$

$$\{(T_{yy})_1\}^- = \frac{4A (y - y_0)^2 \{3x^2 - (y - y_0)^2\}}{\{x^2 + (y - y_0)^2\}^3} - \frac{B\{x^2 - (y - y_0)^2\}}{\{x^2 + (y - y_0)^2\}^2}$$
$$+ \frac{3C\{x^4 + (y - y_0)^4 - 6x^2(y - y_0)^2\}}{\{x^2 + (y - y_0)^4\}^4},$$

$$\{(T_{xy})_1\}^- = \frac{8A x (y - y_0) \{x^2 - (y - y_0)^2\}}{\{x^2 + (y - y_0)^2\}^3} + \frac{2B x (y - y_0)}{\{x^2 + (y - y_0)^2\}^2}$$
$$+ \frac{12C x (y - y_0) \{(y - y_0)^2 - x^2\}}{\{x^2 + (y - y_0)^2\}^4} \qquad [6]$$

where the subscript 1 refers to the first system of stresses; subscripts $+$ and $-$ refer to the inhomogeneity and matrix respectively and the constants A, B. and C have following values

$$A = -\frac{\mu_1 \mu_2 r'^2 (\epsilon_1 - \epsilon_2)}{\mu_2 + \mu_1 k_2},$$

$$B = - \frac{2\,\mu_1\,\mu_2\,r'^{\,2}\,(\epsilon_1 + \epsilon_2)}{2\,\mu_1 - \mu_2 + k_1\,\mu_2},$$

$$C = - \frac{\mu_1\,\mu_2\,r'^{\,4}\,(\epsilon_1 - \epsilon_2)}{\mu_1 + k_1\,\mu_2},$$

INHOMOGENEITY IN AN INFINITE STRIP

We shall now consider inhomogeneity in a two-dimensionol infinite elastic strip (Fig. 1). The edges of the strip are given by $Y = \pm a$. The centre of inhomogeneity is taken at $(0, Y_0)$ and as before the inhomogeneity in the absence of matrix undergoes a prescribed deformation $(\epsilon_1 x, \epsilon_2 y)$. The elastic constants of inhomogeneity and matrix are the same as in the previous section. On the edges of the strip normal and shearing tractions should be zero. In order to satisfy the edge conditions, we superimpose on the first stress system a second stress system which is obtained by solving a strip problem (containing no inhomogeneity) in which the edges $y = \pm a$ are subjected to the surface tractions $-\{(T_{yy})_1\}^-$ and $-\{(T_{xy})_1\}^-$
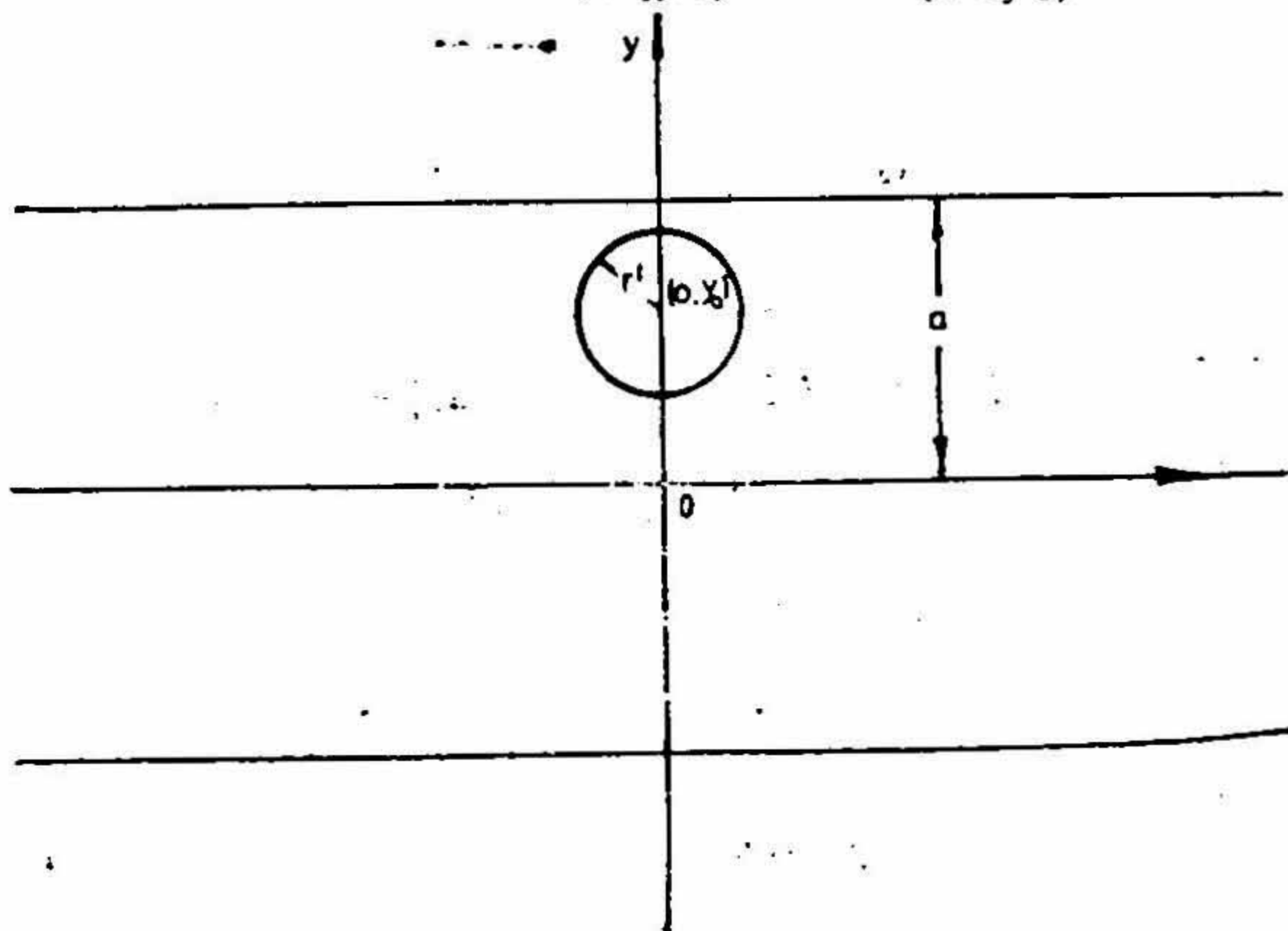

FIG. 1

Infinite Strip with Circular Hole and the Coorninate System.

The second system may be obtained as follows :

The solution of the equations of equilibrium in two-dimensional case in terms of biharmonic function Φ is known [5]. The exponential Fourier transform of a function $f(x,y)$, denoted by $\bar{f}(y, \xi)$ is defined as

$$\bar{f}(y,\xi) = \int_{-a}^{a} e^{i\xi x} f(x, y)\, dx$$

Assuming that $e^{i\xi}$ ($\partial^r \Phi / \partial x^r$) ($r = 0, 1, 2, 3$) vanish as $x \to \pm\infty$ and taking the exponential Fourier transform of the biharmonic equation in Φ and the corresponding stresses in terms of Φ, we get

$$\overline{\Phi} = (P + Qy)e^{-|\xi|y} + (R + Sy)e^{|\xi|y}$$

$$\overline{(T_{xx})}_2 = (P\xi^2 + Qy\xi^2 - 2Q\xi)e^{-|\xi|y} + (R\xi^2 + Sy\xi^2 + 2S\xi)e^{|\xi|y}$$

$$\overline{(T_{yy})}_2 = -\xi^2\{(P + Qy)e^{-|\xi|y} + (R + Sy)e^{|\xi|y}\}$$

and

$$\overline{(T_{xy})}_2 = i\xi[\{-\xi P - Q(y\xi - 1)\}e^{-|\xi|y} + \{R\xi + S(1 + y\xi)\}e^{|\xi|y}] \qquad [7]$$

where bar denotes the Fourier transform; subscript 2 refers to the second system of stresses and P, Q, R, and S are functions of ξ and are to be determined with the help of boundary conditions.

The boundary conditions are

$$\{(T_{yy})_2\}_{y=\pm a} = -[\{(T_{yy})_1\}^-]_{y=\pm a}, \quad \{(T_{xy})_2\}_{y=\pm a} = -[\{(T_{xy})_1\}^-]_{y=\pm a}.$$

It may be seen that

$$\overline{[\{(T_{yy})_1\}^-]}_{y=a} = \pi e^{(y_0-a)\xi}[2A\xi\cdot(y_0 - a) + B\xi + \tfrac{1}{2}c\xi^3]$$

$$\overline{[\{(T_{yy})_1\}^-]}_{y=-a} = \pi e^{-(a+y_0)\xi}[-2A\xi^2(a + y_0) + B\xi + \tfrac{1}{2}c\xi^3]$$

$$\overline{[\{(T_{xy})_1\}^-]}_{y=a} = i\pi e^{(y_0-a)\xi}[2A\xi\{1 + \xi(y_0 - a)\} + B\xi + \tfrac{1}{2}c\xi^3]$$

$$\overline{[\{(T_{xy})_1\}^-]}_{y=-a} = i\pi e^{-(a+y_0)\xi}[2A\xi\{\xi(y_0 + a) - 1\} - B\xi - \tfrac{1}{2}c\xi^3]$$

The constants P, Q, R and S come out to be

$$P = -\frac{\pi}{\Delta(\xi)}(2B + C\xi^2)[e^{-2a\xi}e^{\xi y_0}\{\sin h\, 2\xi a + 2a\xi(1 + 2a\xi)e^{2\xi a}\}$$

$$- e^{-\xi y_0}\{\sin h\, 2\xi a + 2a\xi \cos h\, 2\xi a\}] - \frac{4\pi A}{\Delta(\xi)}[e^{-2a\xi}e^{\xi y_0}\{\xi y_0 \sin h\, 2\xi a$$

$$+ 2a\xi(\xi y_0 - 2a^2\xi^2 + 2ay_0\xi^2)e^{2a\xi}\} + e^{-\xi y_0}\{2ay_0\xi^2 \cos h\, 2\xi a$$

$$+ (2a^2\xi^2 + \xi y_0)\sinh 2\xi a\}]$$

$$Q = -\frac{\pi\xi}{\Delta(\xi)}(2B + C\xi^2)[4a\xi e^{\xi y_0} - 2e^{-\xi y_0}\sin h\, 2\xi a]$$

$$- \frac{4\pi\xi A}{\Delta(\xi)}[e^{-2a\xi}e^{\xi y_0}\{(4ay_0\xi^2 + 2a\xi - 4a^2\xi^2)e^{2a\xi} - \sin h\, 2\xi a\}$$

$$+ e^{-\xi y_0}\{(2\xi y_0 - 1)\sin h\, 2\xi a + 2a\xi \cos h\, 2\xi a\}]$$

$$R = -\frac{\pi}{\Delta(\xi)} (2B + C\xi^2) \left[e^{\xi y_0}(\sin h\, 2\xi a + 2a\xi \cos h\, 2\xi a) \right.$$

$$\left. - e^{-2a\xi}e^{-\xi y_0}\{(1 + 2d\xi) \sin h\, 2\xi a + 4a^2\xi^2 e^{2a\xi} + 2a\xi \cos h\, 2\xi a\} \right]$$

$$+ \frac{4\pi A\xi}{\Delta(\xi)}[e^{\xi y_0} \{(y_0 - 2a^2\xi) \sin h\, 2\xi a + 2a\xi y_0 \cos h\, 2\xi a\}$$

$$+ e^{-2a\xi}e^{-\xi y_0}\{y_0 \sin h\, 2\xi a + 2a\xi e^{2\xi a}(y_0 + 2a\xi y_0 + 2a^2\xi)\}]$$

$$\frac{2\pi\xi}{\Delta(\xi)} (2B + C\xi^2)[e^{\xi y_0} \sin h\, 2\xi a - 2a\xi e^{-\xi y_0}] \qquad [8]$$

$$S = + \frac{4\pi A\xi}{\Delta(\xi)} [e^{\xi y_0} \{(2\xi y_0 + 1) \sin h\, 2\xi a - 2a\xi \cos h\, 2\xi a\}$$

$$+ e^{-2a\xi}e^{-\xi y_0}\{\sin h\, 2\xi a + 2a\xi e^{2a\xi}(2a\xi + 2\xi y_0 - 1)\}]$$

where $\Delta(\xi) = 4\xi \{\sin^2 h\, 2\xi a - 4a^2\xi^2\}$. On substituting these values of P, Q, R and S in (7) and using inversion formula, we get

$$(T_{yy})_2 = - \frac{1}{\pi} \int_0^\infty \xi^2 \{(P + Qy)e^{-\xi y} + (R + Sy)e^{\xi y}\} \cos(\xi x) d\xi,$$

$$(T_{xx})_2 = \frac{1}{\pi} \int_0^\infty \{(P\xi^2 + Qy\xi^2 - 2Q\xi)e^{-\xi y} + (R\xi^2 + Sy\xi^2 + 2S\xi)e^{\xi y}\} \cos(\xi x) d\xi,$$

$$(T_{xy})_2 = \frac{1}{\pi} \int_0^\infty \xi[- \{P\xi + Q(y\xi - 1)\} e^{-\xi y} + \{R\xi + S(1 + y\xi)\} e^{\xi y}] \sin (\xi x) d\xi. \qquad [9]$$

Resultant stresses in the strip for inhomogeneity and matrix are given by

$$(T_{yy})^- = \{(T_{yy})_1\}^- + (T_{yy})_2 \text{ and } (T_{yy})^+ = \{(T_{yy})_1\}^+ + (T_{yy})_2 \text{ and similarly for}$$

$$(T_{xx})^-, (T_{xx})^+, (T_{xy})^- \text{ and } (T_{xy})^+. \qquad [10]$$

It may be seen that on substituting $y = \pm a$ in (10), $(Txy)^+$ come out to be zero. Also as $a \to \infty$, $(T_{yy})_2 = (T_{xx})_2 = (T_{xy})_2 = 0$ and we get the results of circular inhomogeneity in an infinite medium. Continuity of normal and shearing stresses at the equilibrium boundary and discontinuity in displacements can also be verified.

The problem of arbitrary distribution of loads on the edges of the strip can also be solved by the above method.

NUMERICAL EVALUATION OF INTEGRALS AND DISCUSSION

It may be noted that the infinite integrals in (9) are all convergent. The singularities of integrands in (9) can arise from the zeros of $\Delta(\xi)$ which are at $\xi = 0$. However in the limit as $\xi \to 0$, the integrands tend to some finite limit.

The order of convergence of the integrands in (9) is given by

$$O(\xi^4 e^{|y/a|+|y_0/a|-2)\xi})\text{ as }\xi\to\infty \tag{11}$$

The upper limit of infinite integrals in (9) may be taken as some finite vnlue of ξ which is not arbitrary and depends upon the resuli in (11). The integrals can be evaluated by using Filon's method [6] which is more suitable because of the presence of oscillatory functions sin (ξx) and cos (ξx).

Graphs of T_{rr}, $T_{\theta\theta}$ and $T_{\theta r}$ (r, θ are the polar co-ordinates) versus θ at the equilibrium boundary have been drawn in Figs. 2, 3, and 4 respectively for the case $\epsilon_1=\epsilon_2=\epsilon$ and $\mu_1=\mu_2=\mu$ and $K_1=K_2=K$. Centre of the inclusion is taken at the origin. From symmetry considerations θ has been given values from 0 to $\pi/2$ only. When the ratio r'/a is 0.1, stresses in the strip due to circular inclusion are almost the same as for circular inclusion in the infinite medium.
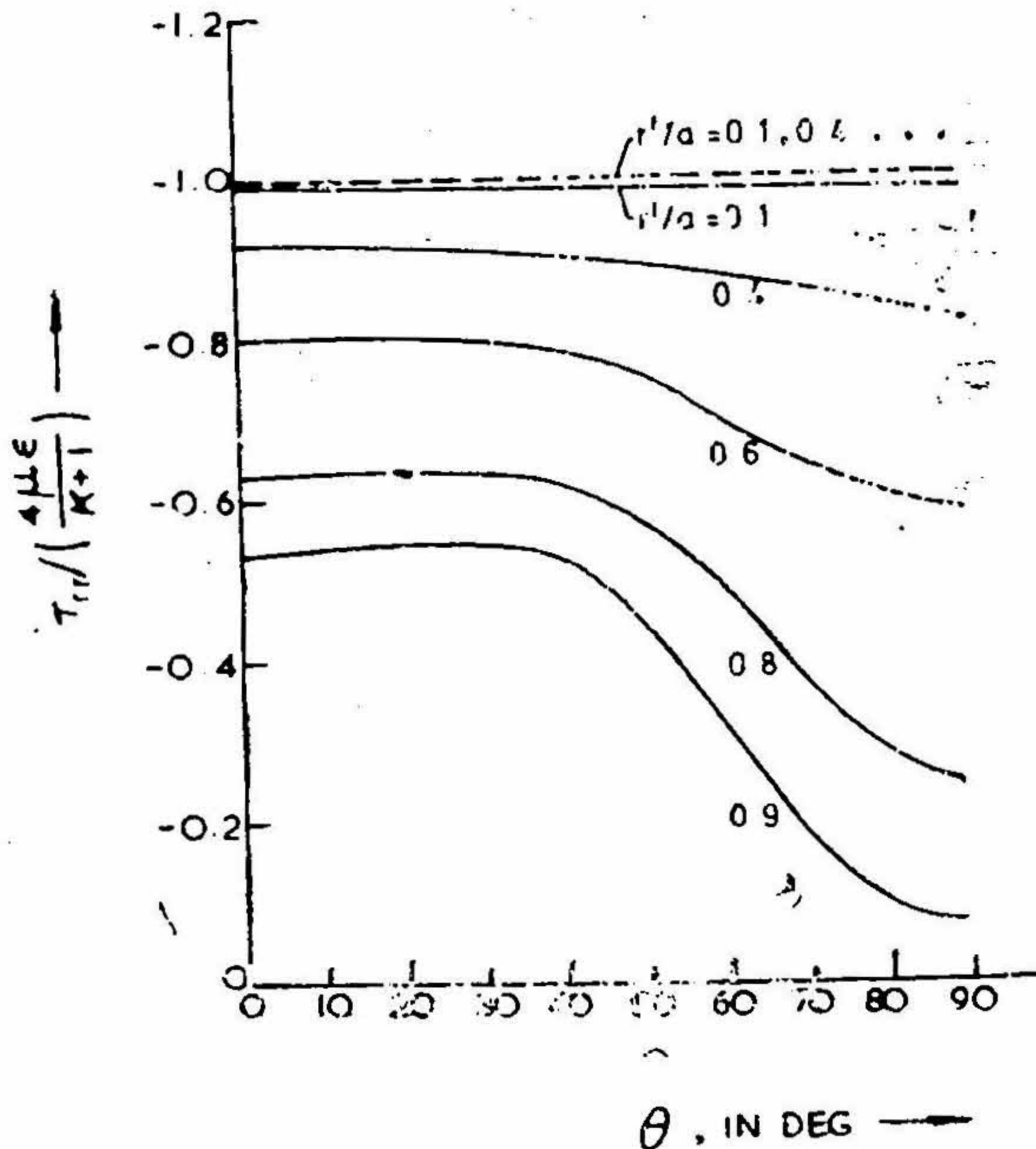

FIG. 2

T_{rr} Versus θ on the Equilibrium Boundary for the Case $\epsilon_1=\epsilon_2=\epsilon$. Dotted Line is the Graph for Circular Inclusion in an Infinite Medium.
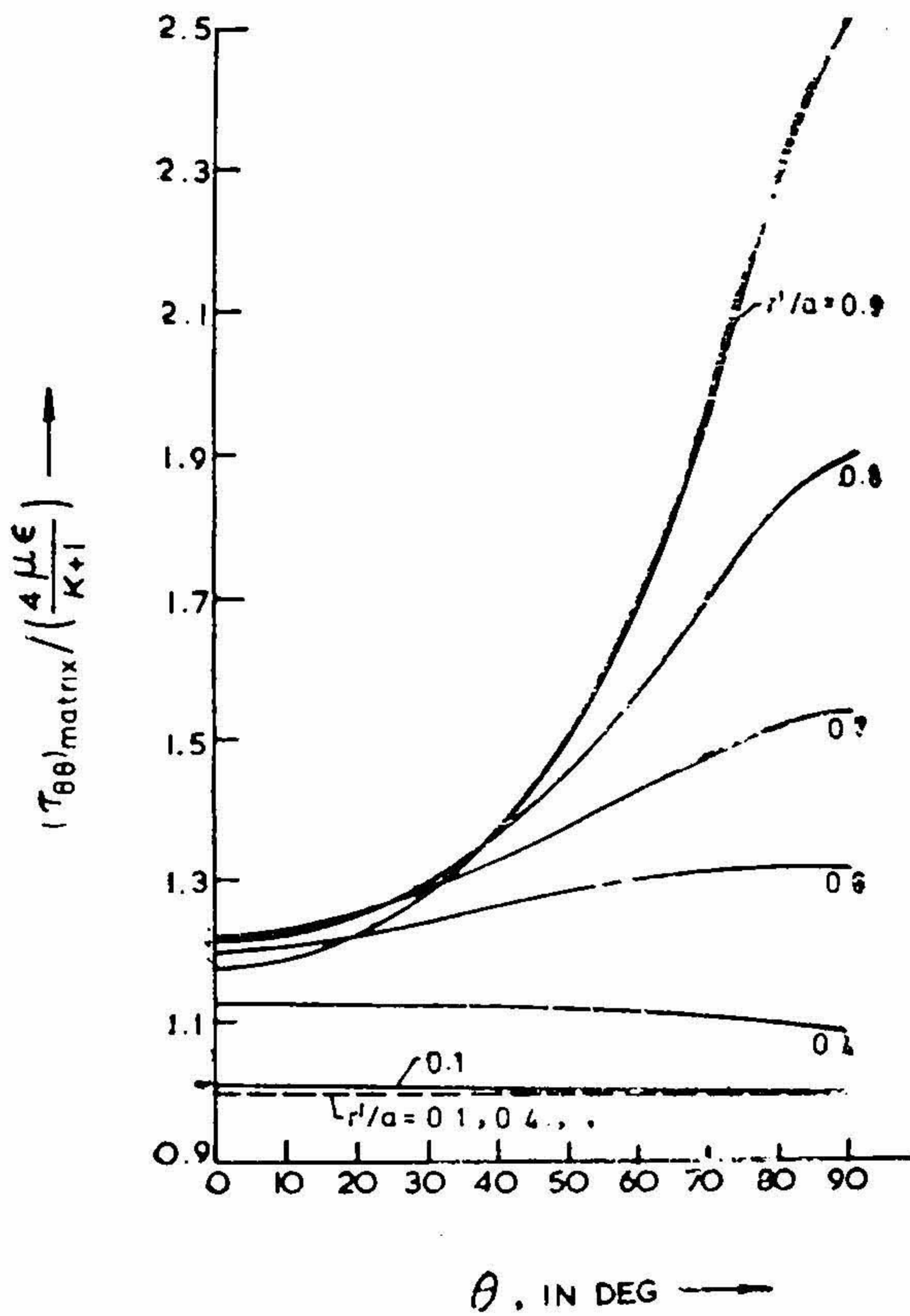
FIG. 3

$(\tau_{\theta\theta})_{matrix}$ Versus θ on the Equilibrium Boundary for the Case $\epsilon_1 = \epsilon_2 = \epsilon$. Dotted Line is the Graph of Circular Inclusion in an Infinite Medium.

FIG. 4

τ_{rr} Versus θ on the Equilibrium Boundary for the Case $\epsilon_1 = \epsilon_1 = \epsilon$. Dotted Line is the Graph for Circular Inclusion in an Infinite Medium.
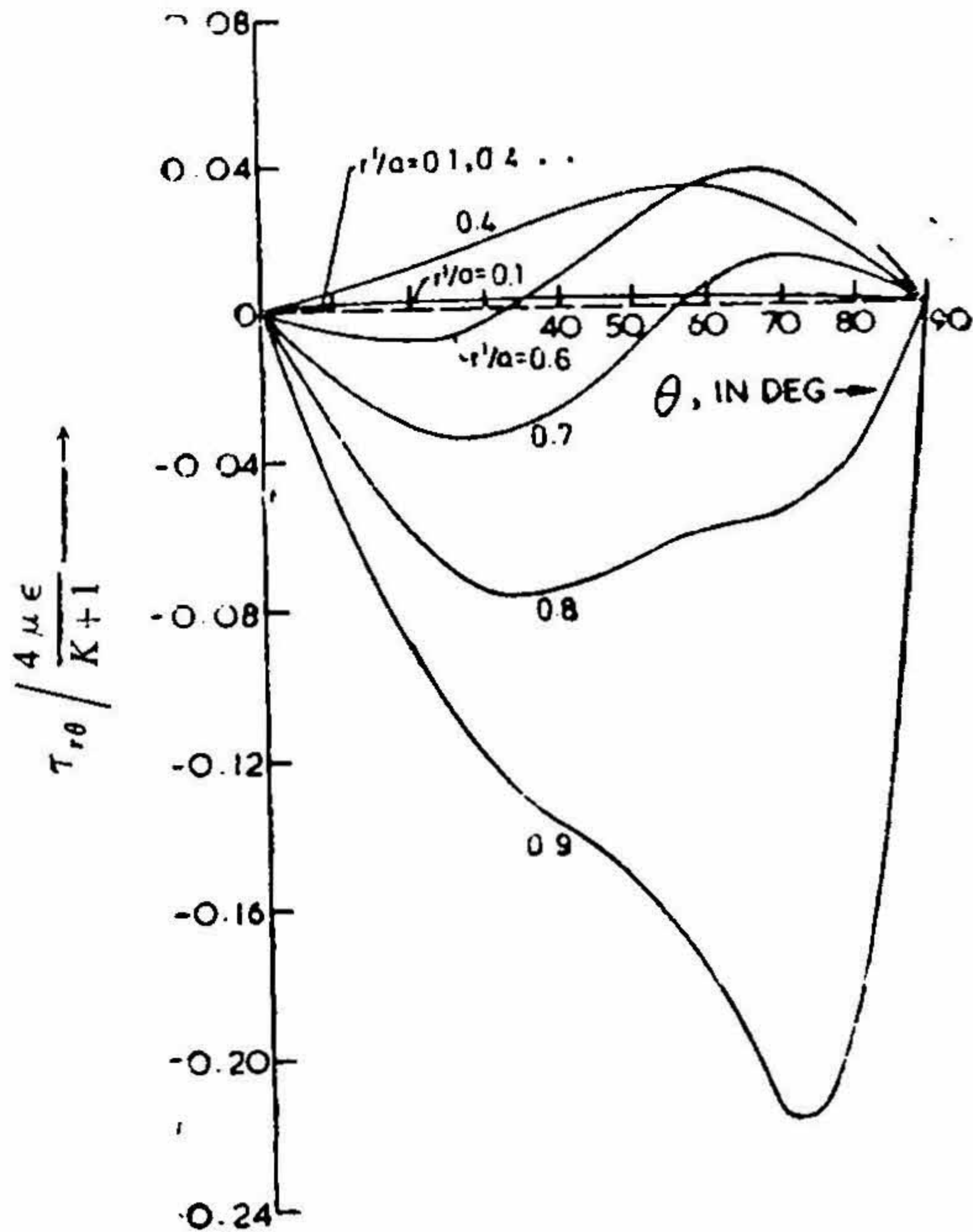
REFERENCES

1. Eshelly, J. D.   ..   ..   *Proc. R. Soc.*, 1957, **241**, 376.

2. Jaswon, M. A. and Bhargava R. D. ..   *Proc. Camb. Phil. Soc.*, 1961, **57**, 669.

3. Bhargava, R. D.   ..   ..   *Appl. Scient. Res.*, 1963, **11**, 80.

4. Bhargava, R. D. and Gupta, S.C. ..   *Bull. Pol. Des. Scie.*, 1969, **17**, 239.

5. Muskhelishvili, N. I.   ..   Some basic problems of the mathematical theory of elasticity. P. Noordhoff, Ltd. Gromingen, 1953.

6. Zdenek Kopal   ..   ..   Numertical analysis N. Y. Wiley, 2nd ed (1961).