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RE-PRINTS

LARGE DEFLECTIONS OF SIMPLY SUPPORTED BEAMS

BY

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SUMMARY

Large deflections of simply supported beams have been studied when the transverse loading consists of a uniformly distributed load plus a centrally concentrated load under the two cases, (1) the reactions are vertical, (2) the reactions are normal to the bent beam together with frictional forces. The solutions are obtained by the use of power series expansions. This method is applicable to all cases of symmetrical loading in beams.

INTRODUCTION

In the classical theory of the deflection of beams we start with the Bernoulli-Euler relation connecting the bending moment in the beam with the curvature, but neglect all terms of degree higher than the first in the expression for the curvature. Results based on this simple theory however cannot be applied in the case of large deflections as they may lead to unconservative errors. Barten (1)[2] and Bisshopp and Drucker (2) provided large deflection solutions for the cantilever with a concentrated load at the free end. Large deflection solution for the simply supported beam with a central concentrated load was given by Conway (3) and some formulas for beam columns have been recently proved by Saelman (4). In all these cases, the equations are directly integrated leading to solutions expressible in terms of elliptic integrals. Rohde (5) has expanded the slope in a power series of the arc length to study the large deflections of a cantilever with uniformly distributed load. This method is found to be of wide applicability in the solution of large deflection problems and can be very well employed even in the problems treated in References 1–4.

In the present paper we are concerned with the large deflections of simply supported beams. Using the method of power series expansions, we study the deflections of beams carrying a uniformly distributed load plus a central concentrated load, in the following two cases:

(1) when the reactions at the ends are assumed vertical,
(2) when the reactions at the ends are assumed normal to the bent beam together with tangential frictional forces opposing deflections.

The problem in (3) is clearly a particular case of that treated here.

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[2] The boldface numbers in parentheses refer to the references appended to this paper.

We take beams of uniform cross-section. The following basic assumptions are made: (1) the deformations are elastic; (2) the bending does not alter the length of the beams; and (3) the supports are at a fixed distance apart and at the same horizontal level.

Case 1

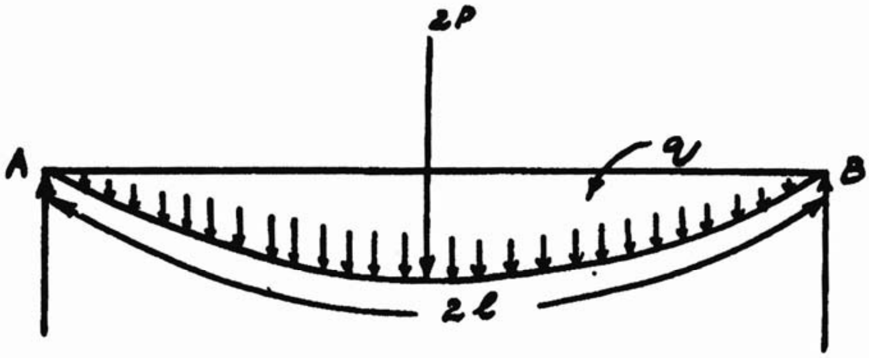

FIG. 1.

The Bernoulli-Euler relation, *viz.*,

$$\frac{1}{\rho} = \frac{d\phi}{ds} = \frac{M}{B}, \tag{1}$$

leads to

$$B\frac{d^2\phi}{ds^2} = \frac{dM}{ds} = [q(l - s) + P]\cos\phi \tag{2}$$

where $B = \frac{EI}{1 - \sigma^2}$, s = arc length measured from A, q = load intensity, $2P$ = central concentrated load, ϕ = inclination and $2l$ = length of the beam. The boundary conditions are

$$\begin{aligned} \phi &= 0 \quad \text{at} \quad s = l; \\ \frac{d\phi}{ds} &= 0 \quad \text{at} \quad s = 0, s = 2l; \\ \phi &= \alpha \quad \text{at} \quad s = 0. \end{aligned} \tag{3}$$

The nonlinear equation (2) does not lend itself to any direct solution. Put

$$\phi = a_0 + a_1 s + a_2 s^2 + \cdots. \tag{4}$$

From (3) we have

$$a_0 = \alpha, \quad a_1 = 0$$

so that

$$\phi = \alpha + \beta$$

where

$$\beta = a_2 s^2 + a_3 s^3 + \cdots. \tag{5}$$

We may write from (2)

$$B(2a_2 + 6a_3 s + \cdots) = [q(l - s) + P]\cos(\alpha + \beta). \tag{6}$$

Expanding $\cos\beta$, $\sin\beta$ in the right hand side and comparing coefficients, we find

$$a_2 = \frac{(ql + P)\cos\alpha}{2B}, \quad a_3 = -\frac{q\cos\alpha}{6B},$$

$$a_4 = \frac{(ql + P)^2 \sin\alpha\cos\alpha}{24B^2},$$

$$a_5 = \frac{q(ql + P)\sin\alpha\cos\alpha}{30B^2}.$$

$$\cdots\cdots\cdots$$

Using the condition $\phi = 0$ at $s = l$ and introducing the non-dimensional parameters

$$K_u = \frac{ql^3}{B}, \quad K_c = \frac{Pl^2}{B}, \tag{7}$$

$$-\alpha = \begin{cases} \dfrac{K_u\cos\alpha}{3} + \dfrac{K_c\cos\alpha}{2} - \dfrac{K_u^2\sin\alpha\cos\alpha}{72} \\ -\dfrac{K_uK_c\sin\alpha\cos\alpha}{20} - \dfrac{K_c^2\sin\alpha\cos\alpha}{24} + \cdots. \end{cases} \tag{8}$$

For a first approximation

$$\alpha = -\frac{K_u}{3} - \frac{K_c}{2}$$

which is in agreement with the small deflection theory. The maximum deflection y_m is given by

$$\begin{aligned} \frac{1}{l}y_m &= \frac{1}{l}\int_0^l \sin\phi\, ds = \frac{1}{l}\int_0^l (\sin\alpha\cos\beta + \cos\alpha\sin\beta)ds \\ &= \sin\alpha + \left(\frac{K_u}{8} + \frac{K_c}{6}\right)\cos^2\alpha \\ &\qquad + \left(-\frac{K_u^2}{60} - \frac{17}{360}K_uK_c - \frac{K_c^2}{30}\right)\sin\alpha\cos^2\alpha + \cdots. \end{aligned} \tag{9}$$

In the simple theory

$$\frac{y_{\max}}{l} = \frac{y_s}{l} = \alpha + \left(\frac{K_u}{8} + \frac{K_c}{6}\right) = -\frac{5}{24}K_u - \frac{K_c}{3}.$$

Choosing a definite ratio of $\frac{K_u}{K_c} = n$, we may determine K_u and K_c for different assumed values of α from Eq. 8 and y_m from Eq. 9. $\frac{y_m}{2l}, \frac{y_s}{2l}$ and $\frac{y_m}{y_s}$ can be plotted against K_u or K_c. Putting $K_c = 0$ in Eqs. 8 and 9, we have the solution for a beam with a uniformly distributed load. If $K_u = 0$, Eqs. 8 and 9 provide the solution for the case of a centrally concentrated load, treated in a quite different way in Conway's paper (3).

Case 2

When the forces at the ends consist of reactions normal to the bent beam R plus tangential forces $\mu R (\mu = \tan \lambda$ is the coefficient of friction), we have the case shown in Fig. 2. The resultant vertical and horizontal

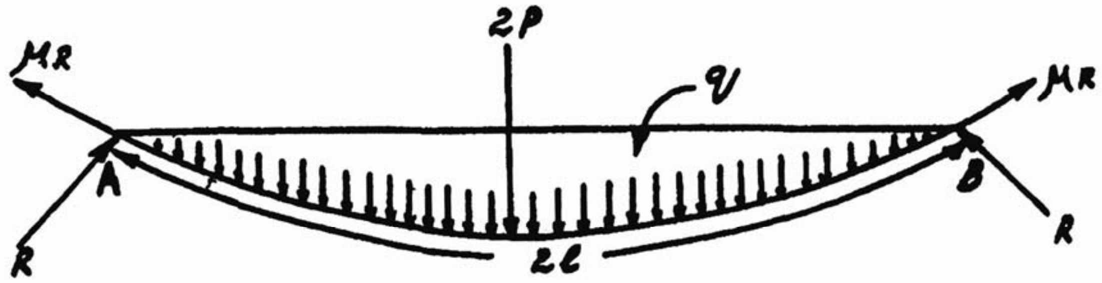

FIG. 2.

forces

$$ql + P = R\cos\gamma + \mu R \sin\gamma,$$
$$Q = R\sin\gamma - \mu R\cos\gamma,$$

giving

$$Q = (ql + P)\tan(\gamma - \lambda).$$

The bending moment equation leads to

$$B\frac{d^2\phi}{ds^2} = (ql + P - qs)\cos\phi + \tan(\gamma - \lambda)(ql + P)\sin\phi. \quad (10)$$

The boundary conditions are

$$\begin{aligned} \phi &= 0 \quad \text{at} \quad s = l; \\ \frac{d\phi}{ds} &= 0 \quad \text{at} \quad s = 0, s = 2l; \\ \phi &= \gamma \quad \text{at} \quad s = 0. \end{aligned} \quad (11)$$

Proceeding as before, put

$$\phi = b_0 + b_1 s + b_2 s^2 + \cdots.$$

Here $b_0 = \gamma$, $b_1 = 0$ so that

$$\phi = \gamma + \delta$$

where

$$\delta = b_2 s^2 + b_3 s^3 + \cdots.$$

Using the same procedure as in Case 1, we obtain

$$\phi = \begin{cases} \gamma + \dfrac{(ql + P)\cos\lambda}{2B\cos(\gamma - \lambda)} s^2 - \dfrac{q\cos\gamma}{6B} s^3 \\ \quad - \dfrac{(ql + P)^2 \sin\lambda\cos\lambda}{24B^2\cos^2(\gamma - \lambda)} s^4 + \cdots. \end{cases}$$

From the condition $\phi = 0$ at $s = l$ we have

$$-\gamma = K_u \left\{ \frac{\cos\lambda}{2\cos(\gamma - \lambda)} - \frac{\cos\gamma}{6} \right\} + K_c \left\{ \frac{\cos\lambda}{2\cos(\gamma - \lambda)} \right\}$$
$$+ K_u^2 \left\{ - \frac{\sin\lambda\cos\lambda}{24\cos^2(\gamma - \lambda)} + \frac{\cos\gamma\sin\lambda + 3\sin\gamma\cos\lambda}{120\cos(\gamma - \lambda)} - \frac{\sin\gamma\cos\gamma}{180} \right\} + \cdots.$$

The maximum deflection $y_m = \int_0^l \sin\phi \, ds$ leads to

$$\frac{y_m}{l} = \sin\gamma + K_u \left(\frac{\cos\gamma\cos\lambda}{6\cos(\gamma - \lambda)} - \frac{\cos^2\gamma}{24} \right) + K_c \cdot \frac{\cos\gamma\cos\lambda}{6\cos(\gamma - \lambda)} + \cdots.$$

In the simple theory, the maximum deflection y_s is given by

$$\frac{y_s}{l} = \gamma + K_u \left(\frac{\cos\lambda}{6(\cos\lambda + \gamma\sin\lambda)} - \frac{1}{24} \right) + \frac{K_c}{6} \left(\frac{\cos\lambda}{\cos\lambda + \gamma\sin\lambda} \right)$$

where γ is a root of the equation

$$-\gamma = \frac{(K_u + K_c)\cos\lambda}{2(\cos\lambda + \gamma\sin\lambda)} - \frac{K_u}{6}.$$

Proceeding as in Case 1, it is possible to compute the values of y_m for different values γ for any definite ratio of K_u, K_c. These values may be employed for plotting y_m against K_u or K_c.

Acknowledgment

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