Stress Concentration in Post-Tensioned Prestressed Concrete Beams

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The use of the multiple Fourier method to analyze the stress distribution in the end regions of a post-tensioned prestressed concrete beam has been shown. The multiple Fourier method demonstrated here is a relatively new method for solving those problems for which the "Saint Venant principle" is not applicable.

The actual three-dimensional problem and a two-dimensional simplified representation of it are treated. The two-dimensional case is treated first and rather completely to gain further experience with multiple Fourier procedures. The appropriate Galerkin Vector for the three-dimensional case is found and the required relations between the arbitrary functions are stated.

1. INTRODUCTION

In post-tensioned prestressed concrete beams the tension steel applies longitudinal compressive forces to the concrete. In general the resultant of the forces will be an axial force and a moment. The determination of concrete stresses at a considerable distance from either end of the beam is a simple problem in elementary mechanics but the critical stresses are in the end regions where elementary theory is not applicable.

No doubt the method of finite differences is the simplest from theoretical considerations for such problems and a start was made using this method. But the labour of computations became so great that the method was not considered feasible until one of the large digital computers became available. The multiple Fourier method used by the senior author on the problem of compression of a cylinder [Pickett (1944)] and on other problems has also been applied in this paper.

The first simplification was to treat the beam as infinite in length. This introduces only a negligible error for stresses in the end region if the length-depth and length-width ratios are, say, ten or more. The next simplification was to consider the problem as two-dimensional. Since this would probably cause considerable error it was done primarily to gain experience with multiple Fourier methods on a relatively simple problem preliminary to attacking the three-dimensional problem. The general procedure for the two-dimensional case will now be given.

2. Analysis

Two-dimensional Case

The differential equation for the case is

$$\nabla^2 \nabla^2 \phi = 0. \tag{1}$$

Expressions for stresses are

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 \phi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \tag{2}$$

The boundary conditions are

$$\left.\begin{aligned} \sigma_x\Big|_{x=0} &= -f(y), \qquad \tau_{xy}\Big|_{x=0} = 0 \\ \sigma_y\Big|_{y=\pm b} &= 0, \qquad \tau_{xy}\Big|_{y=\pm b} = 0 \\ \sigma_x\Big|_{\infty} &= -\frac{P}{2b} - \frac{3My}{2b^3} \\ \text{where} \\ P &= \int_{-b}^{b} f(y)\,dy, \quad M = \int_{-b}^{b} f(y)\,y\,dy. \end{aligned}\right\} \tag{3}$$

It may be verified by substitution that the following expression satisfies the differential equation and the boundary conditions for τ_{xy}.

$$\begin{aligned} \phi = -\frac{Py^2}{4b} &+ \sum_{1,2,3}^{\infty} \frac{A_n \cos\frac{n\pi y}{b}}{\left(\frac{n\pi}{b}\right)^2}\left[1 + \frac{n\pi x}{b}\right] e^{-\frac{n\pi x}{b}} \\ -\frac{My^3}{4b^3} &+ \sum_{1,3,5}^{\infty} \frac{B_m \sin\frac{m\pi y}{2b}}{\left(\frac{m\pi}{2b}\right)^2}\left[1 + \frac{m\pi x}{2b}\right] e^{-\frac{m\pi x}{2b}} \\ &+ \int_0^{\infty} \frac{C\cos ax}{a^2\cosh ab}\left[ay\sinh ay - (1 + ab\coth ab)\cosh ay\right] da \\ &+ \int_0^{\infty} \frac{D\cos ax}{a^2\sinh ab}\left[ay\cosh ay - (1 + ab\tanh ab)\sinh ay\right] da \end{aligned} \tag{4}$$

It will be observed that the first and third lines of the expression are even functions of y and are for the symmetrical part of the loading. The second and fourth lines of the expression are odd functions of y and are for the anti-symmetrical part of the loading. These two parts can be treated separately since this is a linear problem and superposition can be applied. Detail results for symmetrical loading have been reported by the junior author in a recent paper. [Iyengar (1955).]

The two series [the A-series in line 1 and the B-series in line 2 of Equation (4)] are for the purpose of expressing the difference in σ_x at $x = 0$ and at $x \to \infty$ by Fourier series. Unfortunately both these series give a boundary stress σ_y at $y = \pm b$. The two integrals (lines 3 and 4) are for the purpose of removing any σ_y stress that appears at $y = \pm b$; the integral with the C-function removes the boundary stress produced by the A-series and the D-function removes those produced by the B-series. This is done by arbitrarily setting $\sigma_y = 0$ at $y = \pm b$ giving the following equations :

$$\sum_{1,2}^{\infty} \left(-1\right)^{n} A_n \left[1 - \frac{n\pi x}{b}\right] e^{-\frac{n\pi x}{b}} - \int_0^{\infty} C \cos ax \left[1 + \frac{2ab}{\sinh 2ab}\right] da = 0 \quad (5)$$

$$\sum_{1,3,5}^{\infty} \left(-1\right)^{\frac{m-1}{2}} B_m \left[1 - \frac{m\pi x}{2b}\right] e^{-\frac{m\pi x}{2b}} - \int_0^{\infty} D \cos ax \left[1 - \frac{2ab}{\sinh 2ab}\right] da = 0 \quad (6)$$

Expressions for the C and D functions are obtained by taking the Fourier cosine transforms of these equations

$$\left[1 + \frac{2ab}{\sinh 2ab}\right] C = \frac{4b}{\pi} \sum_{1,2,3}^{\infty} \left(-1\right)^{n} A_n \frac{(n\pi)\,(ab)^2}{[(ab)^2 + (n\pi)^2]^2} \quad (7)$$

$$\left[1 - \frac{2ab}{\sinh 2ab}\right] D = \frac{4b}{\pi} \sum_{1,3,5}^{\infty} \left(-1\right)^{\frac{m-1}{2}} B_m \frac{\left(\frac{m\pi}{2}\right) \left(ab\right)^2}{\left[\left(ab\right)^2 + \left(\frac{m\pi}{2}\right)^2\right]^2} \quad (8)$$

If the C and D functions are given by these expressions then the boundary conditions at $y = \pm b$ will be satisfied but the integrals with these functions give a contribution to σ_x at $x = 0$ making it necessary for the A and B series to remove this difference as well as provide the required difference between σ_x at $x = 0$ and $x \to \infty$.

Setting $\sigma_x = -f(y)$ at $x = 0$ gives

$$\sum_{1,2,3}^{\infty} A_n \cos \frac{n\pi y}{b} + \sum_{1,3,5}^{\infty} B_m \sin \frac{m\pi y}{2b} = f(y) - \frac{P}{2b} - \frac{3}{2} \frac{My}{b^3}$$

$$+ \int_0^\infty [\, ay \sinh ay + (1 - ab \coth ab)\cosh ay \,] \frac{C da}{\cosh ab}$$

$$+ \int_0^\infty [\, ay \cosh ay + (1 - ab \tanh ab)\sinh ay \,] \frac{D da}{\sinh ab} \tag{9}$$

Expressions for A_n and B_m are found by taking the finite Fourior cosine and sine transforms respectively of this equation. The result is

$$A_n = I_n + (-1)^n 4 \int_0^\infty \frac{ab\,(n\pi)^2 \tanh ab}{[(ab)^2 + (n\pi)^2]^2} C da \tag{10}$$

$$B_m = I_m + (-1)^{\frac{m-1}{2}} 4 \int_0^\infty \frac{ab \left(\frac{m\pi}{2}\right)^2 \coth ab}{\left[\left(ab\right)^2 + \left(\frac{m\pi}{2}\right)^2\right]^2} D da \tag{11}$$

where

$$I_n = \frac{1}{b}\int_{-b}^{b} \left[f(y) - \frac{P}{2b} \right] \cos \frac{n\pi y}{b} dy \tag{12}$$

$$I_m = \frac{1}{b}\int_{-b}^{b} \left[f(y) - \frac{3My}{2b^3} \right] \sin \frac{m\pi y}{2b}\, dy \tag{13}$$

Equation (7) expressing C in terms of the A's and equation (10) expressing A_n in terms of C and the loading $f(y)$ are theoretically sufficient for the evaluation of each A_n and the complete determination of the C-function. The method has been explained more fully by Iyengar (1955). In like manner equations (8) and (11) are sufficient for the evaluation of each B_m and the determination of the D-function. The detailed results of this evaluation will be reported in a later paper. However, a few more steps in the procedure will be given here.

The substitution of C from equation (7) into equation (10) gives

$$A_n = I_n + 16\pi^2 n^2 \sum_{1,2,3}^{\infty} (-1)^{s+n} s A_s K(s,n) \tag{14}$$

where

$$K(s,n) = \int_0^\infty \frac{x^3 \tanh x dx}{\left[1 + \frac{2x}{\sinh 2x}\right]\left[x^2 + (s\pi)^2\right]^2 \left[x^2 + (n\pi)^2\right]^2} \tag{15}$$

In like manner substitution of D from equation (8) into equation (11) gives

$$B_m = I_m + 16\pi^2 \left(\frac{m}{2}\right)^2 \sum_{1,3,5}^{\infty} \left(-1\right)^{\frac{r+m-2}{2}} \left(\frac{r}{2}\right) B_r L\left(r,m\right) \tag{16}$$

where

$$L(r,m) = \int_0^\infty \frac{x^3 \coth x \, dx}{\left[1 - \frac{2x}{\sinh 2x}\right] \left[x^2 + \left(\frac{r\pi}{2}\right)^2\right]^2 \left[x^2 + \left(\frac{m\pi}{2}\right)^2\right]^2} \tag{17}$$

The integrals K and L may be evaluated in the following way :

$$K(s,n) = J(s,n) - \int_0^\infty \left[1 - \frac{\tanh x}{1 + \frac{2x}{\sinh 2x}}\right] \frac{x^3 \, dx}{[x^2 + (s\pi)^2]^2 [x^2 + (n\pi)^2]^2} \tag{18}$$

$$L(r,m) = J\left(\frac{r}{2}, \frac{m}{2}\right) - \int_0^\infty \left[1 - \frac{\coth x}{1 - \frac{2x}{\sinh 2x}}\right] \frac{x^3 \, dx}{\left[x^2 + \left(\frac{r\pi}{2}\right)^2\right]^2 \left[x^2 + \left(\frac{m\pi}{2}\right)^2\right]^2} \tag{19}$$

where

$$J(g,h) = \int_0^\infty \frac{x^3 \, dx}{[x^2 + (g\pi)^2]^2 \, [x^2 + (h\pi)^2]^2} \tag{20}$$

$$J(g,h) = \frac{1}{\pi^4 [g^2 - h^2]^2} \left[\frac{g^2 + h^2}{g^2 - h^2} \log \frac{g}{h} - 1\right] \tag{21}$$

The integrals in equations (18) and (19) are readily evaluated numerically because of the rapid convergence of the integrands to zero with increase of x. However, for good accuracy in the final determination of stresses throughout the end region of the prestressed beam a large number of these integrals may be needed. If the number of terms used in the A-series is p then $p(p+1)/2$ different combinations of s and n in equation (18) are required. With the required number of $K(s,n)$ evaluated from equation (18) then as many equations as terms in the A-series are required, may be written from equation (14). These may be readily solved. The B-series may be treated in like manner using equation (16) after the required number of $L(r,m)$ have been determined from equation (19).

With the required number of A's and B's determined the stresses may be evaluated for any point desired. For example from equations (2), (3), (7) and (8) the stress σ_x is given by

$$\sigma_x = -\frac{P}{2b} - \sum_{1,2,3}^{\infty} A_n \left[\cos \frac{n\pi y}{b} \left(1 + \frac{n\pi x}{b}\right) e^{\frac{-n\pi x}{b}} - n(-1)^n F_n\right]$$

$$-\frac{3My}{2b^3} - \sum_{1,3,5}^{\infty} B_m \left[\sin \frac{m\pi y}{2b} \left(1 + \frac{m\pi x}{2b}\right) e^{\frac{-m\pi x}{2b}} - \left(\frac{m}{2}\right) (-1)^{\frac{m-1}{2}} G_m\right] \tag{22}$$

where

$$F_n = 4b^3 \int_0^\infty \frac{[ay \sinh ay + (1 - ab \coth ab) \cosh ay]\, a^2 \cos a x da}{[(ab)^2 + (n\pi)^2]^2 [\cosh ab + ab/\sinh ab]} \tag{23}$$

$$G_m = 4b^3 \int_0^\infty \frac{[ay \cosh ay + (1 - ab \tanh ab) \sinh ay]\, a^2 \cos a x da}{\left[(ab)^2 + \left(\frac{m\pi}{2}\right)^2 \right]^2 \left[\sinh ab - ab/\cosh ab\right]} \tag{24}$$

The integrands in equations (23) and (24) converge to zero sufficiently rapidly for numerical integration to be possible, especially if $| y | < b$.

Three-dimensional Case

The differential equation is

$$\nabla^2 \nabla^2 F = O \tag{25}$$

where

$$F = i\, F_x + j\, F_y + k\, F_z. \tag{26}$$

Expressions for stresses are given by (Westergaard (1952))

$$\sigma_x = 2(1-\mu)\frac{\partial}{\partial x}\nabla^2 F_x + \left(\mu \nabla^2 - \frac{\partial^2}{\partial x^2}\right)\nabla . F \tag{27}$$

$$\sigma_y = 2(1-\mu)\frac{\partial}{\partial y}\nabla^2 F_y + \left(\mu \nabla^2 - \frac{\partial^2}{\partial y^2}\right)\nabla . F \tag{28}$$

$$\sigma_z = 2(1-\mu)\frac{\partial}{\partial z}\nabla^2 F_z + \left(\mu \nabla^2 - \frac{\partial^2}{\partial z^2}\right)\nabla . F \tag{29}$$

$$\tau_{xy} = (1-\mu)\left(\frac{\partial}{\partial y}\nabla^2 F_x + \frac{\partial}{\partial x}\nabla^2 F_y\right) - \frac{\partial^2}{\partial x \partial y}\nabla . F \tag{30}$$

$$\tau_{yz} = (1-\mu)\left(\frac{\partial}{\partial z}\nabla^2 F_y + \frac{\partial}{\partial y}\nabla^2 F_z\right) - \frac{\partial^2}{\partial y \partial z}\nabla . F \tag{31}$$

$$\tau_{zx} = (1-\mu)\left(\frac{\partial}{\partial x}\nabla^2 F_z + \frac{\partial}{\partial z}\nabla^2 F_x\right) - \frac{\partial^2}{\partial z \partial x}\nabla . F \tag{32}$$

The boundary conditions are

$$\begin{aligned}
&\text{at } x = 0, && \sigma_x = 0 - f(x, y), && \tau_{xy} = 0, && \tau_{xz} = 0 \\
&\text{at } y = \pm b, && \sigma_y = 0, && \tau_{xy} = 0, && \tau_{yz} = 0 \\
&\text{at } z = \pm c, && \sigma_z = 0, && \tau_{xz} = 0, && \tau_{yz} = 0 \\
&\text{at } x \to \infty, && \sigma_x = -\frac{P}{4bc} - \frac{3M_1 y}{2cb^3} - \frac{3M_2 z}{2bc^3}
\end{aligned} \tag{33}$$

where

$$P = \int_{-b}^{b} \int_{-c}^{c} f\,dy\,dz, \quad M_1 = \int_{-b}^{b} \int_{-c}^{c} fy\,dy\,dz, \quad M_2 = \int_{-b}^{b} \int_{-c}^{c} fz\,dy\,dz \tag{34}$$

It may be verified by substitution that the following expressions satisfy the differential equation and the boundary conditions for the shear stresses :

$$F_x = \Sigma\,\Sigma\, A_{mn} \cos\frac{m\pi z}{c} \cos\frac{n\pi y}{b} [2\mu + \gamma_{mn}\, x]\, \gamma_{mn}^{-3}\, e^{-\gamma_{mn} x} \tag{35}$$

$$F_y = \Sigma \int_0^{\infty} \frac{bB_m\, \beta^{-3} \cos\frac{m\pi z}{c} \cos ax}{\cosh \beta b} [\beta y \cosh \beta y - (2\mu + \beta b \coth \beta b) \sinh \beta y]\, da \tag{36}$$

$$F_z = \Sigma \int_0^{\infty} \frac{cC_n\, \gamma^{-3} \cos\frac{n\pi y}{b} \cos ax}{\cosh \gamma c} [\gamma z \cosh \gamma z - (2\mu + \gamma c \coth \gamma c) \sinh \gamma z]\, da \tag{37}$$

where

$$\gamma_{mn} = \sqrt{\left(\frac{m\pi}{c}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

$$\beta = \sqrt{a^2 + \left(\frac{m\pi}{c}\right)^2}$$

$$\gamma = \sqrt{a^2 + \left(\frac{n\pi}{b}\right)^2}$$

The foregoing takes care of only the variation of the symmetrical part of the applied load. A function must be added which will satisfy the relation,

$$\sigma_x = -\frac{P}{4bc} - \frac{3M_1 y}{2cb^3} - \frac{3M_2 z}{2bc^3}$$

and will not contribute to the other stresses. Functions similar to equations (35), (36) and (37) must be superimposed to take care of the unsymmetrical part of the applied load.

It will be observed that F_x is for the purpose of giving the proper value of σ_x at $x = 0$,

F_y to make $\sigma_y = 0$ at $y = \pm b$ and
F_z to make $\sigma_z = 0$ at $z = \pm c$

3. Conclusions

The necessary theoretical formulas have been obtained for the mathematical solution for stresses in both the two-dimensional and the three-dimensional problems of post-tensioned prestressed concrete.

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References

Iyengar, K. T. S. (1955) : *On a two-dimensional problem in the end-block design of post-tensioned prestressed concrete beams. First Congress on Theoretical and Applied Mechanics, Kharagpur, Proceedings* P. 107

Pickett, G. (1944) : *Application of the Fourier method to the solution of certain boundary problems in the theory of elasticity. Journal of Applied Mechanics*, September, 1944, Vol. II, A-176.

Westergaard. (1952) : *Theory of elasticity and plasticity. New York : John Wiley & Sons, Inc.*