2nd Congress on Theoretical and Applied Mechanics, Delhi, 87- 94, (1956)
Stress Analysis of Deep Beams
K. T. Sundara Raja Iyengar

ANALYSIS OF DEEP BEAMS

By

K. T. SUNDARA RAJA IYENGAR

INDIAN INSTITUTE OF SCIENCE, BANGALORE, INDIA

ABSTRACT : In a beam whose depth is comparable to its span, the distribution of bending and shear stresses differs appreciably from those given by the ordinary flexural theory. In this paper, a general solution for the onalysis of a rectangular, single-span beam, under symmetrical loading is developed. The Multiple Fourier procedure is employed, using four series by which it has been possible to satisfy all boundary conditions and the resulting relations among the co-efficients are derived.

1. Introduction :

Beams whose depths are comparable to their spans often arise in many practical constructions, such as the walls of bunkers, foundation walls and in Resnforced Concrete hipped-plate construction, etc. The elementary theory of flexure fails to give the correct distribution of bending and shear stresses in such a beam, Continuous beams of this type have been analysed by F. Dischinger[1] and on this is based the information published by Portland Cement Association[2]. Recently some work has been published on the analysis of stresses in single span beams. Li Chow, Conway and Winter[3] have analysed single span beams under different loadings. By treating this as a plane-stress problem in Elasticity, they have employed the method of finite diferences to get the stress function. The same method has been used by Uhlmann[4] who employs Richardson's method of successive approximation[5] to solve the several equations. Guzman and Luisoni have applied Galerkin's variational method to obtain an approximate solution to the same problem[6].

2. Method of Solution :

A simply supported beam under a typical loading is shown in Figure 1.

It is assumed that conditions are such as to permit a two-dimensional analysis. Then we have to determine a stress function ϕ satisfying the equation (neglecting body forces).

$$\nabla^4\phi = 0 \qquad \ldots \qquad (1)$$

and the stresses are given by

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2},\ \sigma_y = \frac{\partial^2\phi}{\partial x^2},\ \tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} \qquad \ldots \qquad (2)$$

The prescribed boundary conditions are

$$\begin{aligned} \text{on} \quad & y=b, \ \sigma_y=f_1(x) \text{ [given]} \\ & y=-b, \ \sigma_y=f_2(x) \text{ [given]} \\ & x=\pm a, \ \sigma_x=0 \\ \text{and} \quad & \left.\begin{array}{l} x=\pm a \\ y=\pm b \end{array}\right\} \quad \tau_{xy}=0 \end{aligned} \qquad \ldots \quad (3)$$

The main difficulty in this kind of problem is to satisfy all the boundary conditions. By using the usual Fourier Series solution the boundary conditions at the top and bottom edges can be satisfied and the boundary conditions for σ_x at the two vertical edges are not satisfied. Conway and others[7] have used another stress function to eliminate the normal stresses on the vertical edges due to the first stress function. The second stress function is got by employing the principle of least work.

An exact solution for the Equation (1) can be given by employing the Multiple Fourier Method used by Pickett[8] on the problem of Compression of a Cylinder and on other problems. The same method has been used by the author for a two dimensional problem in the design of prestressed concrete[9,10] It is possible, by using this method to satisfy all the boundary conditions.

It may be verified by substitution that the following expression for ϕ satisfies the differential equation (1) and the boundary conditions for τ_{xy}.

$$\begin{aligned} \phi = -\frac{Px^2}{2a} &+ \sum_{m=1.2.3..}^{\infty} \frac{\cos\frac{m\pi x}{a}}{\left(\frac{m\pi}{a}\right)^2}\Bigg[\frac{Am}{\cosh\frac{m\pi b}{a}}\left\{\frac{m\pi y}{a}\sinh\frac{m\pi y}{a} - \left(1+\frac{m\pi b}{a}\coth\frac{m\pi b}{a}\right)\cosh\frac{m\pi y}{a}\right\} \\ &+ \frac{B_m}{\sinh\frac{m\pi b}{a}}\left\{\frac{m\pi y}{a}\cosh\frac{m\pi y}{a} - \left(1+\frac{m\pi b}{a}\tanh\frac{m\pi b}{a}\right)\sinh\frac{m\pi y}{a}\right\}\Bigg] \\ &+ \sum_{n=1.2.3...}^{\infty} \frac{\cos\frac{n\pi y}{b}}{\left(\frac{n\pi}{b}\right)^2} \frac{C_n}{\cosh\frac{n\pi a}{b}}\left[\frac{n\pi x}{b}\sinh\frac{n\pi x}{b} - \left(1+\frac{n\pi a}{b}\coth\frac{n\pi a}{b}\right)\cosh\frac{n\pi x}{b}\right] \\ &+ \sum_{N=1.3.5...}^{\infty} \frac{\sin\frac{N\pi y}{2b}}{\left(\frac{N\pi}{2b}\right)^2} \frac{DN}{\cosh\frac{N\pi a}{2b}}\left[\frac{N\pi x}{2b}\sinh\frac{N\pi x}{2b} - \left(1+\frac{N\pi a}{2b}\coth\frac{N\pi a}{2b}\right)\cosh\frac{N\pi x}{2b}\right] \end{aligned} \qquad \ldots \quad (4)$$

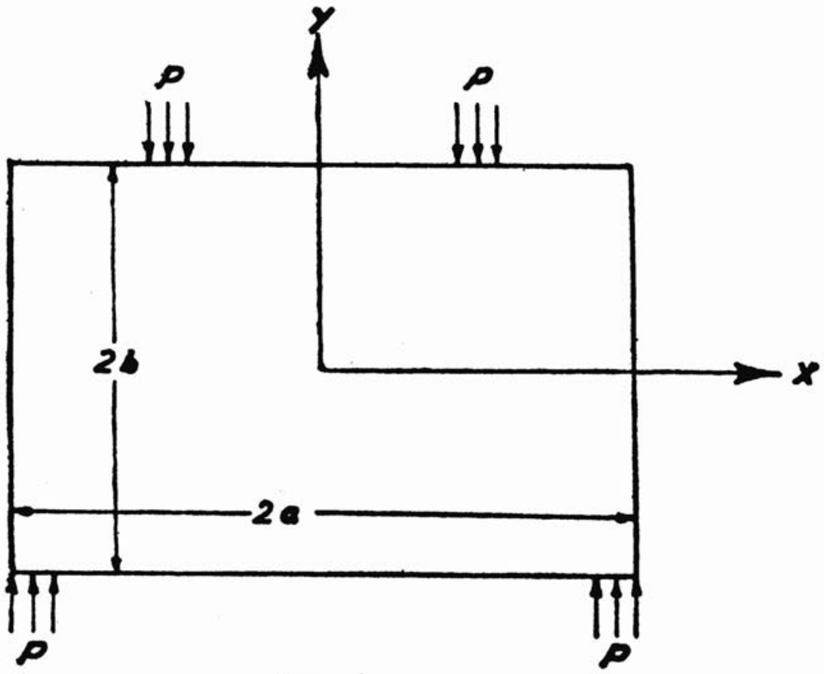

FIG. 1

and the stress components are given by

$$\sigma_x = \sum_{m=1,2,3\ldots}^{\infty} \cos\frac{m\pi x}{a}\left[\frac{A_m}{\cosh\frac{m\pi b}{a}}\left\{\frac{m\pi y}{a}\sinh\frac{m\pi y}{a} + \left(1 - \frac{m\pi b}{a}\coth\frac{m\pi b}{a}\right)\cosh\frac{m\pi y}{a}\right\}\right.$$

$$\left. + \frac{B_m}{\sinh\frac{m\pi b}{a}}\left\{\frac{m\pi y}{a}\cosh\frac{m\pi y}{a} + \left(1 - \frac{m\pi b}{a}\tanh\frac{m\pi b}{a}\right)\sinh\frac{m\pi y}{a}\right\}\right]$$

$$- \sum_{n=1,2,3\ldots}^{\infty} \cos\frac{n\pi y}{b}\,\frac{C_n}{\cosh\frac{n\pi a}{b}}\left[\frac{n\pi x}{b}\sinh\frac{n\pi x}{b} - \left(1 + \frac{n\pi a}{b}\coth\frac{n\pi a}{b}\right)\cosh\frac{n\pi x}{b}\right]$$

$$- \sum_{n=1,3,5\ldots}^{\infty} \sin\frac{n\pi y}{2b}\,\frac{D_n}{\cosh\frac{n\pi a}{2b}}\left[\frac{n\pi x}{2b}\sinh\frac{n\pi x}{2b} - \left(1 + \frac{n\pi a}{2b}\coth\frac{n\pi a}{2b}\right)\cosh\frac{n\pi x}{2b}\right] \qquad \ldots (5)$$

I2

$$\sigma_y = -\frac{P}{a} - \sum_{m=1.2.3...}^{\infty} \cos\frac{m\pi x}{a}\left[\frac{Am}{\cosh\frac{m\pi b}{a}}\left\{\frac{m\pi y}{a}\sinh\frac{m\pi y}{a} - \left(1+\frac{m\pi b}{a}\coth\frac{m\pi b}{a}\right)\cosh\frac{m\pi y}{a}\right\}\right.$$

$$\left. + \frac{Bm}{\sinh\frac{m\pi b}{a}}\left\{\frac{m\pi y}{a}\cosh\frac{m\pi y}{a} - \left(1+\frac{m\pi b}{a}\tanh\frac{m\pi b}{a}\right)\sinh\frac{m\pi y}{a}\right\}\right]$$

$$+\sum_{n=1.2.3...}^{\infty} \cos\frac{n\pi y}{b}\ \frac{C_n}{\cosh\frac{n\pi a}{b}}\left[\frac{n\pi x}{b}\sinh\frac{n\pi x}{b} + \left(1-\frac{n\pi a}{b}\coth\frac{n\pi a}{b}\right)\cosh\frac{n\pi x}{b}\right]$$

$$+\sum_{n=1.3.5...}^{\infty} \sin\frac{n\pi y}{2b}\ \frac{D_n}{\cosh\frac{n\pi a}{2b}}\left[\frac{n\pi x}{2b}\sinh\frac{n\pi x}{2b} + \left(1-\frac{n\pi a}{2b}\coth\frac{n\pi a}{2b}\right)\cosh\frac{n\pi x}{2b}\right] \quad \ldots \quad (6)$$

and

$$\tau_{xy} = \sum_{m=1.2.3...}^{\infty} \sin\frac{m\pi x}{a}\left[\frac{Am}{\cosh\frac{m\pi b}{a}}\left\{\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\right.\right.$$

$$\left. - \frac{m\pi b}{a}\coth\frac{m\pi b}{a}\sinh\frac{m\pi y}{a}\right\}$$

$$\left. + \frac{Bm}{\sinh\frac{m\pi b}{a}}\left\{\frac{m\pi y}{a}\sinh\frac{m\pi y}{a} - \frac{m\pi b}{a}\tanh\frac{m\pi b}{a}\cosh\frac{m\pi y}{a}\right\}\right]$$

$$+\sum_{n=1.2.3....}^{\infty} \sin\frac{n\pi y}{b}\ \frac{Cn}{\cosh\frac{n\pi a}{b}}\left[\frac{n\pi x}{b}\cosh\frac{n\pi x}{b} - \frac{n\pi a}{b}\coth\frac{n\pi a}{b}\sinh\frac{n\pi x}{b}\right]$$

$$-\sum_{n=1.3.5....}^{\infty} \cos\frac{n\pi y}{2b}\ \frac{D_n}{\cosh\frac{n\pi a}{2b}}\left[\frac{n\pi x}{2b}\cosh\frac{n\pi x}{2b} - \frac{n\pi a}{2b}\coth\frac{n\pi a}{2b}\sinh\frac{n\pi x}{2b}\right] \quad \ldots \quad (7)$$

The above solution refers to the case when the loading function on the beam is symmetrical in x. Similar solution can also be obtained for an unsymmetrical case.

It will be observed that the A and B series are chosen so as to give the prescribed normal loading on $y = \pm b$. Both of these series will give a boundary stress σ_x on $x = \pm a$. The two other series that is C and D series are chosen for the purpose of removing these stresses on $x = \pm a$, the C-series remove the boundary stress produced by the A-series and the D-series remove those produced by the B-series. This is done by arbitrarily setting $\sigma_x = 0$ on $x = \pm a$, giving the following equation,

$$\sum_{m=1.2.3...}^{\infty} \cos m\pi \left[\frac{A_m}{\cosh\frac{m\pi b}{a}} \left\{ \sinh\frac{m\pi y}{a} \frac{m\pi y}{a} + \left(1 - \frac{m\pi b}{a}\coth\frac{m\pi b}{a}\right)\cosh\frac{m\pi y}{a} \right\} \right.$$

$$\left. + \frac{B_m}{\sinh\frac{m\pi b}{a}} \left\{ \frac{m\pi y}{a}\cosh\frac{m\pi y}{a} + \left(1 - \frac{m\pi b}{a}\tanh\frac{m\pi b}{a}\right)\sinh\frac{m\pi y}{a} \right\} \right]$$

$$\pm \sum_{n=1.2.3..}^{\infty} \cos\frac{n\pi y}{b}\, C_n \left[1 + \frac{2n\pi\frac{a}{b}}{\sinh 2n\pi\frac{a}{b}}\right]$$

$$+ \sum_{n=1.3.5...}^{\infty} \sin\frac{n\pi y}{2b}\, D_n \left[1 + \frac{n\pi\frac{a}{b}}{\sinh\frac{n\pi a}{b}}\right] = 0 \qquad \dots \quad (8)$$

By taking the finite Fourier transform of this equation we have the two relations,

$$\left[1 + \frac{2n\pi\frac{a}{b}}{\sinh 2n\pi\frac{a}{b}}\right] C_n = \frac{1}{b}\sum_m \int_{-b}^{b} (-1)^{m-1} \frac{A_m}{\cosh\frac{m\pi b}{a}} \left\{ \frac{m\pi y}{a}\sinh\frac{m\pi y}{a} \right.$$

$$\left. + \left(1 - \frac{m\pi b}{a}\coth\frac{m\pi b}{a}\right)\cosh\frac{m\pi y}{a} \right\} \cos\frac{n\pi y}{a}\, dy \qquad \dots \quad (9)$$

and

$$\left[1 + \frac{n\pi\frac{a}{b}}{\sinh n\pi\frac{a}{b}}\right] D_n = \frac{1}{b}\sum_m \int_{-b}^{b} (-1)^{m-1} \frac{B_m}{\sinh\frac{m\pi b}{a}} \left\{ \frac{m\pi y}{a}\cosh\frac{m\pi y}{a} \right.$$

$$\left. + \left(1 - \frac{m\pi b}{a}\tanh\frac{m\pi b}{a}\right)\sinh\frac{m\pi y}{a} \right\} \sin\frac{n\pi y}{2b}\, dy \qquad \dots \quad (10)$$

After simplification, we have

$$\left[1 + \frac{2n\pi\frac{a}{b}}{\sinh 2n\pi\frac{a}{b}}\right] C_n + \sum_{m=1.2.3...}^{\infty} (-1)^{m+n} \frac{4}{\pi}\left(\frac{a}{b}\right)^3 \frac{mn^2 \tanh m\pi\frac{b}{a}}{\left[m^2 + \left(\frac{na}{b}\right)^2\right]^2} A_m = 0 \qquad \dots \quad (11)$$

$$\left[1+\frac{n\pi\frac{a}{b}}{\sinh n\pi\frac{a}{b}}\right]D_n+\sum_{m=1}^{\infty}(-1)^{m+(n-1)/2}\ \frac{4}{\pi}\left(\frac{a}{b}\right)^3\frac{m\left(\frac{n}{2}\right)^2\coth\frac{m\pi b}{a}}{\left[m^2+\left(\frac{an}{2b}\right)^2\right]^2}\ B_m=0 \qquad \ldots \quad (12)$$

Similarly by taking the boundary conditions on $y=\pm b$, that is,

$$\sigma_y=f_1(x) \quad \text{on} \quad y=b$$
$$=f_2(x) \quad \text{on} \quad y=-b$$

and after simplification we get two more equations giving the relationship between A_m, C_m and B_m, D_n. They are

$$\left[1+\frac{2m\pi\frac{b}{a}}{\sinh 2m\pi\frac{b}{a}}\right]A_m+\sum_{n=1,2,3.}^{\infty}(-1)^{m+n}\ \frac{4}{\pi}\ \frac{a}{b}\ \frac{m^2n\ \tanh n\pi\frac{a}{b}}{\left[m^2+\left(\frac{na}{b}\right)^2\right]^2}\ C_n=\frac{K_m+L_m}{2} \qquad \ldots \quad (13)$$

$$\left[1-\frac{2m\pi\frac{b}{a}}{\sinh 2m\pi\frac{b}{a}}\right]B_m+\sum_{n=1,3,5\ldots}^{\infty}(-1)^{m+(n-1)/2}\ \frac{4a}{\pi b}\ \frac{\frac{m^2n}{2}\ \tanh\frac{n\pi a}{2b}}{\left[m^2+\left(\frac{na}{2b}\right)^2\right]^2}\ D_n=\frac{K_m-L_m}{2} \qquad \ldots \quad (14)$$

where K_m and L_m depend on the nature of the given stress distribution on $y=\pm b$.

For this σ_y at $y=b$ is taken in the form of a Fourier series, i.e.,

$$f_1{}^*=-\frac{P}{a}+\sum_{m=1,2,3}^{\infty}K_m\cos\frac{m\pi x}{a} \qquad \ldots \quad (15)$$

and σ_y at $y=-b$ as

$$f_2{}^*=-\frac{P}{a}+\sum_{m=1,2,3}^{\infty}L_m\cos\frac{m\pi x}{a} \qquad \ldots \quad (16)$$

and K_m and L_m are found in the usual manner, thus,

$$K_m=\frac{1}{a}\int_{-a}^{a}\sigma_y\cos\frac{m\pi x}{a}\,dx \qquad \text{on } y=+b$$

$$L_m=\frac{1}{a}\int_{-a}^{a}\sigma_y\cos\frac{m\pi x}{a}\,dx \qquad \text{on } y=-b \qquad \ldots \quad (17)$$

Equations (11) to (14) are theoretically sufficient for the evaluation of all the A_m, B_m, C_n and D_n, and then the stresses are calculated from equations (5) to (7). It is thus seen that all the boundary conditions are satisfied exactly and the solutions are exact in the sense that stresses are expressed by series and become exact in limit as more terms are used. If the beam is loaded only at the bottom edge the same solution holds good except for f_1^* which will be zero. Abramyan[11], by taking a stress function similar to Eqn. (4) has proved that the system has a unique solution. He has considered a rectangle symmetrically loaded along its edges and hence the function contains only two sets of series.

To calculate the coefficients for a particular ratio a/b, we can take a finite number of terms in the series and the resulting set of simultaneous equations (Equations (11) to (14)) can be solved using desk calculator. Stresses can be calculated from equations (5) to (7). Detailed results of calculations for stresses for a beam whose depth is equal to the span under different loadings will be reported in a later paper.

Acknowledgements :

The work reported in this paper is a part of the research work of the author on the application of Multiple Fourier Methods to two and three dimensional problems in Engineering Design, The author records his sincere thanks to Prof. Geraled Pickett for introducing him to the Fourier Methods and under whose supervision the work was started, It is a pleasure for the author to thank Prof. N. S. Govinda Rao, Professor of Civil and Hydraulic Engineering, Indian Institute of Science, for his kind interest in this investigation.

References :

1. Dischinger, F. (1932), "Beitrag Zur Theorie der Halbscheibe und des wandartigen Tragers", Publications, International Association for Bridge and Structural Engg., Zurich, Vol. 1,
2. Portland Cement Association, "Design of Deep Girders", Pamphlet No. ST 66.
3. Li Chow, H. D. Conway and G. Winter, (1953), "Stresses in Deep Beams", Trans. A. S. C. E., Vol. 118. Pp. 686-702.
4. Uhlmann, H. L. B. (1952), "The theory of girder walls with special reference to reinforced concrete design", Structural Engg. Aug. Pp. 172-181.
5. Richardson, L. F. (1910), "Approximate arithmetical solution by finte differences of physical problems involving differential equations", Phil. Trans. Roy. Sec., London. A. Vol. 210,
6. Guzman A. M. and Luisoni, C. J. (1948), "Solucion variacional del problema de la viga rectangular simplemente apoyada de gran altura", Ciencia y Tecnica, Buenos Aires, Argentina. Vol. 111.

7. Conway, Li-Chow H. D. and Morgan, G. W. (1959), "Analysis of Deep Beam", Journal of Appl. Mech., Vol. 18, No. 2, June 1951, Pp. 163-173.

8. Pickett, G. (1944), "Application of the Fourier Method to the solution of certain boundary problems in the theory of elasticity", Journal of App. Mechanics, Vol. 11, A176.

9. Sundara Raja Iyengar, K. T., (1955), "On a two dimensional problem in the end-block design of post-tensioned prestressed concrete beams", Proc. of First congress on Theoretical and Applied Mechanics, Kharagpur, Pp. 107-112.

10. Pickett G., and Sundara Raja Iyengar K. T., (1956), "Stress concentration in post-tensioned prestressed concrete beams", Journal of Technology, Vol. 1, No. 2, 105-112.

11. Abramyan B. L., (1955), "On a case of the plane problem of the theory of elasticity for a rectangle", Akad. Nauk. Armyan. SSR, Dok 121, Pp. 193-198.