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ON A TWO-DIMENSIONAL PROBLEM IN THE END-BLOCK DESIGN OF POST-TENSIONED PRESTRESSED CONCRETE BEAMS

By

K. T. SUNDARA RAJA IYENGAR

INDIAN INSTITUTE OF SCIENCE, BANGALORE.

Introduction :

In the construction of post-tensioned prestressed concrete units, whatever be the process employed in creating the prestressing forces, high concentrations of stresses occur around the anchorages of the cables due to the discontinuous distribution of forces on the end surfaces. These prestressing forces are normally concentrated on certain bearing surfaces to be dealt with. Hence it becomes imperative for the designer to know the distribution of stress towards the ends of beams, to ensure safety against failure of concrete under excessive tensile stresses. Such failures usually appear as horizontal cracks developing during the post-tensioning stage.

Magnel [1] has treated this subject by an empirical method. He has assumed the distribution of vertical stress on a horizontal plane of the form of a 'cubic parabola' and the shear stress diagram to be a fourth degree curve. Guyon [2] has treated this problem analytically. Starting with an infinite half plane he has obtained an approximate solution for a rectangular semi-infinite prism. Recently Christodoulides [3] has obtained a photoelastic solution for the same problem for the case of two cables with anchorages symmetrically arranged about the centre line of the end section.

Most of the investigations that have been effected till now have been done under the following general assumptions :

(1) The problem permits a two dimensional analysis.

(2) The effect of the cable-ducts are ignored.

(3) The stresses induced by the cables would become normal *i.e.*, the stresses are almost entirely longitudinal, in a distance equal to the beam depth.

2. Statement of the Problem.

The problem considered in this study is the case of a beam post-tensioned by two cables with anchorages symmetrically arranged about the centre line of the end section (Fig. 1).

The first two of the above assumptions are made here also. Though the third assumption may be valid (according to St. Venant's Principle) it is felt unnecessary. The following derivation is not restricted by that assumtion.

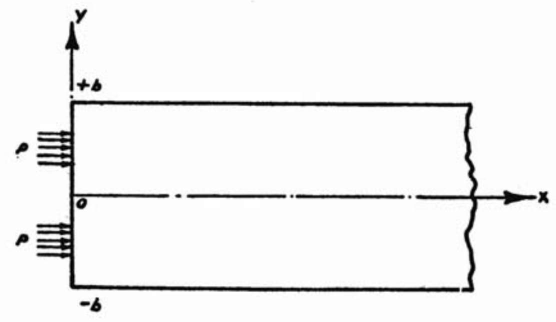

FIG. 1

Hence we have a plane problem in which we have to determine the stress function ϕ satisfying the equation

$$\frac{\partial^4\phi}{\partial x^4}+2\frac{\partial^4\phi}{\partial x^2 dy^2}+\frac{\partial^4\phi}{\partial y^4}=0 \qquad (1)$$

The solution of the problem is then subject to the boundary conditions, *i.e.*,

$$\begin{aligned}
\sigma_y &= 0 \quad \text{at} \quad y=\pm b\\
\tau_{xy} &= 0 \quad \text{at} \quad y=\pm b \text{ and } x=0\\
\sigma_x &= -p(y) \quad \text{at} \quad x=0\\
&= -\frac{P}{b} \quad \text{at } x\to\infty \qquad \ldots \qquad \ldots \qquad \ldots \qquad (2)
\end{aligned}$$

where $p(y)$ is given.

The stresses are then given by

$$\begin{aligned}
\sigma_x &= \frac{\partial^2\phi}{\partial y^2}\\
\sigma_y &= \frac{\partial^2\phi}{\partial x^2}\\
\tau_{xy} &= -\frac{\partial^2\phi}{\partial x\partial y} \qquad \ldots \qquad \ldots \qquad \ldots \qquad (3)
\end{aligned}$$

3. Solution.

$$\phi=\frac{-P}{2b}y^2+\sum_{n=1,2,3}^{\infty}\frac{B_n\cos\frac{n\pi y}{b}}{\left(\frac{n\pi}{b}\right)^2}\left[1+\frac{n\pi x}{b}\right]e^{\frac{-n\pi x}{b}}$$

$$+\int_0^{\infty}\frac{A(\alpha)\cos\alpha x}{\alpha^2\cosh\alpha b}\left[\alpha y\sinh\alpha y-(1+\alpha b\coth\alpha b)\cosh\alpha y\right]d\alpha \quad \ldots\ldots(4)$$

satisfies the equation (1).

Then the stresses are

$$\sigma_x = \frac{-P}{b} - \sum_{n=1,2,3}^{\infty} B_n \cos\frac{n\pi y}{b}\left[1+\frac{n\pi x}{b}\right] e^{\frac{-n\pi x}{b}}$$

$$+\int_0^\infty \frac{A(\alpha)\cos\alpha x}{\cosh\alpha b}\left[\alpha y\sinh\alpha y + (1-\alpha b\coth\alpha b)\cosh\alpha y\right] d\alpha \quad \text{......(5)}$$

$$\sigma_y = \sum_{n=1,2,3}^{\infty} B_n \cos\frac{n\pi y}{b}\left[-1+\frac{n\pi x}{b}\right] e^{\frac{-n\pi x}{b}}$$

$$-\int_0^\infty \frac{A(\alpha)\cos\alpha x}{\cosh\alpha b}\left[\alpha y\sinh\alpha y - (1+\alpha b\coth\alpha b)\cosh\alpha y\right] d\alpha \quad \text{.........(6)}$$

and

$$\tau_{xy} = -\Sigma B_n \sin\frac{n\pi y}{b}\cdot\frac{n\pi x}{b}\cdot e^{\frac{-n\pi x}{b}}$$

$$+\int_0^\infty \frac{A(\alpha)\sin\alpha x}{\cosh\alpha b}\left[\alpha y\cosh\alpha y - \alpha b\coth\alpha b\sinh\alpha y\right] d\alpha \quad \text{......(7)}$$

It can be seen that from (7)

$$\tau_{xy}=0 \quad \text{on} \quad y=\pm b \text{ and } x=0$$

since from (2), $\sigma_y = 0$ on $y=\pm b$

we have

$$\sum B_n \cos n\pi\left(-1+\frac{n\pi x}{b}\right) e^{\frac{-n\pi x}{b}} + \int_0^\infty \left(1+\frac{2\alpha b}{\sinh 2\alpha b}\right) A(\alpha)\cos\alpha x\, d\alpha = 0$$

giving [4]

$$A(\alpha)\left[1+\frac{2\alpha b}{\sinh 2\alpha b}\right] = \sum\frac{2}{\pi}\int_0^\infty B_n(-1)^n\left(1-\frac{n\pi x}{b}\right) e^{\frac{-n\pi x}{b}} \cos\alpha x\, dx \quad \text{......(8)}$$

Using the following results *i.e.*,

$$\int_0^\infty e^{\frac{-n\pi x}{b}} \cos\alpha x\, dx = \frac{\frac{n\pi}{b}\left[\alpha^2+\left(\frac{n\pi}{b}\right)^2\right]^2}{\left[\alpha^2+\left(\frac{n\pi}{b}\right)^2\right]^2}$$

and

$$\int_0^\infty \frac{n\pi x}{b} e^{\frac{-n\pi x}{b}} \cos \alpha x \, d\alpha = \frac{\left[\left(\frac{n\pi}{b}\right)^2 - \alpha^2\right] \frac{n\pi}{b}}{\left[\alpha^2 + \left(\frac{n\pi}{b}\right)^2\right]^2}$$

we obtain

$$\left[1 + \frac{2\alpha b}{\sinh 2\alpha b}\right] A(\alpha) = \frac{4}{\pi} \sum \frac{B_n(-1)^n \alpha^2 \frac{n\pi}{b}}{\left[\alpha^2 + \left(\frac{n\pi}{b}\right)^2\right]^2} \quad \ldots \quad \ldots \quad (9)$$

To satisfy the last boundary condition of (2), σ_x at $x=0$ is taken in the form of Fourier Series *i.e.*,

$$-p(y) = -P/b - \Sigma I_n \cos \frac{n\pi y}{a}$$

Then from equation (5) we have

$$-\frac{P}{b} - \Sigma I_n \cos \frac{n\pi y}{b} = -P/b - \Sigma B_n \cos \frac{n\pi y}{b}$$

$$+ \int_0^\infty \frac{A(\alpha)}{\cosh \alpha b} \left[\alpha y \sinh \alpha y + (1 - \alpha b \coth \alpha b) \cosh \alpha y\right] d\alpha \ \ldots\ldots(10)$$

Thus we have

$$B_n = I_n + \frac{1}{b} \int_{-b}^{b} \int_0^\infty \frac{A(\alpha)}{\cosh \alpha b} \left[\alpha y \sinh \alpha y + (1 - \alpha b \coth \alpha b) \cosh \alpha y\right] d\alpha \cos \frac{n\pi y}{b} dy$$

$$\ldots\ldots\ldots\ldots(11)$$

where

$$\frac{1}{b} \int_{-b}^{b} p(y) \cos \frac{n\pi y}{b} dy = I_n$$

we can write

$$B_n = I_n + \int_0^\infty F(\alpha, n) \, A(\alpha) \, d\alpha$$

$$= I_n + \sum_{m=1,2,3\ldots}^{\infty} K_{mn} B_m \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

where

$$K_{mn} = \frac{4}{\pi}(-1)^m \int_0^\infty \frac{F(\alpha, n) \alpha^2 \frac{m\pi}{b} d\alpha}{\left[\alpha^2 + \left(\frac{m\pi}{b}\right)^2\right]^2 \left[1 + \frac{2\alpha b}{\sinh 2\alpha b}\right]}$$

and

$$F(\alpha, n) = \frac{1}{b} \int_{-b}^{b} \frac{1}{\cosh \alpha b} \left[\alpha y \sinh \alpha y + (1 - \alpha b \coth \alpha b) \cosh \alpha y \right] \cos \frac{n\pi y}{b} \, dy$$

$$= \frac{(-1)^n \, 4\alpha b \, (n\pi)^2 \tanh \alpha b}{[(\alpha b)^2 + (n\pi)^2]^2}$$

giving

$$\frac{K_{mn}}{16\pi^2 mn^2 (-1)^{m+n}} = \int_0^\infty \frac{(\alpha b)^3 \tanh \alpha b \, d(\alpha b)}{\left[1 + \frac{2\alpha b}{\sinh 2\alpha b}\right] \left[(\alpha b)^2 + (m\pi)^2\right] \left[(\alpha b)^2 + (n\pi)^2\right]^2} \quad \ldots(13)$$

This can be put in the form

$$\frac{K_{mn}}{16\pi^2 mn^2 (-1)^{m+n}} = \int_0^\infty \frac{(\alpha b)^3 \, d(\alpha b)}{[(\alpha b)^2 + (m\pi)^2]^2 [(\alpha b)^2 + (n\pi)^2]^2}$$

$$- \int_0^\infty \left[1 - \frac{\tan \alpha b}{1 + \frac{2\alpha b}{\sinh 2\alpha b}}\right] \frac{(\alpha b)^3 \, d(\alpha b)}{[(\alpha b)^2 + m^2\pi^2]^2 [(\alpha b)^2 + n^2\pi^2]^2} \quad \ldots(14)$$

The first integral of this Right Hand Side can be evaluated completely and we have

when $m \neq n$

$$\int_0^\infty \frac{(\alpha b)^3 \, d(\alpha b)}{[(\alpha b)^2 + m^2\pi^2]^2 [(\alpha b)^2 + n^2\pi^2]^2} = -\frac{1}{2} \frac{m^2\pi^2 + n^2\pi^2}{(m^2\pi^2 - n^2\pi^2)^3} \log \frac{n^2}{m^2} - \frac{1}{(m^2\pi^2 - n^2\pi^2)} \quad \ldots(15)$$

and when $m = n$

$$\text{the integral} = \frac{1}{12 m^4 \pi^4}$$

To calculate the second integral, the following formula for numerical integration is used.

$$\int_0^\infty f(y) \, dy = \Big[14y_0 + 64y_{0\cdot2} + 24y_{0\cdot4} + 64y_{0\cdot6} + 28y_{0\cdot8} + 64y_{1\cdot0} + 24y_{1\cdot2}$$

$$+ 64y_{1\cdot4} + 28y_{1\cdot6} + 64y_{1\cdot8} + 24y_{2\cdot0} + \cdots\cdots + 640y_{14} \Big] \frac{2}{45} \quad \ldots(16)$$

After simplification, equation (12) reduces to a set of simultaneous equations of the form

$$B_n[1 - K_{nn}] = I_n + \Sigma(K_{mn} \, B_m) \qquad [m \neq n] \quad \ldots(17)$$

Taking six terms, the set of six simultaneous equations has been solved by employing Crout's method and we have the coefficients, B_1, B_2,.........B_6 in terms of I_s' for the case of a symmetrical case of loading.

$$
\begin{aligned}
B_1 &= 1.156167 I_1 - 0.071679 I_2 + 0.038738 I_3 - 0.023592 I_4 + 0.015612 I_5 - 0.010911 I_6 \\
B_2 &= -0.143374 I_1 + 1.087804 I_2 - 0.055466 I_3 + 0.037203 I_4 - 0.026184 I_5 + 0.019204 I_6 \\
B_3 &= 0.116216 I_1 - 0.083198 I_2 + 1.058327 I_3 - 0.041912 I_4 + 0.031159 I_5 - 0.023839 I_6 \\
B_4 &= -0.094369 I_1 + 0.074404 I_2 - 0.055882 I_3 + 1.042481 I_4 - 0.0329981 I_5 + 0.026025 I_6 \\
B_5 &= 0.078061 I_1 - 0.065461 I_2 + 0.051932 I_3 - 0.041247 I_4 + 1.033072 I_5 - 0.026856 I_6 \\
B_6 &= -0.065467 I_1 + 0.057613 I_2 - 0.047678 I_3 + 0.039038 I_4 - 0.032227 I_5 + 1.026808 I_6
\end{aligned} \qquad \ldots(18)
$$

For a particular case as shown in Fig. 2 (the case considered by Christodoulides in reference [3]), we have

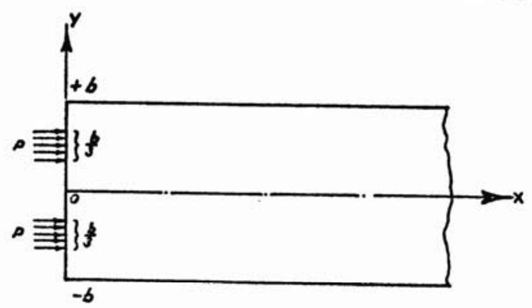

Fig. 2

$$
I_n = \frac{2}{b} \int_{b/3}^{2b/3} \frac{3p}{b} \cos \frac{n\pi y}{b}\, dy
= \frac{6p}{n\pi b}\left[\sin \frac{2n\pi}{3} - \sin \frac{n\pi}{3} \right] \qquad \ldots(19)
$$

giving

$$
I_1 = I_3 = I_5 = I_6 = 0,
$$

$$
I_2 = -\frac{3\sqrt{3}P}{\pi b}, \quad I_4 = \frac{3\sqrt{3}P}{2\pi b}.
$$

Then from (18) we have

$$
\begin{aligned}
B_1 &= 0.099046\ P/b, & B_2 &= -1.768447\ P/b \\
B_3 &= 0.102947\ P/b, & B_4 &= 0.739061\ P/b \\
B_5 &= 0.074160\ P/b, & B_6 &= -0.063007\ P/b
\end{aligned} \qquad \ldots(20)
$$

After knowing the coefficients B'_s for the particular position of loading, the stresses are calculated from equations (5), (6) and (7).

The problem treated hitherto is only a two-dimensional approximation to the actual three-dimensional one. The stresses vary along the breadth and the ducts for the cables may also affect the distribution of stresses.

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