THE EQUILIBRIUM OF A SELF-GRAVITATING INCOMPRESSIBLE FLUID SPHERE WITH MAGNETIC FIELD

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1. The study of the problem of equilibrium and stability of a fluid sphere under its own gravitation and an imposed magnetic field has been initiated by Chandrasekhar and Fermi[1] with a view to understand the problem of magnetic stars. Since then a number of workers[2-9] have contributed significantly towards the understanding of this problem. In the present paper, we discuss the problem of magnetic fields that can prevail in an axi-symmetric configuration in a coherent manner. This has enabled us to characterise a new class of axi-symmetric magnetic fields, in addition to point out some immediate generalisations of the fields already discussed. In the latter part of the paper, we consider the equilibrium of axi-symmetric incompressible fluid configuration under the effect of the new class of magnetic fields.

2. Equations of the Problem

The equation of equilibrium is

$$0 = - \operatorname{grad}(p - \rho V) + \vec{L}, \tag{2.1}$$

where ρ, p are the density and hydrostatic pressure of the liquid, V, the gravitational potential and $\vec{L}$, the Lorentz force given by

$$\vec{L} = \vec{j} \times \vec{H}, \tag{2.2}$$

$\vec{j}$ and $\vec{H}$ being the current vector and magnetic field.

The electromagnetic properties are governed by the Maxwell equations, which in the steady case, in which we are interested, reduce to

$$\operatorname{curl} \vec{H} = 4\pi \vec{j} \tag{2.3}$$

$$\operatorname{div} \vec{H} = 0 \tag{2.4}$$

$$\vec{E} = 0, \tag{2.5}$$

where we have taken the electrical conductivity to be infinite. We may note that our assumption of infinite conductivity is justified on the grounds that in case of finite conductivity the current system cannot be steady.

From (2.2) and (2.3)

$$\vec{L} = \frac{1}{4\pi} (\text{curl } \vec{H}) \times \vec{H} \tag{2.6}$$

From (2.1), the equilibrium will be possible if

$$\text{curl } \vec{L} = 0 \tag{2.7}$$

i.e., the Lorentz force is obtainable from a potential Φ and

$$\vec{L} = -\text{ grad } \Phi. \tag{2.8}$$

Using (2.8) in (2.1), it reduces to

$$p - \rho V + \Phi = K, \text{ a constant.} \tag{2.9}$$

We may rewrite (2.7) as

$$\text{curl } [(\text{curl } \vec{H}) \times \vec{H}] = 0 \tag{2.7'}$$

Thus (2.9) is the condition to be satisfied by a magnetic field that can prevail in an incompressible fluid.

An arbitrary deformation of an incompressible body can be realised by applying at each point of it a displacement $\vec{\xi}$. Without loss of generality, we may assume the displacement to be irrotational, so that a scalar point function ψ exists such that

$$\vec{\xi} = \text{grad } \psi \tag{2.10}$$

Besides, for an incompressible fluid

$$\text{div } \vec{\xi} = 0 \tag{2.11}$$

so that

$$\Delta \psi = 0. \tag{2.12}$$

The regular solution of (2.12) is

$$\psi = \Sigma \psi_n r^n P_n (\mu), \ \mu = \cos \theta, \tag{2.13}$$

where $P_n (\mu)$ is the Legendre polynomial of order n, and μ the angle which the radius vector makes with axis of symmetry.

If a sphere of radius R be deformed to

$$r_s = R \sum_{n=0}^{\infty} \epsilon_n P_n(\mu), \ \epsilon_0 = 1 \tag{2.14}$$

then on comparing the values of ξ_r at $r = R$ obtained from (2.13) and (2.14), we have

$$\psi = R \sum_{n=1}^{\infty} \frac{\epsilon_n}{n} \frac{r^n}{R^{n-1}} P_n(\mu) \tag{2.15}$$

$$\xi_r = R \sum_{n=1}^{\infty} \epsilon_n \left(\frac{r}{R}\right)^{n-1} P_n(\mu) \tag{2.16}$$

and

$$\xi_\theta = -R \sin \theta \sum_{n=1}^{\infty} \frac{\epsilon_n}{n} \left(\frac{r}{R}\right)^{n-1} P'_n(\mu), \tag{2.17}$$

where dash denotes derivation with respect to μ.

Under the assumption of infinite conductivity there exists[1] a simple relation

$$\delta \vec{H} = \text{curl} (\vec{\xi} \times \vec{H_0}) \tag{2.18}$$

between the displacement $\vec{\xi}$ and the corresponding change $\delta \vec{H}$ in the magnetic field, where $\vec{H_0}$ is the initial magnetic field. In writing (2.18) we have neglected the quantities of the order of square of $|\vec{\xi}|$.

3. BOUNDARY CONDITIONS

We shall denote the quantities referred to an inner and an outer point of the configuration by superscripts (i) and (e) respectively.

If there are no currents outside the configuration,

$$\text{div} \ \vec{H}^{(e)} = 0 \tag{3.1}$$

$$\text{curl} \ \vec{H}^{(e)} = 0. \tag{3.2}$$

Hence the external magnetic field can be obtained from a scalar potential W such that

$$\vec{H}^{(e)} = -\text{grad} \ W \tag{3.3}$$

and

$$\Delta W = 0. \tag{3.4}$$

In an axi-symmetric case, the general solution of (3.4) is

$$W = \sum_{n=1}^{\infty} A_n \frac{1}{r^{n+1}} P_n(\mu). \tag{3.5}$$

Similarly

$$V^{(e)} = \sum_{n=1}^{\infty} B_n \frac{1}{r^{n+1}} P_n(\mu). \tag{3.6}$$

The boundary conditions for the gravitational potential are:

$$V^{(i)} = V^{(e)} \tag{3.7}$$

$$\frac{\partial V^{(i)}}{\partial n} = \frac{\partial V^{(e)}}{\partial n} \tag{3.8}$$

on the surface of the configuration, where ∂/∂_n denotes differentiation along the outward drawn normal to the boundary.

The boundary conditions on the magnetic field can be expressed vectorially in the form

$$\vec{H}^{(e)} - \vec{H}^{(i)} = 4\pi \vec{j}^* \times \vec{n}, \tag{3.9}$$

where $\vec{j}^*$ is the surface current vector per unit width of the surface layer.

From (3.9), we have

$$\vec{j}^* = \frac{1}{4\pi} \vec{n} \times (\vec{H}^{(e)} - \vec{H}^{(i)}). \tag{3.10}$$

When there is no surface current

$$\vec{H}^{(e)} = \vec{H}^{(i)} \tag{3.11}$$

i.e., $\vec{H}$ is continuous across the boundary of the configuration.

At the boundary of the configuration, the total pressure P, namely the sum of the hydrostatic pressure and the magnetic pressure, is continuous. Hence on the boundary of the configuration

$$p + \frac{1}{8\pi} [|\vec{H}^{(i)}|]^2 = \frac{1}{8\pi} [|\vec{H}^{(e)}|]^2. \tag{3.12}$$

Thus in the absence of the surface currents, the boundary is defined by

$$p = 0, \tag{3.13}$$

but when the surface currents are present

$$p = \frac{1}{8\pi} \{[|\vec{H}^{(e)}|]^2 - [|\vec{H}^{(i)}|]^2\}$$

$$= 2\pi (j^*)^2 + j^* H_{j\perp}{}^{(i)}, \tag{3.14}$$

where $H_{j\perp}$ is the component of the magnetic field perpendicular to $\vec{j}^*$ and tangential to the surface and j^* is the magnitude of the surface current per unit width of the layer. Thus in this case the hydrostatic pressure is not zero on the boundary.

4. MAGNETIC FIELDS IN AN AXI-SYMMETRIC CONFIGURATION

Chandrasekhar and Prendergast[8] have reduced to a simple form the condition (2.7′) as applicable to the axi-symmetric cases. We shall put this condition in still more suitable form, which enables us to enlist few more cases of magnetic fields that can prevail in an axi-symmetric configuration.

We shall take I_r, I_θ, I_ϕ as the unit vector at a point of the configuration, forming a right-handed system as usual. In view of (2.4) we may express the most general axi-symmetric field as

$$\vec{H} = T(r, \theta) I_\phi + \text{curl} [P(r, \theta) I_\phi],$$

$$= -\frac{1}{r} \frac{\partial}{\partial \mu} (\sqrt{1 - \mu^2} P) I_r - \frac{1}{r} \frac{\partial}{\partial r} (rP) I_\theta + T I_\phi, \tag{4.1}$$

where P and T are arbitrary functions of r and θ or $\mu = \cos \theta$.

From (2.3) we have

$$4\pi j_r = (\text{curl } \vec{H})_r = -\frac{1}{r} \frac{\partial}{\partial \mu} (T \sqrt{1 - \mu^2}) \tag{4.2}$$

$$4\pi j_\theta = (\text{curl } \vec{H})_\theta = -\frac{1}{r} \frac{\partial}{\partial r} (rT) \tag{4.3}$$

and

$$4\pi j_\phi = (\text{curl } \vec{H})_\phi = -\frac{1}{r^2} X, \tag{4.4}$$

where

$$X = \frac{\partial}{\partial r} \left(r^2 \frac{\partial P}{\partial r} \right) + \frac{\partial}{\partial \mu} \left\{ (1 - \mu^2) \frac{\partial P}{\partial \mu} \right\} - \frac{P}{1 - \mu^2}. \tag{4.5}$$

Also, from (2.6), we have

$$4\pi L_r = -\frac{T}{r}\frac{\partial}{\partial r}(rT) - \frac{X}{r^3}\frac{\partial}{\partial r}(rP) \tag{4.6}$$

$$4\pi L_\theta = \frac{X}{r^3}\frac{\partial}{\partial\mu}(P\sqrt{1-\mu^2}) + \frac{T}{r}\frac{\partial}{\partial\mu}(T\sqrt{1-\mu^2}) \tag{4.7}$$

and

$$4\pi L_\phi = -\frac{1}{r^3\sqrt{1-\mu^2}}\frac{\partial(r\sqrt{1-\mu^2}T,\ r\sqrt{1-\mu^2}P)}{\partial(r,\ \mu)}. \tag{4.8}$$

The condition (2.7) now gives

$$\frac{\partial}{\partial\mu}(r\sqrt{1-\mu^2}L_\phi) = 0 \tag{4.9}$$

$$\frac{\partial}{\partial r}(r\sqrt{1-\mu^2}L_\phi) = 0 \tag{4.10}$$

$$\frac{\partial}{\partial r}(rL_\theta) - \frac{\partial}{\partial\theta}(L_r) = 0. \tag{4.11}$$

From (4·9) and (4·10)

$$L_\phi = \frac{\text{const.}}{r(1-\mu^2)^{\frac{1}{2}}}$$

In order that L_ϕ be finite everywhere

$$\text{Const.} = 0$$

so that

$$L_\phi = 0. \tag{4.12}$$

Using (4.6), (4.7) in (4.11), we have

$$\frac{\partial\left(\dfrac{X}{r^3\sqrt{(1-\mu^2)}},\ r\sqrt{1-\mu^2}P\right)}{\partial(r,\ \mu)}$$

$$= \frac{1}{r^3(1-\mu^2)^2}\left[r\mu\frac{\partial}{\partial r}\{r^2(1-\mu^2)T^2\} + (1-\mu^2)\frac{\partial}{\partial\mu}\{r^2(1-\mu^2)T^2\}\right] \tag{4.13}$$

We can easily verify that we can conveniently write (4.13) as

$$\frac{\partial\left(\dfrac{X}{r^3\sqrt{(1-\mu^2)}} + \dfrac{G}{r^2(1-\mu^2)},\ r\sqrt{1-\mu^2}\,P\right)}{\partial(r,\ \mu)} = 0, \tag{4.14}$$

where

$$2G\left(Pr\sqrt{1-\mu^2}\right) = \frac{d\left[r^2\left(1-\mu^2\right)T^2\right]}{d\left[Pr\sqrt{1-\mu^2}\right]}. \tag{4.15}$$

From (4.14) we conclude that

$$\frac{X}{r^3\sqrt{(1-\mu^2)}} + \frac{G}{r^2\left(1-\mu^2\right)} = \phi\left(r\sqrt{1-\mu^2}\,P\right), \tag{4.16}$$

where ϕ is an arbitrary function of $r\sqrt{1-\mu^2}P$ and $(4\cdot15)$ determines G when P and T are known.

The above treatment indicates that there may be a considerable simplification if we put

$$Pr\sqrt{1-\mu^2} = \eta \tag{4.17}$$

and

$$Tr\sqrt{1-\mu^2} = \zeta. \tag{4.18}$$

In terms of η and ζ, we have

$$H_r = -\frac{1}{r^2}\frac{\partial\eta}{\partial\mu},\ H_\theta = -\frac{1}{r\sqrt{1-\mu^2}}\frac{\partial\eta}{\partial r},\ H_\phi = \frac{\zeta}{r\sqrt{1-\mu^2}} \tag{4.19}$$

$$4\pi j_r = -\frac{1}{r^2}\frac{\partial\zeta}{\partial\mu},\ 4\pi j_\theta = -\frac{1}{r\sqrt{1-\mu^2}}\frac{\partial\zeta}{\partial r},\ 4\pi j_\phi = -\frac{\bar{X}}{r\sqrt{1-\mu^2}} \tag{4.20}$$

$$\left.\begin{aligned}
4\pi L_r &= -\frac{1}{r^2\left(1-\mu^2\right)}\left[\bar{X}\frac{\partial\eta}{\partial r} + \zeta\frac{\partial\zeta}{\partial r}\right]\\[4pt]
4\pi L_\theta &= \frac{1}{r^3\sqrt{1-\mu^2}}\left[\bar{X}\frac{\partial\eta}{\partial\mu} + \zeta\frac{\partial\zeta}{\partial\mu}\right]\\[4pt]
4\pi L_\phi &= \frac{1}{r^3\sqrt{1-\mu^2}}\frac{\partial\left(\eta,\,\zeta\right)}{\partial\left(r,\,\mu\right)}
\end{aligned}\right\} \tag{4.21}$$

$$\frac{d\zeta^2}{d\eta} = 2G\left(\eta\right) \tag{4.22}$$

and

$$\bar{X} + G\left(\eta\right) = r^2\left(1-\mu^2\right)\phi\left(\eta\right), \tag{4.23}$$

where

$$\bar{X} = \frac{\partial^2\eta}{\partial r^2} + \frac{1-\mu^2}{r^2}\frac{\partial^2\eta}{\partial\mu^2}. \tag{4.24}$$

The functions P and T used by Chandrasekhar and Prendergast[8] are related to the present η and ζ through the following relations:

$$\eta = P\widehat{\omega}^2, \; \zeta = T\widehat{\omega}^2, \; \widehat{\omega} = r\sqrt{1-\mu^2}. \tag{4.25}$$

The general magnetic field (4.19) which can be associated with an axi-symmetric configuration is governed by the equations (4.22) and (4.23), which are the main equations for discussions in the next section.

5. Special Types of Magnetic Fields which can be Associated with Axi-Symmetric Configurations

(a) *Force-free fields.*—The force-free fields are characterised by

$$\vec{L} = 0 \tag{5.1}$$

Hence in view of (4.21) the equations determining η and ζ reduce to

$$\frac{\partial^2\eta}{\partial r^2} + \frac{1-\mu^2}{r^2}\frac{\partial^2\eta}{\partial\mu^2} + G(\eta) = 0 \tag{5.2}$$

and

$$2G(\eta) = \frac{d\zeta^2}{d\eta} \tag{5.3}$$

We may point out a simple case of a force-free field by taking

$$\left.\begin{array}{c} G(\eta) = 0 \\ \\ \zeta = 0 \end{array}\right\} \tag{5.4}$$

so that

In this case the magnetic field is purely poloidal and then the regular solution of (5.2) is

$$\eta = \eta_0 r^{n+1}(1-\mu^2)^{\frac{1}{2}} P_n^1(\mu), \tag{5.5}$$

where η_0 is an arbitrary constant and $P_n^1(\mu)$ is the associated Legendre polynomial of first kind, first order and the nth degree. Here

$$\vec{j} = 0 \tag{5.6}$$

$$H_r = -\eta_0 n(n+1) r^{n-1} P_n(\mu)$$

$$H_\theta = \eta_0(n+1) r^{n-1}(1-\mu^2)^{\frac{1}{2}}\frac{d}{d\mu}[P_n(\mu)]. \tag{5.7}$$

Next simpler case will evidently be to take $G(\eta) = k$, a constant. But in this case

$$\eta = -\frac{k}{2}r^2 + (\text{solution of } \overline{X} = 0) \tag{5.8}$$

but this is unsuitable for in this case H_θ will be unbounded on the axis of symmetry.

The case

$$G(\eta) = a^2\eta \tag{5.9}$$

has been introduced by Lüst and Schlüter[10] and has been discussed thoroughly by Chandrasekhar.[2]

Here

$$\zeta = a\eta \tag{5.10}$$

and

$$\eta = \eta_0 r^{\frac{1}{2}} J_{n+\frac{1}{2}}(ar)(1-\mu^2)^{\frac{1}{2}} P_n^1(\mu) \tag{5.11}$$

In this case the magnetic field is a combination of toroidal and poloidal field and is given by

$$\left.\begin{aligned}
H_r &= -\eta_0 n(n+1) r^{-3/2} J_{n+\frac{1}{2}}(ar) P_n(\mu), \\
H_\theta &= -\eta_0 \frac{1}{r} \frac{d}{dr}[r^{\frac{1}{2}} J_{n+\frac{1}{2}}(ar)] P_n^1(\mu), \\
H_\phi &= \eta_0 a r^{-\frac{1}{2}} J_{n+\frac{1}{2}}(ar) P_n^1(\mu)
\end{aligned}\right\} . \tag{5.12}$$

The solution (5.11) can be reduced to that of Chandrasekhar if we set

$$\eta = Pr^2(1-\mu^2) \tag{5.13}$$

and use the relation

$$C_n^{3/2}(\mu) = -(1-\mu^2)^{-\frac{1}{2}} P^1_{n+1}(\mu) \tag{5.14}$$

between the Gegenbauer polynomials and the associated Legendre polynomials.

We may note that any other choice of $G(\eta)$ will render the equation (5.2) non-linear and hence may be intractable.

(β) *Poloidal Fields.*—These fields are obtained by taking $\zeta = 0$, *i.e.*, $G(\eta) = 0$ in (4.23), which consequently reduces to

$$\frac{\partial^2\eta}{\partial r^2} + \frac{1-\mu^2}{r^2}\frac{\partial^2\eta}{\partial\mu^2} = r^2(1-\mu^2)\phi(\eta). \tag{5.15}$$

We have already discussed the case $\phi(\eta) = 0$. We shall begin by taking $\phi(\eta) = a$ so that (5.15) reduces to

$$\frac{\partial^2\eta}{\partial r^2} + \frac{1-\mu^2}{r^2}\frac{\partial^2\eta}{\partial\mu^2} = ar^2(1-\mu^2). \tag{5.16}$$

We find that

$$\eta = Y(r)(1-\mu^2) \tag{5.17}$$

is a solution of (5·16), where

$$r^2 Y'' - 2Y = ar^4 \tag{5.18}$$

The regular solution of (5.18) is

$$y = a_0 r^2 + \frac{a}{10} r^4 \tag{5.19}$$

so that

$$\eta = \left(a_0 r^2 + \frac{a}{10} r^4 \right) (1 - \mu^2) \tag{5.20}$$

and

$$\left. \begin{array}{l} H_r = 2 \left(a_0 + \frac{a}{10} r^2 \right) \mu, \ H_\theta = -2 \left(a_0 + \frac{a}{5} r^2 \right) \sqrt{1 - \mu^2} \\ H_\phi = 0 \end{array} \right\} \tag{5.21}$$

$$j_r = 0, \ j_\theta = 0, \ J_\phi = -\frac{a}{4\pi} r (1 - \mu^2)^{\frac{1}{2}}. \tag{5.22}$$

The case studied by Ferraro[4] is obtained on taking

$$a_0 = -\frac{1}{6} a R^2 \tag{5.23}$$

He finds that in this case when there is no surface current, *i.e.*, when $\vec{H}^{(i)} = \vec{H}^{(e)}$, the equilibrium configuration is an oblate spheroid with ellipticity

$$\frac{\epsilon}{R} = -\frac{1}{27} \frac{a^2 R^6}{GM^2}. \tag{5.24}$$

Auluck and Kothari[6] have considered the equilibrium configurations with the field corresponding to $a_0 = 0$. In this case the field at the centre is zero and can be realised by superposing on the field in (5.21) a field $-1/3 \, a R^2$ parallel to $\theta = 0$ axis at every point of the configuration and also at each point of the outside space in order to ensure continuity of magnetic field at the boundary of the configuration. They find that the equilibrium configuration is a prolate spheroid with ellipticity.

$$\frac{\epsilon}{R} = \frac{1}{2} \frac{H_0^2 R^4}{GM^2} \ H_0 = \frac{1}{3} a R^2 \tag{5.25}$$

(γ) *Toroidal Field.*—If we take $\eta = 0$, we have only the toroidal field In this case the condition (4.12), *i.e.*, $L_\phi = 0$

or

$$\frac{\partial (\zeta, \eta)}{\partial (r, \mu)} = 0 \tag{5.26}$$

is automatically satisfied, while the condition (4.13) reduces to

$$r\mu \frac{\partial \zeta}{\partial r} + (1 - \mu^2) \frac{\partial \zeta}{\partial \mu} = 0. \tag{5.27}$$

The general solution of (5.27) is

$$\zeta = f(r\sqrt{1 - \mu^2}), \tag{5.28}$$

where f is an arbitrary function.

We shall put

$$\zeta = r^2 (1 - \mu^2) f(r\sqrt{1 - \mu^2}), \tag{5.29}$$

where f is any regular function of $\widehat{\omega}$ to ensure boundedness of the physical quantities.

Then

$$H_\phi = \widehat{\omega} f(\widehat{\omega}) \tag{5.30}$$

$$j_r = \frac{1}{4\pi} \mu \left[\widehat{\omega} \frac{df}{d\widehat{\omega}} + 2f \right], \quad j_\theta = - \frac{\sqrt{1 - \mu^2}}{4\pi} \left[\widehat{\omega} \frac{df}{d\widehat{\omega}} + 2f \right], \quad J_\phi = 0. \tag{5.31}$$

This current system is equivalent to the current system

$$j_z = \frac{1}{4\pi} \frac{1}{\widehat{\omega}} \frac{d}{d\widehat{\omega}} [\widehat{\omega}^2 f(\widehat{\omega})]$$

$$j_{\widehat{\omega}} = 0, \quad j_\phi = 0 \tag{5.32}$$

Recently De[9] has discussed the equilibrium configuration under the influence of magnetic field obtained as a particular case of (5.30) by taking $f(\widehat{\omega}) = H_0$, a constant.

We shall consider the general case (5.30) more thoroughly in §6.

(δ) *Certain Combinations of Toroidal and Poloidal Fields.*—We have seen that the general magnetic field that can prevail in an axi-symmetric configuration is given by (4.19) satisfying the equations (4.22) and (4.23). Recently Prendergast[7] has discussed the simplest case which is equivalent to choosing

$$\phi(\eta) = k, \text{ a constant} \tag{5.33}$$

and

$$G(\eta) = a^2\eta. \tag{5.34}$$

In view of (5.34) we have

$$\zeta = a\eta. \tag{5.35}$$

Also, (4.23) reduces to

$$\frac{\partial^2\eta}{\partial r^2} + \frac{1-\mu^2}{r^2}\frac{\partial^2\eta}{\partial \mu^2} + \alpha^2\eta = kr^2(1-\mu^2). \tag{5.36}$$

The solution of (5.36) will be

$\eta = $ a particular solution of (5.36) + the general solution of

$$\vec{X} + \alpha^2\eta = 0$$

$$= \frac{k}{\alpha^2}r^2(1-\mu^2) + \sum_{n=0}^{\infty} A_n r^{\frac{1}{2}} J_{n+\frac{1}{2}}(\alpha r) P_n^1(\mu) \tag{5.37}$$

from (5.11).

He finds that, if α be suitably chosen, a spherical configuration is stable, boundary being given by $p = 0$. Here $\vec{H}^{(i)}$ vanishes on the boundary and is continuous with $\vec{H}^{(e)} = 0$. This in this case the magnetic field is wholly confined inside the boundary and we have no means to detect it under normal conditions.

We may note here that the above choice of $\phi(\eta)$ and $G(\eta)$ made by Prendergast was motivated by the aim that the equation determining η should be linear, so that its solution is simple.

If we take

$$G(\eta) = \alpha^2\eta, \ \phi(\eta) = \beta^2\eta \tag{5.38}$$

We have again

$$\zeta = \alpha\eta$$

and the equation determining η is

$$\frac{\partial^2\eta}{\partial r^2} + \frac{1-\mu^2}{r^2}\frac{\partial^2\eta}{\partial \mu^2} + [\alpha^2 - \beta^2 r^2(1-\mu^2)]\eta = 0. \tag{5.39}$$

It is evident that the solution of (5.39) will be simplified in $(\widehat{\omega}, z)$ variables. On making this transformation and assuming the solution to be of the form

$$\eta = W(\widehat{\omega}) Z(z) \tag{5.40}$$

we have the following equations to determine W and Z:

$$W'' + [\alpha^2 - k - \beta^2\widehat{\omega}^2] W = 0 \tag{5.41}$$

and

$$Z'' + kZ = 0. \tag{5.42}$$

Whatever the combination of the solutions (5.41) and (5.42) we take, we find that we do not get the regular solutions for $\vec{H}$ and $\vec{j}$.

Thus even if we take G (η) and $\phi(\eta)$ as proportional to η simultaneously we are unable to get the admissible solutions, hence we shall try to satisfy the equation (4.23) in particular manner. This approach allows some suitable choice of G (η) and $\phi(\eta)$ as function of η.

(i) Let us take

$$\eta = a^2 r^2 (1 - \mu^2) \tag{5.43}$$

which is a particular solution of

$$\overline{X} = 0. \tag{5.44}$$

After choosing η as in (5.43) we have from (4.23)

$$G(\eta) = \frac{\eta}{a^2} \phi(\eta) \tag{5.45}$$

so that if we make a suitable choice for $\phi(\eta)$ we fix G (η) and then

$$\zeta = \frac{2}{a^2} [\int \eta \phi(\eta) \, d\eta]^{\frac{1}{2}}. \tag{5.46}$$

Thus if we take

$$\phi(\eta) = a^2 \beta^2 \eta^p \tag{5.47}$$

then

$$G(\eta) = \beta^2 \eta^{p+1} \tag{5.48}$$

and

$$\zeta = \sqrt{\frac{2}{p+2}} \beta \eta^{p+2/2}. \tag{5.49}$$

In this case we have

$$H_r = 2a^2 \mu, \ H_\theta = -2a^2 \sqrt{1-\mu^2}, \ H_\phi = \sqrt{\frac{2}{p+2}} \beta a^{p+2} r^{p+1}(1-\mu^2)^{p+1/2} \tag{5.50}$$

$$\left. \begin{array}{l} j_r = \frac{1}{4\pi} \sqrt{2(p+2)} \, \beta a^{p+2} r^p \mu \, (1-\mu^2)^{p/2} \\[2mm] j_\theta = -\frac{1}{4\pi} \sqrt{2(p+2)} \, \beta a^{p+2} r^p \, (1-\mu^2)^{p+1/2} \\[2mm] j_\phi = 0 \end{array} \right\} . \tag{5.51}$$

We may note here that in this case the magnetic field consists of a constant magnetic field of strength $2a^2$ parallel to the axis of symmetry and a toroidal field H_ϕ whose strength is proportional to ϖ^{p+1}. The current flows parallel to the axis of symmetry and has the strength varying proportional to ϖ^p. The existence of a

surface current sheet is unavoidable in the present case, for a current line running parallel to the axis of symmetry from a point on the surface to another point on the surface can be closed only by continuing it along the surface.

(ii) Let us now take

$$\eta = \alpha^2 r^{2p} (1 - \mu^2)^p. \tag{5.52}$$

On substituting this in (4.23) we get

$$4 (p - 1) \alpha^{2/p} \eta^{p-1/p} + G(\eta) = \frac{\eta^{1/p}}{\alpha^{2/p}} \phi(\eta). \tag{5.53}$$

Let us now put

$$G(\eta) = \eta^{p-1/p} \overline{G}(\eta) \text{ and } \phi(\eta) = \eta^{p-2/p} \overline{\phi}(\eta) \tag{5.54}$$

in (5.53), so that it reduces to

$$4p (p - 1) \alpha^{2/p} + \overline{G}(\eta) = \frac{1}{\alpha^{2/p}} \overline{\phi}(\eta). \tag{5.55}$$

Apparently we have a wide variety of choice for $\overline{G}(\eta)$ and $\overline{\phi}(\eta)$ in (5.55). However, we shall consider here only the following:

i.e.,
$$\left. \begin{array}{l} \overline{G}(\eta) = \beta^2, \ \overline{\phi} = 4p (p - 1) \alpha^{4/p} + \beta^2 \alpha^{2/p} \\[2mm] \alpha = \left(\frac{1}{8p (p - 1)} [\{\beta^4 + 16p (p - 1) \overline{\phi}\}^{\frac{1}{2}} - \beta^2] \right)^{p/2} \end{array} \right\} \tag{5.56}$$

where $\overline{\phi}$ is a positive constant in order that α is real. The last equation in (5.56) determines α in terms of the chosen values of β^2 and $\overline{\phi}$ to be used in (5.52).

$$\zeta = \left(\frac{2p}{2p - 1} \right)^{\frac{1}{2}} \beta \eta^{2p-1/2p}, \tag{5.57}$$

$$\left. \begin{array}{l} H_r = 2p\alpha^2 r^{2p-2} (1 - \mu^2)^{p-1} \mu, \ \ H_\theta = - 2p\alpha^2 r^{2p-2} (1 - \mu^2)^{p-\frac{1}{2}} \\[2mm] H_\phi = \left(\frac{2p}{2p - 1} \right)^{\frac{1}{2}} \beta \alpha^{(2p-1)/p} r^{2p-2} (1 - \mu^2)^{p-1} \end{array} \right\} \tag{5.58}$$

and

$$\left. \begin{array}{l} j_r = \frac{1}{4\pi} \{2p (2p - 1)\}^{\frac{1}{2}} \beta \alpha^{2p-1/p} r^{2p-3} (1 - \mu^2)^{(2p-3)/2} \mu \\[2mm] j_\theta = - \frac{1}{4\pi} \{2p (2p - 1)\}^{\frac{1}{2}} \beta \alpha^{(2p-1)/p} r^{2p-3} (1 - \mu^2)^{p-1} \\[2mm] j_\phi = - \frac{1}{\pi} p (p - 1) \alpha^2 r^{2p-3} (1 - \mu^2)^{(2p-3)/2} \end{array} \right\} \tag{5.59}$$

For regularity of the physical quantities, p is greater than unity.

a4

We may note here that in the new classes of magnetic fields (5.50) and (5.58), ζ is not necessarily a linear function of η as was assumed by Prendergast.

6. EQUILIBRIUM CONFIGURATIONS WITH TOROIDAL FIELD

We shall determine the equilibrium configurations of an incompressible fluid under the action of toroidal field defined by (5.30) choosing, as a particular case,

$$f(\widehat{\omega}) = H_0 + \frac{H_1}{R^2} \widehat{\omega}^2 \tag{6.1}$$

so that

$$H_\phi = H_0 \widehat{\omega} + \frac{H_1}{R^2} \widehat{\omega}^3 \tag{6.2}$$

In order to be able to use (2.9) we shall calculate Φ. Using (5.31) in (2.6) and (2.8) we have

$$\Phi = \text{const.} + \frac{1}{4\pi} \left[H_0^2 \widehat{\omega}^2 + \frac{3}{2} \frac{H_0 H_1}{R^2} \widehat{\omega}^4 + \frac{2 H_1^2}{3 R^4} \widehat{\omega}^6 \right] \tag{6.3}$$

so that

$$\Phi - \frac{1}{8\pi} H_\phi^2 = \text{const.} + \frac{1}{8\pi} \left[H_0^2 \widehat{\omega}^2 + \frac{H_0 H_1}{R^2} \widehat{\omega}^4 + \frac{1}{3} \frac{H_1^2}{R^4} \widehat{\omega}^6 \right] \tag{6.4}$$

$$= \phi_0 - \phi_2 P_2(\mu) + \phi_4 P_4(\mu) - \phi_6 P_6(\mu), \tag{6.5}$$

where

$$\phi_0 = \text{const.} + \frac{1}{8\pi} \left[\frac{2}{3} H_0^2 r^2 + \frac{8}{15} H_0 H_1 \frac{r^4}{R^2} + \frac{16}{105} \frac{H_1^2}{R^4} r^6 \right] \tag{6.6}$$

$$\phi_2 = \frac{1}{8\pi} \left[\frac{2}{3} H_0^2 r^2 + \frac{16}{21} H_0 H_1 \frac{r^4}{R^2} + \frac{16}{63} \frac{H_1^2}{R^4} r^6 \right] \tag{6.7}$$

$$\phi_4 = \frac{1}{8\pi} \left[\frac{8}{35} H_0 H_1 \frac{r^4}{R^4} + \frac{48}{385} \frac{H_1^2}{R^4} r^6 \right] \tag{6.8}$$

and

$$\phi_6 = \frac{1}{8\pi} \left[\frac{16}{693} H_1^2 \frac{r^6}{R^4} \right]. \tag{6.9}$$

We now assume that H_ϕ^2 is so small that the surface of equilibrium configuration differs everywhere from the sphere of radius R by a small quantity of the order of magnitude of H_ϕ^2. We shall neglect the powers of this quantity higher than the first and its product with H_ϕ^2. Let the surface of the equilibrium configuration be

$$r_s = R \left[1 + \sum_{n=1}^{\infty} \epsilon_n P_n(\mu) \right]. \tag{6.10}$$

The gravitational potential of this configuration at an internal point will be given by (11)

$$V^{(i)} = 2\pi G\rho \left(R^2 - \tfrac{1}{3}r^2\right) + 4\pi G\rho R^2 \sum_{n=1}^{\infty} \frac{\epsilon_n}{2n+1} \left(\frac{r}{R}\right)^n P_n(\mu) \qquad (6.11)$$

To our approximation, this reduces to

$$V^{(i)} = \frac{4\pi}{3}\pi\rho R^2 - \frac{8\pi}{3} G\rho R^2 \sum_{n=1}^{\infty} \frac{n-1}{2n+1} \epsilon_n P_n(\mu), \qquad (6.12)$$

on the surface of the configuration.

In this case

$$H_r^{(i)} = 0, \; H_\theta^{(i)} = 0, \; H_\phi = \widehat{\omega} f(\widehat{\omega}).$$

Hence at the boundary

$$H_n^{(i)} = 0$$

and from continuity of normal component

$$H_n^{(e)} = 0$$

everywhere on the surface of the configuration; consequently

$$H_r^{(e)} = H_\theta^{(e)} = 0$$

everywhere on the surface of configuration

from (3.5) it is clear that $H_\phi^{(e)} = 0$.

Thus H_ϕ is discontinuous across the boundary of the configuration, and therefore there is a surface current present, whose direction lies in the meridian plane. From (3.10) the magnitude of the surface current per unit width of the current-sheet is given by

$$j^* = \frac{1}{4\pi} [H_\phi^{(i)}]_{\text{surface}}$$

$$= \frac{R}{4\pi} [H_0 \sin\theta + H_1 \sin^3\theta]. \qquad (6.13)$$

This shows that the boundary of the configuration will be defined by taking $P^{(i)} = P^{(e)}$, i.e., from (2.9) by taking

$$\dot{\Phi} - \frac{1}{8\pi} [H_\phi^{(i)}]^2 - \rho V^{(i)} = \text{const.} \qquad (6.14)$$

Using (6.5) and (6.12) in (6.14) and equating the coefficients of various $P_n(\mu)$ polynomials, we get

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$$\epsilon_2 = \frac{5}{32\pi^2\rho^2G}\left[H_0{}^2 + \frac{8}{7}H_0H_1 + \frac{8}{21}H_1{}^2\right] \tag{6.15}$$

$$\epsilon_4 = -\frac{9}{280\pi^2\rho^2G}\left[H_0H_1 + \frac{6}{11}H_1{}^2\right] \tag{6.16}$$

$$\epsilon_6 = \frac{13H_1{}^2}{4620\pi^2\rho^2G} \tag{6.17}$$

and

$$\epsilon_n = 0, \text{ when } n \neq 2, 4, 6. \tag{6.18}$$

In particular, let $H_1 = 0$, then

$$r_s = R\left[1 + \epsilon_2 P_2(\mu), \right. \tag{6.19}$$

where

$$\epsilon_2 = \frac{5H_0{}^2}{32\pi^2\rho^2G}. \tag{6.20}$$

Thus the equilibrium configuration is a prolate spheroid as found by De,[9] but his expression for ϵ_2 does not agree with our expression in (6.20). This is on account of the fact that the value of $V^{(1)}$ taken in his paper is in error due to neglecting the first term in our expression (6.11).

Let $H_0 = 0$, then

$$r_s = R\left[1 + \epsilon_2 P_2(\mu) + \epsilon_4 P_4(\mu) + \epsilon_6 P_6(\mu)\right] \tag{6.21}$$

where

$$\epsilon_2 = \frac{5H_1{}^2}{84\pi^2\rho^2G} \tag{6.22}$$

$$\epsilon_4 = -\frac{27H_1{}^2}{1540\pi^2\rho^2G} \tag{6.23}$$

and

$$\epsilon_6 = \frac{13H_1{}^2}{4620\pi^2\rho^2G}. \tag{6.24}$$

We have chosen this expression for $f(\widehat{\omega})$ quite arbitrarily. The treatment above shows that the method will be applicable to any choice for $f(\widehat{\omega})$ of the type

$$f(\widehat{\omega}) = \Sigma A_n\widehat{\omega}^n. \tag{6.25}$$

Only the mathematical analysis will become more cumbersome.

7. EQUILIBRIUM CONFIGURATION WHEN TOROIDAL AND ALSO POLOIDAL FIELD IS PRESENT

We shall consider the field given by (5.58).

Here

$$-\frac{\partial \Phi}{\partial r} = L_r = \frac{-1}{4\pi} \frac{1}{r^2 (1 - \mu^2)} \left[\bar{X} \frac{\partial \eta}{\partial r} + \zeta \frac{\partial \zeta}{\partial r} \right] \tag{7.1}$$

$$-\frac{\partial \Phi}{r \partial \theta} = L_\theta = \frac{1}{4\pi} \frac{1}{r^3 (1 - \mu^2)^{\frac{1}{2}}} \left[\bar{X} \frac{\partial \eta}{\partial \mu} + \zeta \frac{\partial \zeta}{\partial \mu} \right] \tag{7.2}$$

$$L_\phi = 0 \tag{7.3}$$

so that

$$d\Phi = \frac{\partial \Phi}{\partial r} dr + \frac{\partial \Phi}{\partial \mu} d\mu = \frac{1}{4\pi} \frac{1}{r^2 (1 - \mu^2)} \left[\bar{X} d\eta + \frac{1}{2} d\zeta^2 \right]$$

$$= \frac{1}{4\pi} \phi(\eta) d\eta,$$

on using (4.22) and (4.23).

Hence

$$\Phi = \text{const.} + \frac{1}{4\pi} \int \phi(\eta) d\eta$$

$$= \text{const.} + \frac{1}{8\pi} \frac{p}{p-1} [4p(p-1) a^4 + \beta^2 a^{4p-2/p}] r^{4(p-1)} (1 - \mu^2)^{2(p-1)} \tag{7.4}$$

on using (5.54) and (5.56).

Therefore

$$\Phi - \frac{1}{8\pi} [H_\phi^{(i)}]^2 = \text{constant} + \frac{1}{8\pi} \left[4p^2 a^4 + \frac{p}{(p-1)(2p-1)} \beta^2 a^{4p-2/p} \right]$$

$$\times r^{4(p-1)} (1 - \mu^2)^{2(p-1)} \tag{7.5}$$

Here $H_\phi^{(i)} \neq 0$, while $H_\phi^{(e)} = 0$ in an axi-symmetric case. Therefore, we expect surface currents, the current sheet lying in the meridian plane. Since the condition of continuity on the normal component of the magnetic field is satisfied by taking

$$H_r^{(i)} = H_r^{(e)}, \ H_\theta^{(i)} = H_\theta^{(e)},$$

the boundary of the configuration will be given by

$$p + \frac{1}{8\pi} (H_\phi^{(i)})^2 = 0$$

or from (2.9) by

$$\Phi - \frac{1}{8\pi} (H_\phi^{(i)})^2 - \rho V^{(i)} = \text{const.} \tag{7.6}$$

i.e., by

$$\frac{1}{8\pi} \left[4p^2 a^4 + \frac{p}{(p-1)(2p-1)} \beta^2 a^{4p-2/p} \right] R^{4(p-1)} (1-\mu^2)^{2(p-1)}$$
$$+ \frac{3M^2G}{2\pi R^4} \sum_{n=1}^{\infty} \frac{n-1}{2n+1} \epsilon_n P_n(\mu) = \text{const.} \tag{7.7}$$

where M is the mass of the spherical configuration and we have assumed as in 6 the boundary of the equilibrium configuration to be given by

$$r_s = R \left[1 + \sum_{n=1}^{\infty} \epsilon_n P_n(\mu) \right]. \tag{7.8}$$

Given a, β, p and the initial spherical configuration, we can determine ϵ_n by inserting the values of $(1-\mu^2)^{2p-2}$ in terms of the Legendre polynomials in (7.7) and equating the coefficients of the polynomials of various degrees on its two sides.

Case (i) $p = 2$:

Taking $p = 2$ in (7.7) we have

$$r_s = R [1 + \epsilon_2 P_2(\mu) + \epsilon_4 P_4(\mu)] \tag{7.9}$$

where

$$\epsilon_2 = \frac{40}{63} \frac{R^8}{M^2G} a^3 (8a + \tfrac{1}{3}\beta^2) \tag{7.10}$$

and

$$\epsilon_4 = -\frac{4}{35} \frac{R^8}{M^2G} a^3 (8a + \tfrac{1}{3}\beta^2). \tag{7.11}$$

We may note here that the magnetic field and current in this case are:

$$\left. \begin{array}{l} H_r = 4a^2 r^2 (1-\mu^2) \mu, \ H_\theta = -4a^2 r^2 (1-\mu^2)^{3/2} \\[2mm] H_\phi = \frac{2}{\sqrt{3}} \beta a^{3/2} r^2 (1-\mu^2) \end{array} \right\} \tag{7.12}$$

and

$$j_r = \frac{\sqrt{3}}{2\pi} \beta a^{3/2} r (1 - \mu^2)^{\frac{1}{2}} \mu, \quad j_\theta = -\frac{\sqrt{3}}{2\pi} \beta a^{3/2} r (1 - \mu^2),$$

$$j_\phi = -\frac{2}{\pi} a^2 r (1 - \mu^2)^{\frac{1}{2}}. \tag{7.13}$$

Case (ii): $p = 3$:

When $p = 3$, we have

$$r_s = R \left[1 + \sum_{s=1}^{4} \epsilon_{2s} P_{2s}(\mu) \right], \tag{7.14}$$

where

$$\epsilon_2 = \frac{640}{693} \frac{R^{12}}{M^2 G} a^3 \left(12a + \frac{1}{10} \beta^2 a^{\frac{4}{3}} \right), \tag{7.15}$$

$$\epsilon_4 = -\frac{1728}{5005} \frac{R^{12} a^3}{M^2 G} \left(12a + \frac{1}{10} \beta^2 a^{\frac{4}{3}} \right), \tag{7.16}$$

$$\epsilon_6 = \frac{1664}{17325} \frac{R^{12}}{M^2 G} a^3 \left(12a + \frac{1}{10} \beta^2 a^{\frac{4}{3}} \right). \tag{7.17}$$

and

$$\epsilon_8 = -\frac{544}{45045} \frac{R^{12}}{M^2 G} a^3 \left(12a + \frac{1}{10} \beta^2 a^{\frac{4}{3}} \right). \tag{7.18}$$

The magnetic field and the current in this case are:

$$\left. \begin{array}{l} H_r = 6a^2 r^4 (1 - \mu^2)^2 \mu, \quad H_\theta = -6a^2 r^4 (1 - \mu^2)^{5/2} \\[2mm] H_\phi = \sqrt{\frac{6}{5}} \beta a^{5/3} r^4 (1 - \mu^2)^2 \end{array} \right\} \tag{7.19}$$

and

$$\left. \begin{array}{l} j_r = \frac{\sqrt{30}}{4\pi} \beta a^{5/3} r^3 (1 - \mu^2)^{3/2} \mu, \quad j_\theta = -\frac{\sqrt{30}}{4\pi} \beta a^{5/3} r^3 (1 - \mu^2)^2 \\[2mm] j_\phi = -\frac{6}{\pi} a^2 r^3 (1 - \mu^2)^{3/2} \end{array} \right\} \tag{7.20}$$

8. ENERGY METHOD

We shall now calculate the total change in the magnetic energy in the case considered in 6.

Let us consider a spherical configuration of radius R having an internal magnetic field

$$H_r^{(i)} = H_\theta^{(i)} = 0, \quad H_\phi^{(i)} = H_0 r \sin \theta + \frac{H_1}{R^2} r^3 \sin^3 \theta \tag{8.1}$$

and a surface current whose magnitude is given by

$$j^* = -\frac{R}{4\pi}[H_0 \sin\theta + H_1 \sin.^3\theta]. \tag{8.2}$$

Let us deform the configuration so that its surface becomes

$$r_s = R[1 + \epsilon_2 P_2 + \epsilon_4 P_4 + \epsilon_6 P_6]. \tag{8.3}$$

From (2.16) and (2.17), we find that this deformation will be realised by giving a displacement whose radial and transverse components are

$$\xi_r = \epsilon_2 r P_2(\mu) + \frac{\epsilon_4}{R^2} r^3 P_4(\mu) + \frac{\epsilon_6}{R^4} r^5 P_6(\mu) \tag{8.4}$$

and

$$\xi_\theta = -\sin\theta\left[\frac{\epsilon_2}{2} r P_2{}'(\mu) + \frac{\epsilon_4}{4} \frac{r^3}{R^2} P_4{}'(\mu) + \frac{\epsilon_6}{6} \frac{r^5}{R^4} P_6{}'(\mu)\right]. \tag{8.5}$$

Now the change in the internal magnetic energy density at a point is given by

$$\delta\frac{(|H^{(i)}|)^2}{8\pi} = \frac{1}{4\pi}\vec{H}_0{}^{(i)} \cdot \left[(\vec{H}^{(i)}.\mathrm{grad})\vec{\xi} - (\vec{\xi}.\mathrm{grad})\vec{H}^{(i)}\right]$$

$$= -\frac{1}{8\pi}\left[\xi_r\frac{\partial}{\partial r} + \frac{1}{r}\xi_\theta\frac{\partial}{\partial\theta}\right]\{H_\phi{}^{(i)}\}^2$$

$$= -\frac{1}{4\pi} r\sin\theta\left(H_0{}^2 + \frac{4H_0 H_1}{R^2} r^2\sin^2\theta + \frac{3H_1{}^2}{R^4} r^4\sin^4\theta\right)$$

$$\times (\sin\theta.\xi_r + \cos\theta.\xi_\theta)$$

$$= -\frac{1}{4\pi} r^2\sin^2\theta\left(H_0{}^2 + \frac{4H_0 H_1}{R^2} r^2\sin^2\theta + \frac{3H_1{}^2}{R^4} r^4\sin^4\theta\right)$$

$$\times \left\{\epsilon_2\left(P_2 - \frac{\cos\theta}{2} P_2{}'\right) + \frac{\epsilon_4}{R^2} r^2\left(P_4 - \frac{\cos\theta}{4} P_4{}'\right)\right.$$

$$\left. + \frac{\epsilon_6}{R^4} r^4\left(P_6 - \frac{\cos\theta}{6} P_6{}'\right)\right\}. \tag{8.6}$$

We shall calculate the change in the magnetic energy in two steps: (i) $\delta M_1{}^{(i)}$, the change in the magnetic energy inside the sphere due to change in the magnetic energy density, and (ii) $\delta M_2{}^{(i)}$ the change in the magnetic energy due to deformation between the surface of the sphere and the new surface according to old energy density:

$$\delta M_2{}^{(i)} = \frac{1}{8\pi}\int_{-1}^{1} d\mu \int_{r=R}^{r=r_s} r^2 dr \int_{\phi=0}^{2\pi} d\phi\left[H_0{}^2 r^2\sin^2\theta + \frac{2H_0 H_1}{R^2} r^4\sin^4\theta\right.$$

$$\left. + \frac{H_1{}^2}{R^4} r^6\sin^6\theta\right]$$

$$= -\frac{1}{15} \epsilon_2 R^5 \left(H_0{}^2 + \frac{16}{7} H_0 H_1 + \frac{8}{7} H_1{}^2 \right)$$

$$+ \frac{8}{315} \epsilon_4 R^5 \left(H_0 H_1 + \frac{9}{11} H_1{}^2 \right) - \frac{8}{3003} \epsilon_6 R^5 H_1{}^2 \quad (8.7)$$

and

$$\delta M_1{}^{(4)} = \int_{-1}^{1} d\mu \int_{r=0}^{R} r^2 dr \int_{\phi=0}^{2\pi} \delta \left[\frac{\{|\vec{H}^{(i)}|\}^2}{8\pi} \right] d\phi$$

$$= \frac{1}{15} \epsilon_2 R^5 \left(H_0{}^2 + \frac{16}{7} H_0 H_1 + \frac{8}{7} H_1{}^2 \right)$$

$$- \frac{8}{315} \epsilon_4 R^5 \left(H_0 H_1 + \frac{9}{11} H_1{}^2 \right) + \frac{8}{3003} \epsilon_6 R^5 H_1{}^2.$$

From (8.11) and (8.12)

$$\delta M^{(4)} = \delta M_1{}^{(4)} + \delta M_2{}^{(i)} = 0. \tag{8.8}$$

Thus we see that the change in internal magnetic energy corresponding to volume currents is zero.

We shall now calculate the change in magnetic energy due to surface current, which exerts a surface force $\vec{F}$. We can prove that

$$\vec{F} = \vec{j}^* \times \tfrac{1}{2} (\vec{H}^{(i)} + \vec{H}^{(e)}). \tag{8.9}$$

Since in our case $\vec{H}^{(e)} = 0$,

$$\vec{F} = \tfrac{1}{2} \vec{j}^* \times \vec{H}^{(i)}. \tag{8.10}$$

The change in the magnetic energy corresponding to the surface current is

$$2\pi R^2 \int_{-1}^{1} (F_r \xi_r + F_\theta \xi_\theta) \, d\mu. \tag{8.11}$$

Hence it is only necessary to calculate F_r and F_θ correct to zeroth powers of ϵ_2, ϵ_4, ϵ_6 if we seek the result correct to their first power. To this approximation

$$F_r = \frac{1}{8\pi} R^2 \sin^2 \theta \, (H_0 + H_1 \sin^2 \theta)^2$$

$$F_\theta = F_\phi = 0. \tag{8.12}$$

Substituting these values in (8.11) and evaluating the integral we get change in magnetic energy $\delta M_s^{(4)}$ due to surface currents:

$$\delta M_s^{(4)} = -\frac{1}{15}\,\epsilon_2 R^5\left(H_0^2 + \frac{16}{7}\,H_0 H_1 + \frac{8}{7}\,H_1^2\right)$$

$$+\frac{8}{35}\,\epsilon_4 R^5\left(\frac{1}{9}\,H_0 H_1 + \frac{1}{11}\,H_1^2\right) - \frac{8}{3003}\,\epsilon_6 R^5 H_1^2. \qquad (8.13)$$

Thus, since $\delta M^{(e)} = 0$, the total change in magnetic energy is
$$\delta M = \delta M_s^{(4)}, \qquad (8.14)$$

where ϵ_2, ϵ_4, ϵ_6 have the values determined in (6.15)–(6.17).

If for a moment we put $H_1 = 0$ and ignore the value of ϵ_2 determined in (6.15), then

$$\delta M = -\frac{1}{15}\,\epsilon_2 R^5 H_0^2. \qquad (8.15)$$

Also the change in the potential energy for P_2-deformation is given by

$$\delta\Omega = \frac{3}{25}\,\frac{GM^2}{R}\,\epsilon_2^2. \qquad (8.16)$$

Therefore the change in the total energy δE of the configuration is given by

$$\delta E = -\frac{1}{15}\,\epsilon_2 H_0^2 R^5 + \frac{3}{25}\,\frac{GM^2}{R}\,\epsilon_2^2. \qquad (8.17)$$

Hence on minimising δE for equilibrium, we have

$$\epsilon_2 = \frac{5}{18}\,\frac{H_0^2 R^6}{GM^2} = \frac{5H_0^2}{32\pi^2\rho^2 G} \qquad (8.18)$$

which is same as in (6.20).

In passing we may note that the energy method will not allow us to determine more than one quantity from among ϵ_2, ϵ_4, ϵ_6 as they are not independent, for example in the present case

$$\epsilon_4^2 = \frac{27}{13}\,\epsilon_6\left(\frac{198}{175}\,\epsilon_2 + \frac{2}{7}\,\epsilon_4 - \frac{45}{13}\,\epsilon_6\right). \qquad (8.19)$$

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