# APPROXIMATE ANALYSIS OF NON-LINER SPRING MASS SYSTEMS SUBJECTED TO STEP FUNCTION EXCITATION

*By* M. A. V. Rangacharyulu and P. Srinivasan
(*Department of Mechanical Engineering, Indian Institute of Science, Bangalore-12 India*)

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ABSTRACT

*The paper deals with an approximate method of analysis of undamped non-linear spring mass systems subjected to step function excitation. The non-linear function is replaced by a linear function such that the weighted mean square error is minimum. An appropriate weight function, valid for the interval of approximation, is proposed and an expression for the approximate period is derived. The method is illustrated by an example and the approximate results are compared with the exact results.*

## 1. Introduction

The response of undamped non-linear spring mass systems initially at rest, subjected to step function excitation has been studied by many investigators. Bapat and Srinivasan[1,2] have obtained exact expressions for periods of oscillation for a certain class of non linear systems in terms of well tabulated special functions. They[3] have also extended Panovko's method of direct linearisation and Atkinson's superposition principle to obtain approximate expressions for time periods of the system with arbitrary hardening type spring characteristics. Ariaratnam[7] and Bauer[5] extended Poincare-Lighthill-Kuo method of perturbation to study non-linear systems subjected to pulse excitations. Ergin[6] proposed a bilinear approximation to study undamped problems.

There are many engineering situations, where, the transient response of a non-linear system is of great interest such as for example in the design of shock mounts. The springs used in the shock mounts are invariably non-linear either on account of the material like composite rubber pads in compression or on account of the geometry of spring arrangements. The designer of shock mounts would be interested in the response of the absorbers to step inputs to which it is subjected in application so as to limit the deflection to tolerable values. There are also many applications of this in control theory, when the designer needs, transient response of non-linear system in the design of suitable control systems.

The method of analysis presented here is based on the linearisation of the non-linear function such that the weighted mean square error is minimum. An appropriate weight function is proposed for the interval of approximation and an expression for the approximate period of oscillation is developed. The method is illustrated by an example.

## 2. METHOD OF APPROXIMATION

The governing differential equation of undamped non-linear spring mass systems excited by a stem function can be written as

$$\ddot{x} + f(x) = 0; \quad t < 0$$
$$= F; \quad t > 0 \qquad [1]$$

where $x$ is the displacement from the equilibrium position and $f(x)$ is an odd function given by

$$f(x) = \sum_{i=0}^{n} a_{2i+1} x^{2i+1},$$

$a$'s being positive $F$, the magnitude of the step, is a positive constant. The initial conditions are

$$x\big|_{t=0} = 0 \qquad [2]$$
$$\dot{x}\big|_{t=0} = 0$$

It is known that $x$ always remains positive for the stipulated initial conditions [2] and it oscillates between zero and a maximum value (say $A$, the amplitude), which depends upon the force amplitude $F$. The amplitude, $A$ can be found by analysing the first integral of the system[1,2]. The equation [1] can be written as

$$\ddot{x} + \sum_{i=0}^{n} a_{2i+1} x^{2i+1} - F = 0; \; 0 < x \leq A$$

or

$$\ddot{y} + \phi(A,y) = 0; \; 0 < y \leq 1 \qquad [3]$$

where

$$y = x/A \qquad \text{and}$$

$$\phi\,(A,y) = \sum_{i=0}^{n} a_{2i+1}\, A^{2i}\, y^{2i+1} - \frac{F}{A} \qquad [4]$$

The function $\phi\,(A,y)$ is replaced by a linear function of the form $(K_1 + K_2 y)$ in the interval (0,1) such that the integral

$$I(K_1, K_2) = \int_0^1 [\phi(A,y) - (K_1 + K_2 y)]^2\, w(y)\, dy$$

is minimum. Here $w(y)$ is a positive weight function defined in the interval (0,1). The minimisation of $I\,(K_1, K_2)$ leads to the determination of the constants $K_1$ and $K_2$.

In the present analysis the weight function is chosen as

$$w(y) = 2 - \lambda\, y^{\lambda-2}; \quad \lambda < 2 \qquad [4]$$

where $\lambda$ is a free parameter and the constraint stipulated on ensures the condition that $w(y)$ should be positive in the interval (0,1). The constants $K_1$ and $K_2$ can also be determined independently as $y^{\circ}$ and $y'$ are orthogonal in (0,1) with respect to the weight function (4) and they are given by

$$K_1\,(\lambda, A) = \frac{\int_0^1 \phi\,(A, y)\,(2 - \lambda\, y^{\lambda-2})\, dy}{\int_0^1 (2 - \lambda\, y^{\lambda-2})\, dy} \qquad [5.a]$$

and

$$K_2\,(\lambda, A) = \frac{\int_0^1 \phi\,(A, y)\,(2 - \lambda\, y^{\lambda-2})\, y dy}{\int_0^1 y^2\,(2 - \lambda\, y^{\lambda-2})\, dy} \qquad [5.b]$$

Now the equation [3] can be replaced by

$$\ddot{y} + K_2\,(\lambda, A)\, y + K_1\,(\lambda, A) = 0 \qquad [6]$$

$K_1\,(\lambda, A)$ and $K_2\,(\lambda, A)$ are evaluated from [5.a] and [5.b] and these are given by

$$K_1\,(\lambda, A) = -\left[\frac{F}{A} + \frac{1}{2}\, a_3\, \frac{(\lambda-1)}{(\lambda+2)}\, A^2 + \frac{2}{3}\, a_5\, \frac{(\lambda-1)}{(\lambda+4)}\, A^4 + \cdots\right]$$

$$K_2\,(\lambda, A) = a_1 + \frac{9}{5}\, a_3\, \frac{(\lambda+1)}{(\lambda+3)}\, A^2 + \frac{15}{7}\, a_5\, \frac{(\lambda+1)}{(\lambda+5)}\, A^4 + \cdots$$

The approximate period of oscillation is given by

$$T_a = \frac{2\pi}{\sqrt{[K_2(\lambda, A)]}} = 2\pi\left[[a_1]^{1/2}\left(1 + \frac{9}{5}\frac{a_3}{a_1}\frac{(\lambda+1)}{(\lambda+3)}A^2 + \frac{15}{7}\frac{a_5}{a_1}\frac{(\lambda+1)}{(\lambda+5)}A^4 + \ldots\right)^{1/2}\right]^{-1} \quad [7]$$

## 3. CHOICE OF THE PARAMETER λ – SMALL AMPLITUDE ANALYSIS

One possible way of choosing an appropriate value for λ is by comparing the expressions for the approximate period and the exact period in a manner similar to that given in[8]. The exactexpression $T_e$, for the time period for the system described by [3] can be written as[4].

$$T_e = \sqrt{2}\int_0^1 \frac{dy}{\left[y\int_0^1 \eta(A,\xi)\,d\xi - \int_0^y \eta(A,\xi)\,d\xi\right]^{1/2}} \quad [8]$$

where,

$$\eta(A,\xi) = \sum_{i=0}^{n} a_{2i+1} A^{2i}\xi^{2i+1}$$

Evaluation of the integrals in the integrand reduces [8] to

$$T_e = \frac{2}{\sqrt{a_1}}\int_0^1\left[1 + \frac{1}{2}\frac{a_3}{a_1}A^2(1+y+y^2) + \frac{1}{3}\frac{a_5}{a_1}A^4(1+y+y^2+y^3+y^4)\cdots\right]^{-1/2} \times y^{-1/2}(1-y)^{-1/2}\,dy \quad [9]$$

For small amplitudes the expression in the square brackets of [9] can be expanded in binomial series. Integration after expanding in binomial series and retaining only terms of the order of $A^2$, yield

$$T_e = \frac{2\pi}{\sqrt{a_1}}\left[1 - \frac{15}{32}\frac{a_3}{a_1}A^2 - \ldots\right] \quad [10]$$

The expression for the approximate period (7) can be written as

$$T_a = \frac{a\pi}{\sqrt{(a_1)}}\left[1 - \frac{9}{10}\left(\frac{\lambda+1}{\lambda+3}\right)\frac{a_3}{a_1}A^2 - \ldots\right] \quad [11]$$

Comparison of [10] and [11] suggests that

$$\lambda = \frac{81}{69} \simeq 1.174$$

This is less than 2, thus satisfying the stipulated condition on $\lambda$. The analysis given in this seetion gives an idea of the value of $\lambda$ around which variations can be tried. For a system with cubic restoring force characteristic both [10] and [11] give same value for $\lambda = 81/69$. The following section deals with an example which illustrates the method and brings out the effect of $\lambda$ on the period.

## 4. Example

Consider a cubic spring mass system :

$$\ddot{x} + a_1 x + a_3 x^3 = F \qquad [12]$$

with $x(0) = 0$ and $\dot{x}(0) = 0$.

The exact period of [12] is given by [1]

$$T_e = \frac{4\sqrt{2}\,K(k)}{[4a_1^2 + 8a_1 a_3 A^2 + 3a_3^2 A^4]^{1/4}} \qquad [13]$$

where $K(k)$ is the complete elliptic integral of the first kind, $A$ is the amplitude and $k$ is given by

$$k^2 = \frac{1}{2}\left[1 - \frac{1}{2}\,\frac{(4a_1 + 3a_3 A^2)}{[4a_1^2 + 8a_1 a_3 A^2 + 3a_3^2 A^4]^{1/2}}\right]$$

From [7] the approximate period is given by

$$T_a = 2\pi\left[\sqrt{(a_1)}\sqrt{\left\{1 + \frac{9}{5}\frac{a_3}{a_1}\left(\frac{\lambda+1}{\lambda+3}\right)A^2\right]^{-1} \qquad [14]$$

Here $a_1$ is taken as [11], [13] and [14] are computed for $a_3 = 0.2$, 1.0 and 2.0. Period versus amplitude curves are plotted for different values of $\lambda$. Fgures 1, 2, 3 and 4 show the variation $T_a$[14] with $A$ for different cases.

## 5. Discussion

In the method presented here, the non-linear function is replaced by an equivalent linear function and an approximate period dependent on the maximum displacement (Amplitude) of the system is obtained.

For small amplitudes $\lambda = (81/69) \cong 1.174$ provides a sufficiently accurate approximation to the period for systems with odd type restoring force characteristics. For the example considered here, all the vslues of $\lambda$ i.e., 1, 1.1, 1.174 and 1.2 give almost the exact result for small nonlinearity and small amplitudes (Fig. 1 and 4). The results for $\lambda = 1.0$ and 1.1 are not incorporated in fig. 1 as the approximate results are almost close to the exact ones. $\lambda = 1$ seems to provide better results at large applitudes in all

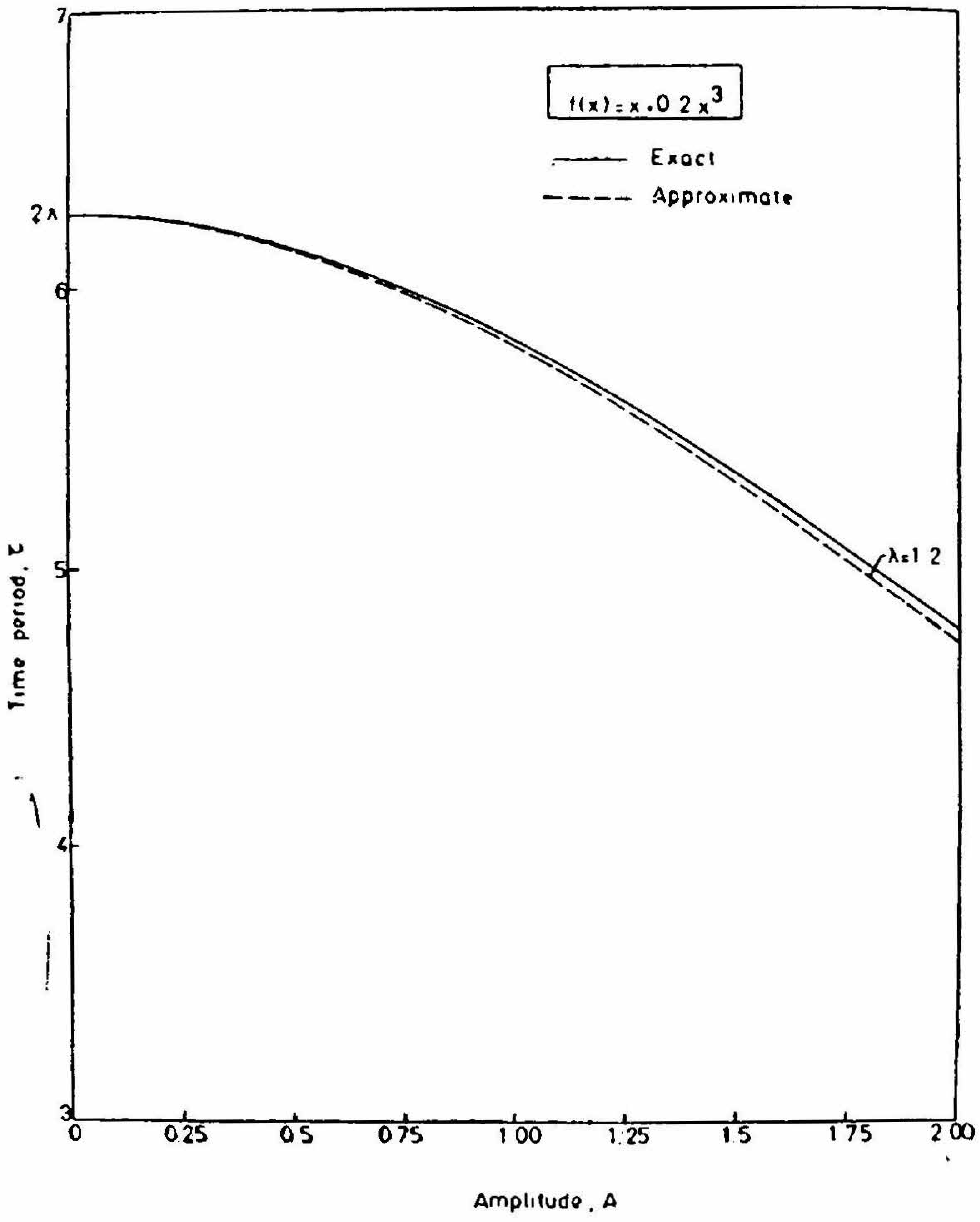

f(x) = x + 0 2 x^3
Exact
Approximate
λ=1 2
7
2π
6
5
4
3
0
0.25
0.5
0.75
1.00
1.25
1.5
1.75
2.00
Time period, τ
Amplitude, A

FIG. 1.

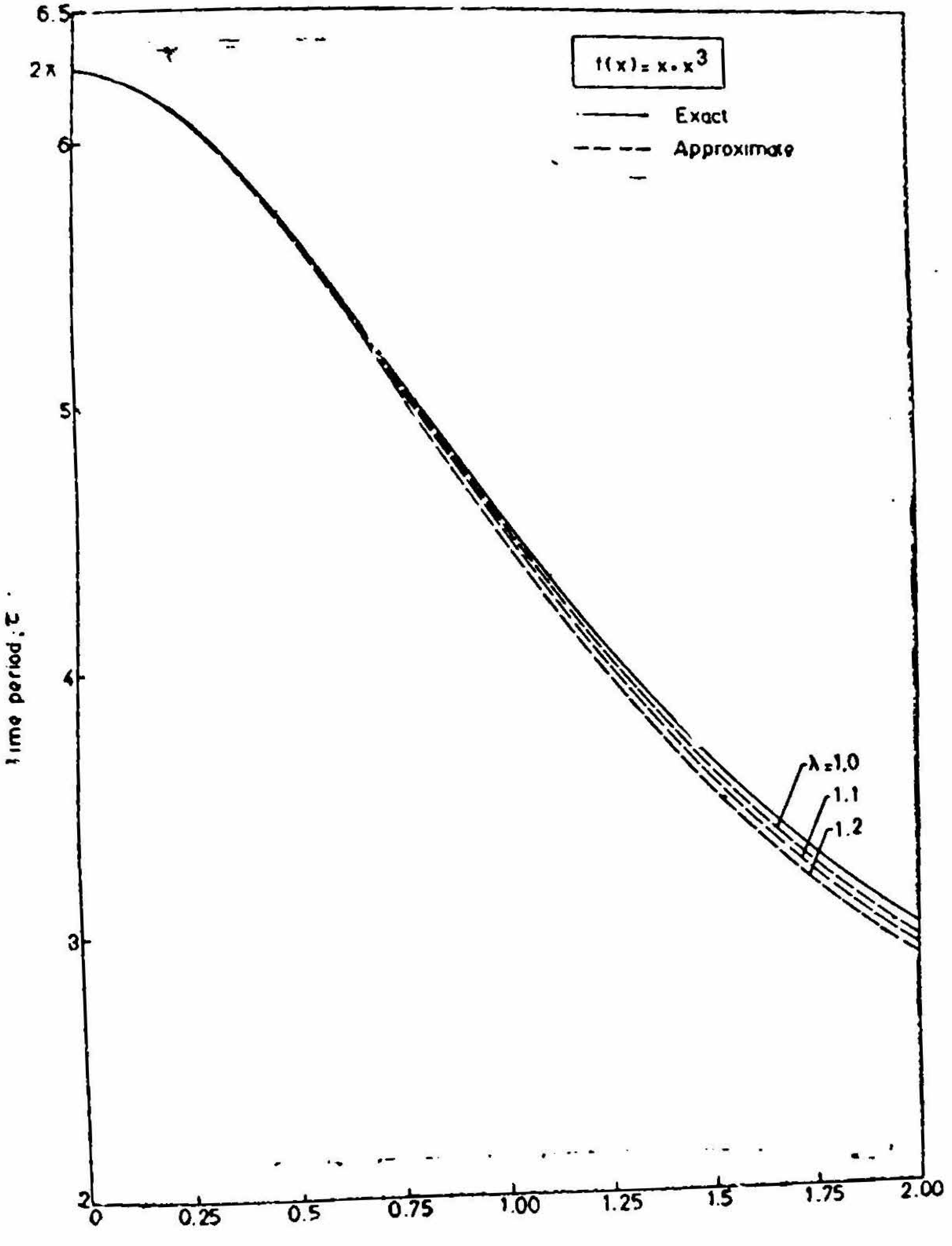
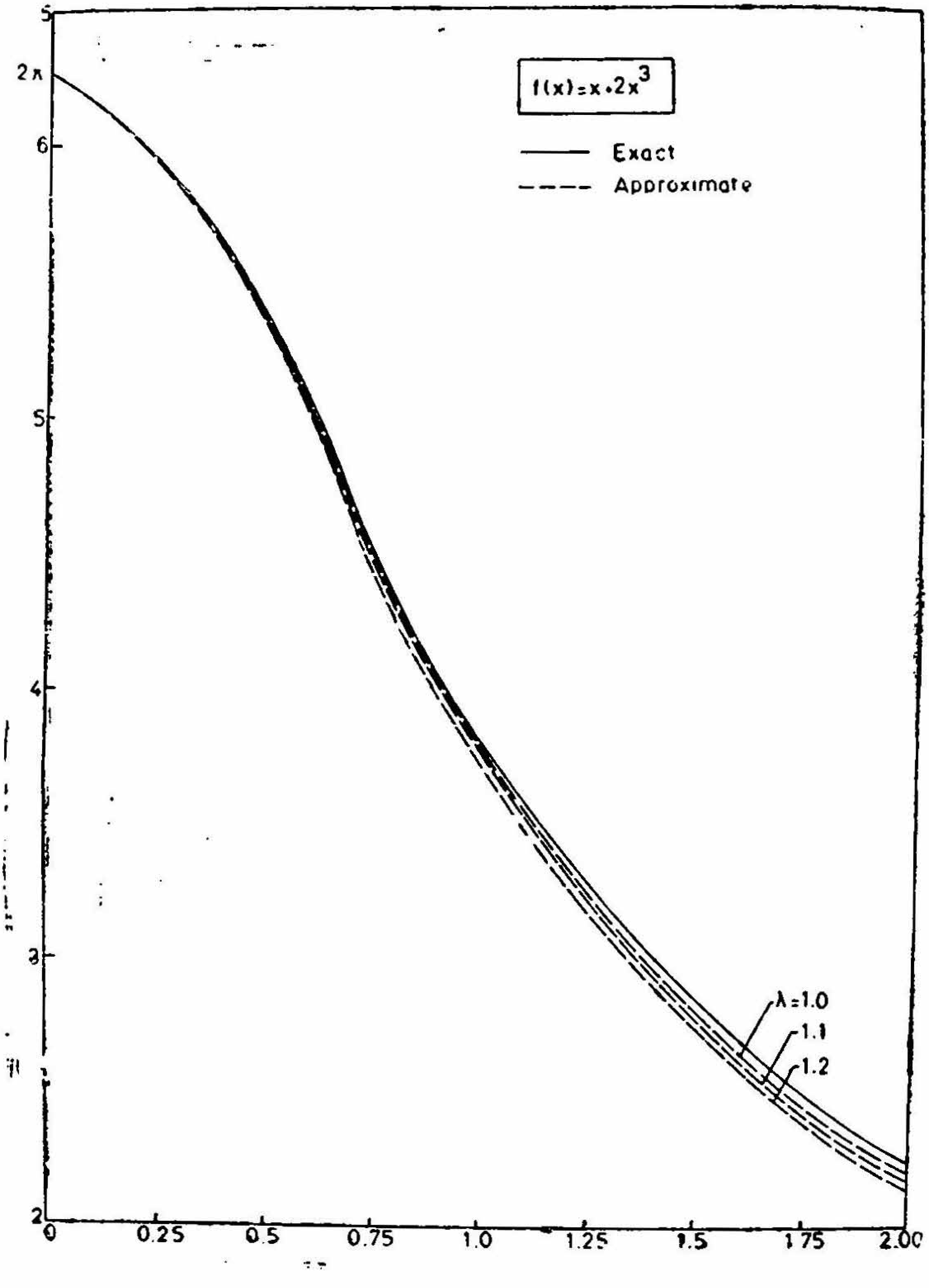

f(x) = x + x^3
Exact
Approximate
λ = 1.0
1.1
1.2
6.5
2π
6
5
4
3
2
0
0.25
0.5
0.75
1.00
1.25
1.5
1.75
2.00
Time period, τ
Amplitude, A

FIG. 2.



$f(x)=x+2x^3$
Exact
Approximate
2π
6
5
4
3
2
0
0.25
0.5
0.75
1.0
1.25
1.5
1.75
2.00
λ=1.0
1.1
1.2
Amplitude, A


FIG. 3.

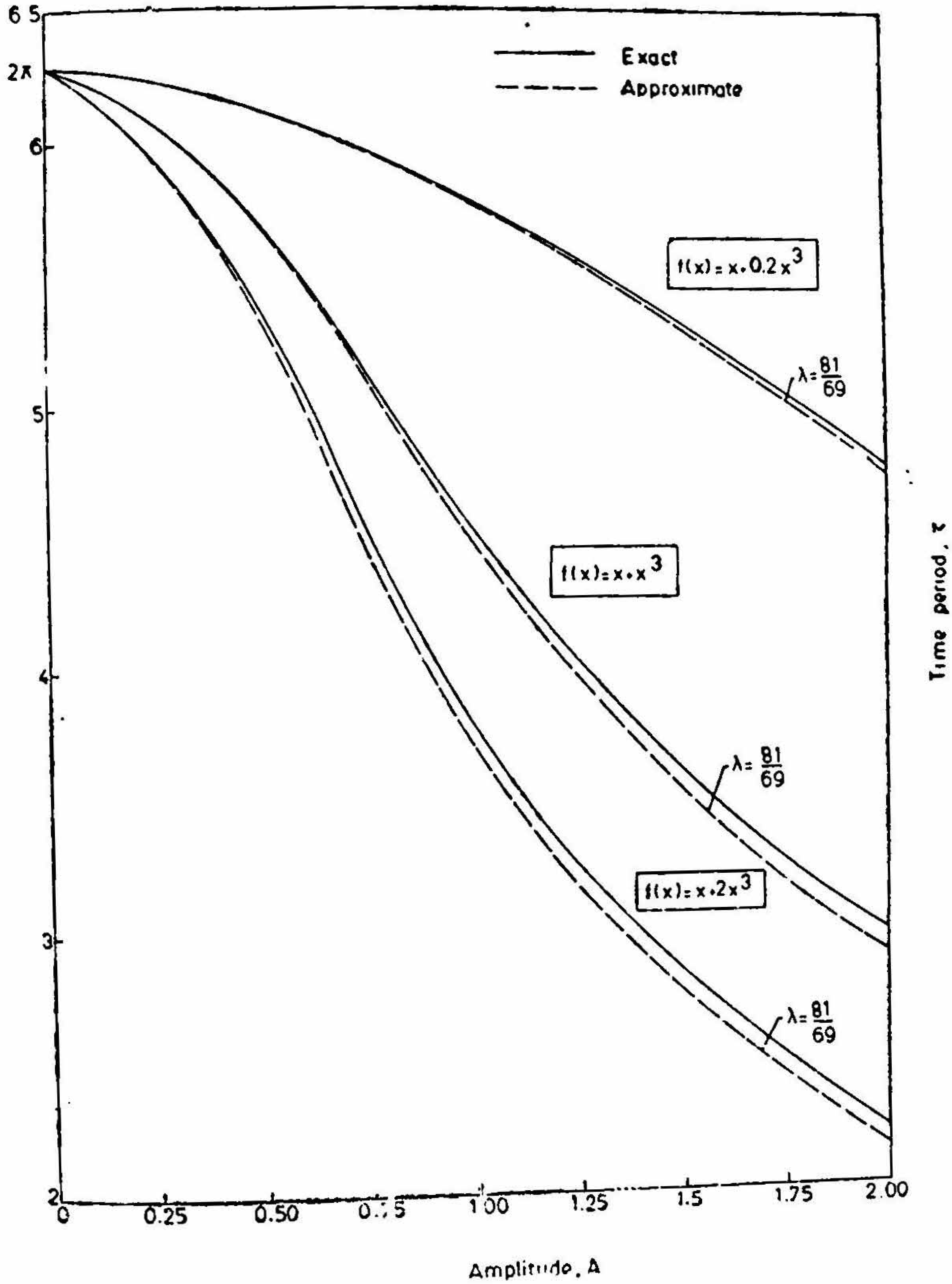

Exact
Approximate
f(x) = x + 0.2x³
λ = 81/69
f(x) = x + x³
λ = 81/69
f(x) = x + 2x³
λ = 81/69
6.5
2π
6
5
4
3
2
0
0.25
0.50
0.75
1.00
1.25
1.5
1.75
2.00
Amplitude, A
Time period, τ

FIG. 4.

the cases considered (Figs. 1,2 and 3). It can be seen that small changes in $\lambda$ will not affect the period very much for small amplitudes. It is wroth-while to attempt to get an expression connecting $\lambda$ and the amplitude. Then it might be poosible to attribute different values of $\lambda$ to different ranges of amplitude.

The method can be applied even if the restoring force characteristic contains even powers of the displacement and can be easily extended to systems if the damping is of viscous nature or coulomb nature. Approximate expressions for maximum displacement and the response time can be arrived at and can be compared with those obtained by other methods. This forms a part of the future work.

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