

# A NEW FORM OF EXPRESSION FOR POWER ON BLADING IN A STAGE OF A TURBOMACHINE BASED ON THE METHOD OF EQUIVALENT REPLACEMENTS

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## ABSTRACT

\* A previous paper<sup>1</sup> presented the investigation of power transfer in a stage of a turbomachine by the method of equivalent replacements and a general scheme of development of mathematical profiles, was given. The manner of development of turbine and compressor profiles and their attendant characteristics were also given in the same paper. In this paper, the method of equivalent replacements is extended to formulate analytical expression for power on blading in a stage of a turbomachine. The expression can be given in such a form that further analysis of stage characteristics can be done easily. A similar form has, till now, been applied sometimes in the theory of axial compressor, with certain limitations. The manner of computing the characteristics and their application to the analysis of stage characteristics of a turbomachine, are presented in this paper.

## NOMENCLATURE

The following nomenclature is used in this paper:

- A thermal equivalent  $\frac{1}{427}$  Cal./kg.m.
- $c$  actual absolute velocity (m./sec.)
- $\vec{c}$  absolute velocity vector.
- $c_u$  Circumferential component of  $c$ .
- $c_a$  axial component of  $c$ .
- $c_{1t}$  theoretical absolute velocity.
- $c_{2(m-1)}$  the absolute exit velocity from preceding stage. [ $c_0^2 = \mu_0 c_{2(m-1)}^2$ ]
- $g$  acceleration due to gravity (m./sec.<sup>2</sup>).
- $G$  total fluid flow rate (kg./sec.).

\* Superscript numbers refer to items in References at the end of the paper.

- $h_0$  stage thermodynamic enthalpy drop along the isentropic path.  
(Cal./kg.).
- $h_0'$   $h_0 + A \cdot \mu_0 \cdot \frac{c_2^2 (m-1)}{2g}$  (Cal./kg.)
- $k_u$  circumferential force coefficient.
- $h_{1,2}$  theoretical work done (or absorbed) per 1 kg. of fluid, from inlet to exit of blades. (kg. m./kg.).
- $(m-1)$  the stage preceding the considered stage,  $m$ .
- $n_1, n_2$  characteristic numbers.
- $P_u$  circumferential turning force.
- $P_{ui}$  impulsive part of  $P_u$ .
- $P_{ur}$  reactive part of  $P_u$ .
- $u$  circumferential velocity (m./sec.).
- $w$  relative velocity (m./sec.).
- $\vec{w}$  relative velocity vector.
- $w_u$  circumferential component of  $w$ .
- $W_{(1)}$  work done (or absorbed) by a fluid flow rate 1 kg./sec. from inlet to exit of blades. (kg. m./sec.).
- $W$  work done (or absorbed) by a fluid flow rate  $G$ . kg./sec. (kg.m./sec.).
- $W_i$  impulsive component of  $W$ .
- $W_r$  reactive component of  $W$ .
- $\Delta w_{ex}$  difference between the circumferential component of  $w$  at exit and inlet of the *exit stream*.
- $\Delta w_{in}$  difference between the circumferential components of  $w$  at inlet and exit of the *inlet stream*.
- $\Delta w_u$  difference between the circumferential components of  $\vec{w}_1$  and  $\vec{w}_2$ , i.e., inlet and exit relative velocity vectors of the *equivalent stream*.
- $\alpha_1, \beta_1$  angles the velocities  $c_1$  and  $w_1$  respectively make with the circumferential direction.
- $\alpha_2, \beta_2$  angles the velocities  $c_2$  and  $w_2$  make with the contrary circumferential direction.

$\bar{\alpha}_2, \bar{\beta}_2$	( $180^\circ - \alpha_2$ ) and ( $180^\circ - \beta_2$ ) respectively.
$\beta_{2b}$	exit angle of real blade to the contrary circumferential direction.
$\eta_{rb}$	efficiency relative to blading.
$\mu_0$	Utilisation factor of absolute exit velocity from preceding stage.
$\rho$	degree of thermal reaction.
$\rho_{ur}$	degree of circumferential force reactivity.
$\rho_{ui}$	degree of circumferential force impulsivity.
$\psi$	blade velocity coefficient.
$\phi$	nozzle velocity coefficient.

## SUBSCRIPTS

1	inlet to blade.
2	exit to blade.
Opt.	optimum value.
$k_u^{st}$	stage $k_u$ .

## INTRODUCTION

IN this paper some considerations are presented which deal with a simplified method of defining the stage of a turbomachine by characteristic numbers and manner of computing the stage characteristics. The simplified method is based on the "method of equivalent replacements", which replaces the real flow in an actual turbomachine stage by a single stream line of "the equivalent stream" concentrating the total mass flow and the real blade by a "mathematical profile". The mathematical profile is an infinitely thin profile having a contour with direction tangents at its inlet and exit coinciding with the inlet and exit relative velocity vectors of the equivalent stream. The general scheme of development of mathematical profiles is given in Appendix. The manner of development of turbine and compressor mathematical profiles and their connected characteristics were covered in a previous paper.<sup>1</sup>

This paper deals with axial turbomachines and characteristic expressions applicable to this type are presented. The degrees of circumferential force—"reactivity  $\rho_{ur}$ ", "impulsivity  $\rho_{ui}$ " and "circumferential force coefficient" in terms of the characteristic numbers are formulated. A simplified expression for power on blading is evaluated, involving the characteristic numbers, the circumferential speed and the mass flow. A few typical axial turbine stages—the theoretical impulse with symmetrical deviation of flow, the congruent and the congruent with under-developed  $\rho_{ur}$ —are analysed with respect to the manner of force interaction,

work output and characteristic mathematical profiles. A discussion of transition of mathematical profiles in the scheme (Appendix) is given along with plots of characteristic criteria of profiles as transition takes place. A general expression to determine design data for optimum operating conditions of a stage is formulated and illustrated by application to a problem. Connection between circumferential force coefficient  $k_u$  and the available energy in a stage is traced. The method of distribution of energies in a multistage axial turbomachine, based, on  $k_u^{st}$  is analysed.

### 1. EXPRESSION FOR POWER ON BLADING IN A STAGE OF A TURBOMACHINE

Work done (or absorbed) per 1 kg. of fluid, flowing around the blades, is expressed by the equation:

$$l_{1,2}^t = \frac{1}{g} [u_1 c_{1u} - u_2 c_{2u}] \quad (1a)$$

and this can also be expressed in the form:

$$l_{1,2}^t = \frac{1}{g} [u_1 (c_1 \cos \alpha_1) - u_2 ((w_2 \cos \bar{\beta}_2 + u_2))] \quad (1)$$

Let

$$c_1 \cos \alpha_1 = n_1 u_1 \quad (2)$$

and

$$w_2 \cos \bar{\beta}_2 = n_2 u_2 \quad (3)$$

$n_1$  and  $n_2$  are pure numbers, whose sign shall be defined as:

$n_1$  would be positive if  $\alpha_1 < 90^\circ$  ( $n_1$  is always positive in an usual turbine stage).

$n_2$  would be positive if  $\beta_2 > 90^\circ$ , i.e.,  $(180 - \beta_2) = \bar{\beta}_2 < 90^\circ$  (see Fig. 1.) ( $n_2$  is always negative under conditions existing in an usual turbine stage).

For a compressor stage  $n_1$  can be negative and  $n_2$  can be positive (see Fig. 1).

Substituting 2 and 3 in 1 we get:

$$l_{1,2}^t = \left[ \frac{n_1 u_1^2}{g} - \frac{(n_2 + 1) u_2^2}{g} \right] \dots \dots \text{kg.m./kg.} \quad (4)$$

If  $u_1 = u_2 = u$  as in an axial stage, equation (4) takes the form:

$$l_{1,2}^t = \left[ \frac{n_1 - n_2 - 1}{g} \right] u^2 \dots \dots \text{kg.m./kg.} \quad (5)$$

Using the expressions (4) and (5), power on blading can be expressed in the form:

$$W_{(1)} = \left[ \frac{n_1 - n_2 - 1}{g} \right] u^2 \dots \dots \text{kg.m./sec.} \quad (6)$$

for a mass flow 1 kg./sec. and power becomes  $G.W_{(1)}$  if the mass flow is  $G$  kg./sec.

$$W = G \cdot \frac{n_1 - n_2 - 1}{g} \cdot u^2 \dots \dots \text{kg.m./sec.} \quad (7)$$

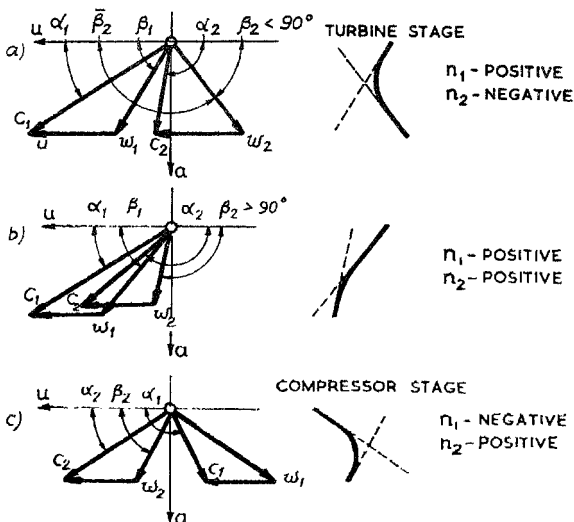


FIG. 1. Sign of stage characteristic numbers  $n_1$  and  $n_2$ . (Vector magnitudes +ve in the direction of vector  $u$ .)

## 2. DEGREE OF CIRCUMFERENTIAL FORCE REACTIVITY AND IMPULSIVITY OF AN AXIAL STAGE

We shall limit ourselves to derive simple and convenient formulæ to calculate the values of: circumferential force reactivity ( $\rho_{ur}$ ) and circumferential force impulsivity ( $\rho_{ui}$ ).<sup>1</sup>

$$\rho_{ur} = \frac{\Delta w_{ue}}{\Delta w_u}$$

For an axial stage:

$$\Delta w_{ue} = (0 - w_2 \cos \bar{\beta}_2) = -n_2 u$$

and

$$\begin{aligned} \Delta w_u &= (w_1 \cos \beta_1 - w_2 \cos \bar{\beta}_2) = (c_1 \cos \alpha_1 - u) - w_2 \cos \bar{\beta}_2 \\ &= (n_1 u - u) - n_2 u = (n_1 - n_2 - 1) u. \end{aligned}$$

Hence degree of circumferential force reactivity is:

$$\rho_{ur} = \frac{-n_2}{n_1 - n_2 - 1} \quad (8)$$

In a similar manner, the degree of circumferential force impulsivity is:

$$\rho_{ui} = \frac{\Delta w_{ui}}{\Delta w_u}$$

Since  $\Delta w_{ui} = w_1 \cos \beta_1 = c_1 \cos \alpha_1 - u = (n_1 - 1) u$ .

$$\rho_{ui} = \frac{n_1 - 1}{n_1 - n_2 - 1} \quad (9)$$

It is clear that it is always:

$$\rho_{ui} + \rho_{ur} = 1. \quad (9a)$$

From this it can easily be observed that equations (8) and (9) are useful and valid only for a certain field of  $u/c_1$  changing. It is possible to consider the reactive manner of force transfer in a stage of a turbomachine, only when the exit stream is present, and similarly of impulsive force transfer, only when the inlet stream be present. If the inlet stream is absent, the total circumferential force is completely reactive and hence  $\rho_{ur} = 1$  and  $\rho_{ui} = 0$ .\* In the absence of the exit stream, the total circumferential force is completely impulsive and hence  $\rho_{ur} = 0$  and  $\rho_{ui} = 1$ .

### 3. CIRCUMFERENTIAL FORCE COEFFICIENT OF STAGE

Equations (6) to (9) are very convenient to analyse the problem of force interaction and energy exchange between flow and blades in any stage of a turbomachine. However, it is possible to simplify the analysis further by introducing the concept of "circumferential force coefficient— $k_u$ " of blades.

$P_u$  the circumferential force is given by the product of flow mass per second and the total change of circumferential component of absolute velocity of the equivalent stream, replacing the real flow.

$$P_u = \frac{G}{g} [c_{1u} - c_{2u}] = \frac{G}{g} [n_1 - n_2 - 1] u \dots \dots \dots \text{kg.} \quad (10)$$

We can express:

$$P_u = \frac{G}{g} \cdot k_u \cdot u \dots \dots \dots \text{kg.} \quad (11)$$

\* (Note; Values  $\rho_{ur} > 1$  or  $\rho_{ui} > 1$  are not absurd values. They show that in the particular case the reactivity or impulsivity is not fully developed.)

and work

$$W = \frac{G}{g} \cdot k_u \cdot u^2 \dots \text{kg.m./sec.} \quad (12)$$

$$k_u = (n_1 - n_2 - 1) \quad (13)$$

shall be defined as the "circumferential force coefficient of blades" of turbo-machine stage. The sense of this name is quite clear and the sign (positive or negative) of  $k_u$  reflects the direction of energy exchange in a stage, the direction of the total circumferential force and its nature as to whether it is reactive and impulsive or only one of them.

The quantities  $G$ ,  $u$ ,  $k_u = (n_1 - n_2 - 1)$  determine the total power developed (or absorbed) by blades. If  $k_u$  is negative, the total circumferential force is a contrary one and the stage is a compressor one, absorbing work. If  $k_u$  is positive, the total force is fair and the stage is a turbine one, developing work. The demarcating regime is characterised (if the wheel is rotating) by  $k_u = (n_1 - n_2 - 1) = 0$ . Under these conditions,  $P_u = 0$  and  $W = 0$  independent of the value of  $G$ . Power on blading will also be zero if  $u = 0$  or  $G = 0$ . (Incidentally we can deduce that any profile of real blades and any wheel speed in absolute vacuum does not absorb power.)

#### 4. CONNECTION BETWEEN THE VALUE OF CIRCUMFERENTIAL FORCE COEFFICIENT AND THE MATHEMATICAL PROFILE OF BLADE

The value of circumferential force coefficient  $k_u$  is determined by the values of  $n_1$ ,  $n_2$ , i.e., the type of streamline of the equivalent stream replacing flow. And the shape of the imaginary infinitely thin mathematical profile of blade depends on the values of  $n_1$ ,  $n_2$ . The mathematical profile can be given by: (a)  $n_1$ ,  $n_2$  and  $\vec{u}$ , or (b)  $\vec{c}_1$ ,  $\vec{w}_2$  and  $\vec{u}$ . The later method (i.e., b) is in common use now for turbo-machine stages. Really if  $\vec{c}_1$  and  $\vec{u}$  are given, the inlet velocity triangle of equivalent stream is given and inlet angle  $\beta_1$  of mathematical profile is specified. Again  $\vec{w}_2$  and  $\vec{u}$  fix the exit triangle and mathematical profile exit angle  $\beta_2 = \beta_{2s}$  is specified. Values of  $n_1$  and  $n_2$  can be determined by expressions (2) and (3):

$$n_1 = \frac{c_1 \cos \alpha_1}{u} \quad (14)$$

and

$$n_2 = \frac{w_2 \cos \beta_2}{u} \quad (15)$$

$k_u$  changes in value, if  $\vec{c}_1$ ,  $\vec{w}_2$ ,  $\vec{u}$  change (i.e., since  $n_1$  and  $n_2$  change) and accordingly all other quantities expressed by  $k_u$ , change.

## 5. IMPULSIVE AND REACTIVE COMPONENTS OF TURNING FORCE AND POWER ON BLADING

If values of  $k_u$ ,  $\rho_{ur}$  (and/or  $\rho_{ui}$ ),  $u$  and  $G$  are known, we can calculate the total energy exchange between flow and blades, and also the force and energy characteristics of the considered stage.

$$\text{Total power on blading: } W = \frac{G}{g} \cdot k_u \cdot u^2 \dots \text{kg.m./sec.} \quad (12)$$

$$\text{Total turning force } P_u = \frac{G}{g} \cdot k_u \cdot u \dots \text{kg.m.} \quad (11)$$

$$\text{Impulsive component of power on blading: } W_t = \rho_{ui} \cdot W = \rho_{ui} \left[ \frac{G}{g} \cdot k_u \cdot u^2 \right] \quad (16)$$

$$\text{and Reactive component of power on blading: } W_r = \rho_{ur} \cdot W = \rho_{ur} \left[ \frac{G}{g} \cdot k_u \cdot u^2 \right] \quad (17)$$

$$\text{Impulsive component of turning force: } P_{ui} = \rho_{ui} \cdot P_u = \rho_{ui} \left[ \frac{G}{g} \cdot k_u \cdot u \right] \quad (18)$$

$$\text{and Reactive component of turning force: } P_{ur} = \rho_{ur} \cdot P_u = \rho_{ur} \left[ \frac{G}{g} \cdot k_u \cdot u \right] \quad (19)$$

We understand the impulsive component of power as the power due to the impulsive component of turning force (*i.e.*, due to the presence of the inlet stream) and reactive component of power as the power due to the reactive component of turning force (*i.e.*, due to the presence of the exit stream). The characteristics of  $W$  and  $P_u$  will depend upon the individual characteristics of the equivalent stream (*i.e.*, the characteristics of inlet and exit streams as to whether they are fair or contrary).

## 6. IMPULSIVE AND REACTIVE COMPONENTS OF POWER ON BLADING FOR SOME TYPICAL TYPES OF TURBOMACHINE AXIAL STAGES

We shall consider a few typical axial stages of a turbomachine and analyse the characteristics of  $W_t$  and  $W_r$ . In all the cases, the scheme of stage (unrolled in the plane of figure), the blades are moving to the left and the flow is admitted to blades from above.



6.1. *The theoretical impulse turbine stage with symmetrical deviation of flow on blading under optimum conditions.—*

In this stage under the given regime:  $\rho = 0$ ,  $\psi = 1$ ,  $\beta_1 = \beta_2$ .

Since

$$\frac{u}{c_1} = \frac{\cos \alpha_1}{2}$$

$$n_1 = \frac{c_1 \cos \alpha_1}{u} = 2.$$

and

$$n_2 = \frac{w_2 \cos \bar{\beta}_2}{u} = \frac{-u}{u} = -1.$$

Degree of circumferential force reactivity

$$\rho_{ur} = \frac{-n_2}{n_1 - n_2 - 1} = 0.5$$

Degree of circumferential force impulsivity

$$\rho_{ui} = \frac{n_1 - 1}{n_1 - n_2 - 1} = 0.5$$

(and again  $\rho_{ui} = 1 - \rho_{ur} = 1 - 0.5 = 0.5$ )

$$k_u = n_1 - n_2 - 1 = 2 - (-1) - 1 = 2$$

(for  $G = 1.0$ )

$$W_{(1)i} = \rho_{ui} \cdot \frac{k_u}{g} \cdot u^2 = 0.5 \cdot \frac{2}{g} \cdot u^2 = \frac{u^2}{g} \dots \text{kg.m./sec.}$$

$$W_{(1)r} = \rho_{ur} \cdot \frac{k_u}{g} \cdot u^2 = \frac{u^2}{g} \dots \text{kg.m./sec.}$$

Total power on blading

$$W_{(1)} = W_{(1)i} + W_{(1)r} = 2 \frac{u^2}{g} \dots \text{kg.m./sec.} \quad (20)$$

Thus in a theoretical impulse turbine stage with symmetrical deviation of flow on blading under optimum conditions, the ratio of impulsive to reactive components of power on blading is:  $W_{(1)i}/W_{(1)r} = 1.0$ . The same value for  $W_{(1)i}/W_{(1)r} = 1.0$  can be realised in a real ( $\rho = 0$  and  $\psi < 1$ ) turbine stage, having unsymmetrical deviation ( $\beta_2' < \beta_1$ ) on blades (see Fig. 2). Such a real stage and the considered theoretical stage are characterised by the values:

$$n_1 = 2; \quad n_2 = -1; \quad \rho_{ur} = 0.5 \quad \text{and} \quad \rho_{ui} = 0.5.$$

6.2. The congruent turbine stage under optimum conditions.—

For this stage (see Fig. 3).

$$n_1 = 1, \quad n_2 = -1, \quad \rho = 0.5 \quad \text{and} \quad k_u = 1.$$

$$\rho_{ur} = \frac{-(-1)}{1 - (-1) - 1} = 1$$

$$\rho_{ut} = \frac{1 - 1}{1 - (-1) - 1} = 0$$

$$W_{(1)} = \frac{k_u}{g} u^2 = \frac{u^2}{g} \tag{21}$$

$$W_{(1)t} = \rho_{ut} W_{(1)} = 0 \tag{22}$$

$$W_{(1)r} = \rho_{ur} W_{(1)} = W_{(1)}$$

and

$$\frac{W_{(1)t}}{W_{(1)r}} = \frac{0}{W_{(1)}} = 0.$$

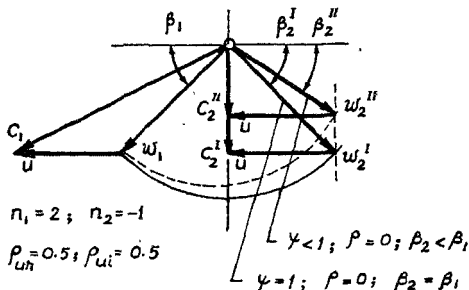
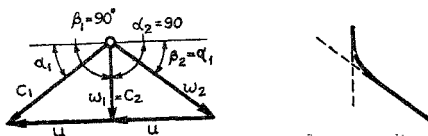


FIG. 2. Theoretical impulse turbine stage with symmetrical deviation of flow on blading under optimum conditions ( $u/c_1 = \cos \alpha_{1/2}$ ).

In a congruent turbine stage under optimum conditions, the total power on blading is reactive. The mathematical profile (see Fig. 3) is characterised by  $\beta_1 = 90^\circ$ , with the exit stream alone. (In the general scheme of mathematical profiles, the profile is disposed in row II in the turbine regime).

6.3. The congruent turbine stage with under-developed circumferential force reactivity.—

The mathematical profiles corresponding to such a stage, in the range  $\cos \alpha_1 < u/c_1 < 2 \cos \alpha_1$ , are disposed in the under right half-row I, in the scheme of profiles.



$$n_1 = 1; \quad n_2 = -1; \quad \rho_{un} = 1; \quad \rho = 0.5; \quad k_u = 1$$

FIG. 3. Congruent turbine stage under optimum conditions ( $u/c_1 = \cos \alpha_1$ ).

For these stages:  $\beta_1 > 90^\circ$  (Fig. 4)

$$1 > n_2 > 0.5;$$

$$n_2 = -n_1;$$

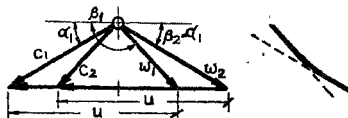
$$\rho = 0.5;$$

$$k_u < 1; \quad \rho_{ur} = 0$$

and

$$\rho_{ur} > 1.$$

(Actually  $\rho_{ur} > 1$  is not absurd, it only indicates that the circumferential force reactivity is under-developed and the total power on blading is reactive.)



$$n_2 = -n_1; \quad \rho = 0.5; \quad k_u < 1; \quad \rho_{un} > 1$$

FIG. 4. Congruent turbine stage with under-developed exit stream ( $\cos \alpha_1 < u/c_1 < 2 \cos \alpha_1$ ).

As  $u/c_1 \rightarrow 2 \cos \alpha_1$ ,  $k_u \rightarrow 0$  &  $W \rightarrow 0$ , the profile moves over from the multi-form row I to the flat profiles in row 0.

6.4. The turbine stage with under-developed circumferential force impulsivity ( $\rho = 0$ ;  $\psi = 1$ ).—

This case is considered to show the wide applicability of the expressions derived earlier. Let us take a particular case,  $n_1 = 2$ ;  $n_2 = 0.25$  (see Fig. 5). The total power on blading is impulsive, the equivalent stream consisting of only the fair inlet stream, which is not fully developed, i.e.,  $\beta_2 > 90^\circ$  ( $\bar{\beta}_2 < 90^\circ$ ).

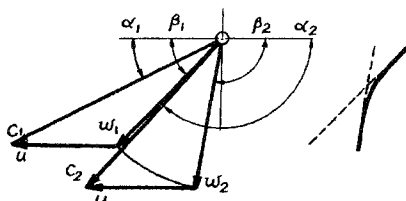
$$k_u = n_1 - n_2 - 1 = 2 - 0.25 - 1 = 0.75$$

$$\rho_{ut} = \frac{n_1 - 1}{n_1 - n_2 - 1} = \frac{2 - 1}{2 - 0.25 - 1} = \frac{1}{0.75} = \frac{4}{3}$$

( $\rho_{ut} > 1$  is not absurd. It merely indicates that  $\rho_{ut}$  is not fully developed and the total power developed is impulsive.) If  $\rho_{ut}$  is fully developed the characteristics of the stage would be:

$$\beta_2 = \bar{\beta}_2 = 90^\circ; n_2 = 0; \rho_{ur} = 0; \rho_{ut} = 1 \text{ and } k_u = 1.$$

We can now state that if  $n_2$  is within the range  $-1$  to  $0$ , the circumferential force impulsivity is under-developed, the circumferential force reactivity is absent and the mathematical profile is disposed in the left under half-row I, as  $n_2 \rightarrow (n_1 - 1)$ ,  $k_u \rightarrow 0$  and  $W_u \rightarrow 0$ .



$$\begin{array}{ll} \rho = 0 & \psi = 1 \\ n_1 = 2 & n_2 = +0.25 \end{array}$$

FIG. 5. Turbine stage with under-developed inlet stream.

## 7. THE NUMBERS $n_1, n_2$ AS CHARACTERISTICS OF THE MATHEMATICAL PROFILE OF BLADE

The development of mathematical profiles, in the general scheme, characterised by  $n_1, n_2$  is analysed in the following sections.

### 7.1. Displacement along the uppermost or the undermost horizontal row IV (Symmetrical profiles).—

Fig. 6 represents the gradual change of velocity triangles, as the mathematical profile is displaced from the middle row IV to the sides. The profiles are symmetrical and velocity vectors  $\vec{w}_1$  and  $\vec{w}_2$  are conjugate,  $\vec{u}$  axis being the imaginary axis.  $\vec{w}_1 = \vec{w}_2$ ,  $\beta_1 = \beta_2$  and  $n_1 + n_2 = 1$ .

$$n_1 + n_2 = 1 \quad (23)$$

For all the profiles in row IV,

$$\rho_{ui} = \frac{n_1 - 1}{n_1 - n_2 - 1} = \frac{n_1 - 1}{n_1 - (1 - n_1) - 1} = \frac{n_1 - 1}{2(n_1 - 1)} = 0.5 \quad (23 a)$$

and

$$\rho_{ur} = \frac{-n_2}{n_1 - n_2 - 1} = \frac{n_1 - 1}{2(n_1 - 1)} = 0.5 \quad (23 b)$$

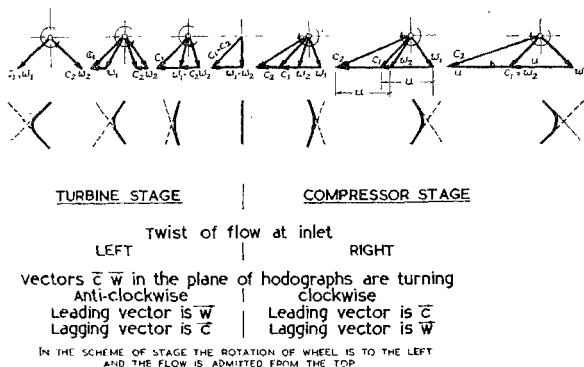


FIG. 6. Displacement along the topmost or bottommost horizontal row IV (Symmetrical profiles).

$\rho_{ui}$  and  $\rho_{ur}$  have these values only when  $(n_1 + n_2) = 1$ , which is the condition for vectors  $\vec{w}_1$  and  $\vec{w}_2$  to be conjugate. If this condition is disturbed,  $\rho_{ui} \neq \rho_{ur}$ , though the equivalent stream is characterised by symmetrical deviation  $\beta_1 = \beta_2$ .

$\vec{w}_1$  and  $\vec{w}_2$  will be conjugate in the following cases:

(1) Theoretical ( $\psi = 1$ ), impulse ( $\rho = 0$ ) stage with symmetrical deviation of flow. ( $\beta_1 = \beta_2$ )

and (2) Real ( $\psi < 1$ ) stage, with symmetrical deviation of flow ( $\beta_1 = \beta_2$ ) and some degree of thermal reaction ( $\rho > 0$ ).

Fig. 7. gives a plot of  $n_2$ ,  $k_u$ ,  $\rho_{ui}$  and  $\rho_{ur}$  with change in  $n_1$  (i.e.,  $n_1$  is the independent variable) for the shift of profile along the topmost and the bottommost horizontal row IV, under the condition  $(n_1 + n_2 = 1)$  (i.e.,  $\vec{w}_1$  and  $\vec{w}_2$  are conjugate).

Straight line 1 gives  $n_2$ , equation being  $n_2 = (1 - n_1)$ .

Straight line 2 gives  $k_u$ , equation being  $k_u = 2n_1 - 2$ .



contrary ( $n_1 < 0$ ): Whereas turbine regime can exist only when the twist of flow is fair and  $n_1 > 1$ .

- 3  $\rho_{st} = \rho_{ur} = 0.5$  and hence the reactive and impulsive parts of turning force and power are equal.
4. Optimum conditions are determined by  $n_1 = 2$  and  $n_2 = -1$  (and, hence,  $u/c_1 = \cos \alpha_1/2$  and  $\beta_2 = \beta_1$ ). These optimum conditions are marked in Fig. 7 by the points A, B on the auxiliary straight 8.

### 7.2. Displacement along any right vertical row of mathematical profiles.--

Any vertical right row can be chosen for investigation. Fig. 8 represents the gradual change of velocity triangles, as the mathematical profile is displaced

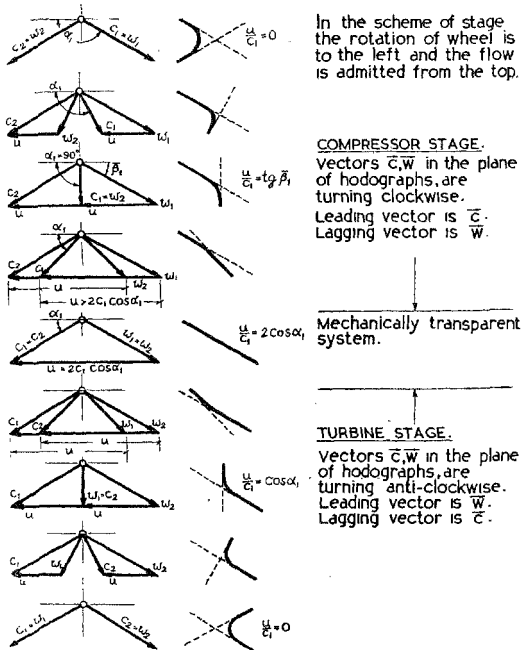


FIG. 8. Displacement along any right vertical row. (Property of congruence maintained.)  
 from the row 0, i.e., the flat profile. As in the general scheme, the flow is directed from the top and  $u$  is directed to the left. Displacement of profile along the vertical

row gives the possibility to maintain Congruence of stage (i.e., Vectors  $\vec{c}_1$ ,  $\vec{w}_2$  and  $\vec{c}_2$ ,  $\vec{w}_1$ , taken in pairs, are conjugate—the  $u$  axis being the imaginary one). Conditions of congruence of stage gives the relation (since  $\alpha_1 = \beta_2$  and  $c_1 = w_2$ )  
 $n_2 = -n_1$

i.e.,

$$(n_1 + n_2) = 0 \quad (24)$$

$$\rho_{ur} = \frac{-n_2}{n_1 - n_2 - 1} = \frac{n_1}{2n_1 - 1} \quad (24 a)$$

$$\rho_{ui} = \frac{n_1 - 1}{n_1 - n_2 - 1} = \frac{n_1}{2n_1 - 1} \quad (24 b)$$

$$k_u = n_1 - n_2 - 1 = 2n_1 - 1 \quad (24 c)$$

The profiles, in the bottommost row IV\* and under row III, are fully developed with both inlet and exit streams ( $u/c_1$  in the range 0 to  $\cos \alpha_1$ ). When  $u/c_1 = \cos \alpha_1$ ;  $n_1 = 1$ ,  $n_2 = -n_1 = -1$ ,  $k_u = 1$ , the inlet stream is absent ( $\rho_{ui} = 0$ ), the exit stream is fully developed [ $\beta_1 = 90^\circ$  and  $\rho_{ur} = 1$ ] and the profile is disposed in the under row II. As the profile is displaced from under row II to 0 ( $u/c_1$  in the range 1 to 0.5),  $\rho_{ur} = 1.0$  and  $\rho_{ui} = 0$  are retained [the exit stream being underdeveloped ( $\beta_1 > 90^\circ$ ) and inlet stream being absent]. In row 0,  $u/c_1 = 0.5$ , the profile is flat ( $\beta_1 = \beta_2$ ),  $n_1 = 0.5$ ,  $n_2 = -0.5$ ,  $k_u = 0$  and expressions (24 a) and (24 b) are not applicable as they give  $\rho_{ur} = \rho_{ui} \rightarrow \infty$ . There is thus a discontinuity, the profile is mechanically transparent, having passed off the turbine regime. There is neither force interaction nor work done (or absorbed).

The transition of the profile into the multiform upper horizontal row I (Compressor regime) from the multiform under horizontal row I (turbine regime) is associated with a sudden change of  $\rho_{ur}$  and  $\rho_{ui}$  from  $\rho_{ur} = 1$  and  $\rho_{ui} = 0$  to  $\rho_{ur} = 0$  and  $\rho_{ui} = 1$ . However the transition is smooth, through the mechanically transparent stage, which we can consider as  $\rho_{ur} = \rho_{ui} = 0$ , with  $k_u = 0$ .

\* Note.—The extreme profiles (disposed in row IV) in Fig. 8, since they are disposed in the horizontal row IV (either topmost or bottommost) should retain the conditions (23 a) and (23 b). As analysed in Section 7.1,  $\vec{w}_1$ ,  $\vec{w}_2$  should be conjugate and as well  $\vec{c}_1$ ,  $\vec{w}_2$  and  $\vec{c}_2$ ,  $\vec{w}_1$  in pairs should be conjugate as seen above. This is possible only if  $\vec{c}_1$  coincides with  $\vec{w}_1$  in the inlet and  $\vec{c}_2$  coincides with  $\vec{w}_2$  in the exit velocity triangles, i.e.,  $u \rightarrow 0$ . When  $u \rightarrow 0$ ,  $n_1 = n_2 \rightarrow \infty$  and these values substituted in (24 a) and (24 b) give  $\rho_{ur} = \rho_{ui} \rightarrow 0.5$ , satisfying the conditions of symmetry, i.e., (23 a) and (23 b). Hence, when  $u \rightarrow 0$ , the profile in row IV retains at the same time the properties of symmetry and congruence, and belongs to the vertical row, as well.





8. GENERAL EXPRESSIONS TO DETERMINE THE OPTIMUM CONDITIONS FOR ANY TURBINE STAGE

It has been shown that optimum conditions, under the two given conditions examined in Section 7, exist when  $n_2 = -1$ , i.e., exit absolute velocity vector  $\vec{c}_2$  is axial. This is the common condition for optimum operation and  $[u/c_1]_{opt}$  can be determined from it. As a matter of fact it is always possible to write

$$n_1 u = c_1 \cos \alpha_1 \tag{2}$$

i.e.,

$$\frac{u}{c_1} = \frac{\cos \alpha_1}{n_1}$$

which under optimum conditions can be written as:

$$\left[ \frac{u}{c_1} \right]_{opt} = \frac{\cos \alpha_1}{n_{1\ opt}} \tag{25}$$

Under optimum conditions, the exit velocity triangle is rectangular,  $\vec{c}_2$  is axial, giving

$$n_2 = n_{2\ opt} = -1 \tag{26}$$

and

$$n_1 = n_{1\ opt} = k_{u\ opt} + n_{2\ opt} + 1 = k_{u\ opt}$$

i.e.,

$$n_{1\ opt} = k_{u\ opt} \tag{26 a}$$

Substituting (26) a, in (25), we have:

$$\left[ \frac{u}{c_1} \right]_{opt} = \frac{\cos \alpha_1}{n_{1\ opt}} = \frac{\cos \alpha_1}{k_{u\ opt}} \tag{27}$$

Again, since

$$\rho_{ur} = \frac{-n_2}{n_1 - n_2 - 1}, \quad \rho_{ur\ opt} = \frac{-n_{2\ opt}}{n_{1\ opt} - n_{2\ opt} - 1}$$

i.e.,

$$\rho_{ur\ opt} = \frac{1}{n_{1\ opt}} \tag{28}$$

Substituting (28) in (27) we have:

$$\left[ \frac{u}{c_1} \right]_{opt} = \rho_{ur\ opt} \cos \alpha_1 = \frac{\cos \alpha_1}{n_{1\ opt}} = \frac{\cos \alpha_1}{k_{u\ opt}} \tag{29}$$

Expression (29) does not give a single solution, unless it is associated with another variable  $c_{1a}/c_{2a}$ , the ratio of the axial components of  $\vec{c}_1$  and  $\vec{c}_2$ . For a given value of  $c_{1a}/c_{2a}$ , expression (29) determines the singular optimum conditions.

Fig. 10 shows the velocity triangles for three turbine stages, chosen from an unlimited number, operating under optimum conditions. *i.e.*,  $\vec{c}_2$  axial,  $\vec{c}_1$  is constant,  $w_1 = w_2$ , *i.e.*,  $\psi = 1$ . Each of the three stages, having the velocity triangles according to Fig. 10, is the most effective one for a certain ratio of  $c_{1a}/c_{2a}$ . Of the three compared stages, *b* is the most profitable, since it is characterised by the least carry over loss ( $c_{1a}/c_{2a} > 1$ ) and the maximum power on blading. Table I gives the characteristics of the three turbine stages represented by the three types of velocity triangles in Fig. 10.

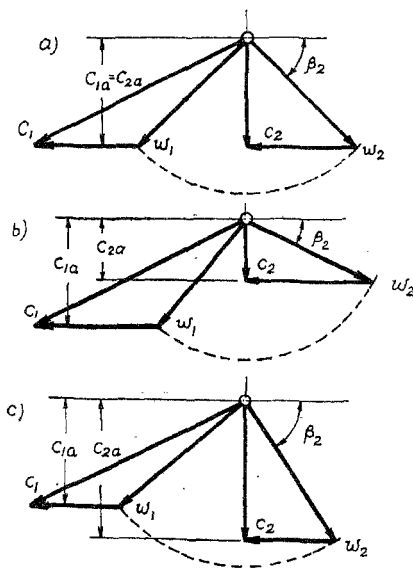


FIG. 10. Turbine stages under optimum conditions ( $n_2 = -1$ ), varying  $c_{1a}/c_{2a}$ .

Problems on design of turbine stages can be worked out by use of expression (29). The following example illustrates the method:

*Example.*—What is the optimum circumferential velocity and the mathematical profile of an axial turbine stage defined by:

stage enthalpy drop:  $h_0 = 28$  Cal/kg.

$$c_0 = 217 \text{ m./sec.}; \phi = 0.95; \psi = 0.9;$$

$$\alpha_1 = 22^\circ (\cos \alpha_1 = 0.927); \rho_{ur} = 0.6 \text{ and } \rho = 0.285.$$

$$\left[ \rho = \frac{h_{02}}{h_0} = \text{degree of thermal reactivity} \right].$$

First determine the theoretical ( $c_{1t}$ ) and actual ( $c_1$ ) discharge velocities from nozzles:

$$\begin{aligned} c_{1t} &= \sqrt{8380(1-\rho) \cdot h_0 + c_0^2} \\ &= \sqrt{8380(1-0.285)28 + 217^2} = 474 \text{ m./sec.} \end{aligned}$$

and

$$c_1 = \phi \cdot c_{1t} = 0.95 \times 474 = 450 \text{ m./sec.}$$

By expression (29), i.e.,

$$\left[ \frac{u}{c_1} \right]_{\text{opt}} = \rho_{ur \text{ opt}} \cdot \cos \alpha_1 = \frac{\cos \alpha_1}{n_1 \text{ opt}} = \frac{\cos \alpha_1}{k_u \text{ opt}}$$

$\rho_{ur}$  is specified as 0.6; i.e.,

$$\left[ \frac{u}{c_1} \right]_{\text{opt}} = 0.6 \times 0.927 = 0.556.$$

Since  $c_1 = 450$  m./sec.; optimum circumferential velocity

$$u_{\text{opt}} = 0.556 \times 450 = 250 \text{ m./sec.}$$

TABLE I

Type	$\frac{u}{\text{m./sec.}}$	$\frac{c_1 \cos \alpha_1}{\text{m./sec.}}$	$\frac{w_2 \cos \beta_2}{\text{m./sec.}}$	$\eta_2$	$\eta_1$	$\rho_{ur}$	$k_u$	$\frac{c_{1a}}{c_{2a}}$	Disposition of mathematical profile in the general scheme
a	150	300	-150	-1	2	0.5	2	1	Under row IV
b	175	300	-175	-1	1.72	0.582	1.72	1.76	Right under half-row III
c	125	300	-125	-1	2.4	0.417	2.4	0.76	Left under half-row III

Fig. 11 shows the velocity triangles. The inlet velocity triangle can be drawn at this stage ( $c_1 = 450$  m./sec.,  $u = 250$  m./sec. and  $\alpha_1 = 22^\circ$ ).  $w_1$  can be read off and  $w_1 = 240$  m./sec.

$w_2$  can be determined since:  $w_2 = \psi \cdot w_{2t} = 0.9 \cdot 353 = 318$  m/sec.

$$w_{2t} = \sqrt{8380 \cdot \rho \cdot h_0 + w_1^2} = \sqrt{8380 \cdot 0.285 \cdot 28 + 240^2} \\ = 353 \text{ m./sec.}$$

The exit triangle can now be drawn. ( $c_2$  is axial for optimum condition.

$\vec{u} = 250$  m./sec. and  $w_2 = 353$  m/sec.)

$c_2$  and  $\beta_2$  can be read off from Fig. 11.

$\beta_2 = 38^\circ 10'$  and  $c_2 = 195$  m./sec.

From the velocity triangles, it can be seen that the mathematical profile will be disposed in the under right half-row III.

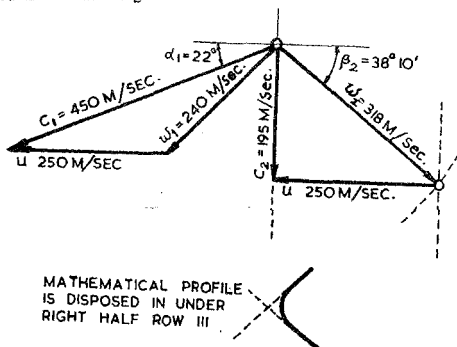


FIG. 11. Turbine stage under optimum conditions.

### 9. CONNECTION BETWEEN CIRCUMFERENTIAL FORCE COEFFICIENT $k_u$ AND AVAILABLE ENERGY OF STAGE

Based on the expressions:

$$W = G \cdot \frac{n_1 - n_2 - 1}{g} \cdot u^2 \dots \text{kgm./sec.} \quad (7)$$

and

$$\dot{W} = \frac{G}{g} \cdot k_u \cdot u^2 \dots \text{kgm./sec.} \quad (12)$$

We can derive the expression connecting  $k_u$  and available energy of stage.

Define efficiency relative to blade (i.e.,  $\eta_{ib}$ ) as

$$\eta_{ib} = \frac{A \cdot I'_{1,2}}{h_0}$$

$$I'_{1,2} = \frac{W}{G} = \frac{1}{g} \cdot k_u \cdot u^2 \dots \text{kgm./kg.} \quad (30)$$

and, since,

$$\eta_{rb} = \frac{A \cdot I'_{1,2}}{h'_0} ; \quad h'_0 = \frac{A}{g} \cdot \frac{k_u}{\eta_{rb}} \cdot u^2 \quad (31)$$

and

$$h'_0 = \frac{1}{4190} \cdot \frac{k_u}{\eta_{rb}} \cdot u^2 \dots \text{Cal./kg.} \quad (32)$$

Let us compare two turbine stages with the same values of  $u$  and  $\eta_{rb}$ :

(a) Impulse stage— $h'_{0i}, \rho = 0, \rho_{ur} = 0.5$  and  $k_u = 2, (n_1 = 2)$ ;

and (b) Congruent stage— $h'_{0r}, \rho = 0.5, \rho_{ur} = 1$  and  $k_u = 1, (n_1 = 1)$ .

In both cases values of  $k_u = n_1$  indicate that they are operating under optimum conditions. ( $n_2 = -1$ ) for particular chosen values of  $c_{1a}/c_{2a}$ .

Using the expression

$$h'_0 = \frac{1}{4190} \cdot \frac{k_u}{\eta_{rb}} \cdot u^2 \quad (32)$$

since  $u$  and  $\eta_{rb}$  remain constant:

$$\frac{h'_{0i}}{h'_{0r}} = \frac{k_{ui}}{k_{ur}} = 2 \quad (33)$$

With  $u$  and  $\eta_{rb}$  remaining the same (that is not always possible), the optimum available energy for an impulse stage with symmetrical deviation of flow ( $\beta_1 = \beta_2$ ) on blades must be twice as in a congruent stage, characterised by  $\rho = 0.5, \rho_{ur} = 1, k_u = 1$ .

*Practically it means that an impulse stage can utilise twice as much enthalpy drop as in a congruent stage, for the same circumferential velocity  $u$ .*

Again for the same stages if  $h'_{0i} = h'_{0r}, u_i \neq u_r$  (and  $\eta_{rbi} = \eta_{rbr}$ ) by expression (32), we can see that:

$$h'_{0i} = \frac{2}{4190 \cdot \eta_{rbi}} \cdot u_i^2 = \frac{1}{4190 \cdot \eta_{rbr}} \cdot u_r^2$$

and hence,

$$2u_i^2 = u_r^2$$

i.e.,

$$u_r = \sqrt{2} \cdot u_i \quad (34)$$

*i.e.*, Development of acceleration by expansion of flow in blading and reactive manner of force transfer from flow to blades by decrease of impulsive manner and increase of circumferential velocity  $u$  (compared to the Impulse stage) to maintain optimum conditions of operation.

Based on expression (32), we can state:

In a turbine with stages I, II, III..... characterised by  $k_{uI}$ ,  $k_{uII}$ ,  $k_{uIII}$ ....., with  $u$  constant and  $\eta_{rb}$  same, the stage available energies are given by:

$$h'_{0I} : h'_{0II} : h'_{0III} : \dots = k_{uI} : k_{uII} : k_{uIII} : \dots \quad (35)$$

If  $u$  differs:

$$h'_{0I} : h'_{0II} : h'_{0III} : \dots = k_{uI} \cdot u_I^2 : k_{uII} \cdot u_{II}^2 : k_{uIII} \cdot u_{III}^2 : \dots \quad (36)$$

If  $k_u$  remains the same ( $\eta_{rb}$  same); different circumferential velocities at the diameter  $d$  of mid-blade length  $u$  are given by:

$$h'_{0I} : h'_{0II} : h'_{0III} : \dots = u_I^2 : u_{II}^2 : u_{III}^2 : \dots \quad (37)$$

*i.e.*,

$$h'_{0I} : h'_{0II} : h'_{0III} : \dots = d_{II}^2 : d_{III}^2 : d_{III}^2 : \dots \quad (38)$$

In the general case, in a multistage turbine characterised by different values of  $u$ ,  $\eta_{rb}$ ,  $k_u$ , we have:

$$\begin{aligned} h'_{0I} : h'_{0II} : h'_{0III} : \dots \\ = \frac{k_{uI} \cdot u_I^2}{\eta_{rbI}} : \frac{k_{uII} \cdot u_{II}^2}{\eta_{rbII}} : \frac{k_{uIII} \cdot u_{III}^2}{\eta_{rbIII}} : \dots \end{aligned} \quad (39)$$

To calculate the necessary available energy (the isentropic enthalpy drop increased by adding the used carry over loss from the preceding stage), we have to choose the values of  $\eta_{rb}$  based on experience. If the values of  $c_2$  and  $\mu_0$  are specified, we can then obtain the isentropic enthalpy drop.

*Example.*—A turbine stage operating under optimum conditions is characterised by:  $k_u = 1.6$ ;  $u = 200$  m/sec.;  $\eta_{rb} = 0.85$ ;  $\mu_0 = 1$  and  $c_{2(m-1)} = 100$  m/sec. Evaluate the work done on blading and determine the necessary enthalpy drop.

Under optimum conditions:  $n_2 = -1$  and  $n_1 = k_u = 1.6$ .

Work done on blading:

$$= A \cdot I_{1,2}' = A \cdot \frac{k_u}{g} \cdot u^2 = \frac{1.6}{4190} \cdot 200^2 = 15.3 \text{ Cal./kg.}$$

Available energy:

$$h_0' = \frac{A \cdot I_{1,2}'}{\eta_{rb}} = \frac{15.3}{0.85} = 18.0 \text{ Cal./kg.}$$

$$h_0' = h_0 + A \cdot \mu_0 \frac{c_2^2}{2g} \frac{c_2^2}{2g} \frac{(m-1)}{2g}$$

i.e.,

$$h_0 = h_0' - \frac{\mu_0 c_2^2}{8380} = 18 - \frac{1 \cdot 100^2}{8380}$$

$$= 16 \cdot 8 \text{ Cal./kg.}$$

If reliable data is available to decide the values of velocity coefficients  $\phi$  and  $\psi$ , the velocity triangles can be drawn for chosen  $\alpha_1$  and  $\rho$ . From the triangles the ratio  $c_{1a}/c_{2a}$  can be determined to examine the given value of  $\eta_v$ .

#### CONCLUSION

Extension of the analysis of the phenomena in the stage of a turbomachine, by the method of equivalent replacements (i.e., the mathematical profile and the equivalent stream replacing the real blade and total flow) gives general expressions to work out problems connected with study of characteristics of stages as well as design. The expressions are simple (devoid of complexity) and are easily manipulated. The methods detailed can easily be extended for the design of a multi-stage turbine, where the complex problem of allocation of energy to the different stages becomes a straight forward application.

Expressing the optimum operational condition (in the circumferential direction) by  $n_2 = -1$  is merely a general one. A singular solution is possible only when this general condition is associated with a particular value of the ratio  $c_{1a}/c_{2a}$ . This aspect of the question has not been brought out by any analysis so far. This paper throws light in this aspect and has outlined a simple method of evaluating stage characteristics. There does not appear any need to complicate expressions to cover any general case of a turbine stage.

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#### APPENDIX

Refer to Fig. 6 of paper "The Method of Equivalent Replacements applied to the Investigation of Force Transfer and Power Exchange in a Stage of a Turbomachine" in this journal.



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