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FLOW OF A CONDUCTING FLUID PAST A ROTATING MAGNETIZED SPHERE

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ABSTRACT

(We have studied the flow characteristics of conducting fluid past a conducting magnetized sphere rotating about the magnetic axis under the assumption that the flow at infinity is uniform and parallel to the axis considering only the first order effect of the magnetic field and conductivity. The main results are that (i) the conductivity of the sphere affects the flow characteristics, (ii) the rotation of the sphere induces the toroidal component of velocity and the magnetic field, (iii) the toroidal component of fluid velocity tends to vanish as the stagnation point is approached, (iv) the toroidal component of vorticity at a large distance from the sphere is affected by rotation only when the magnetic field originating in the sphere has no component due to a dipole, (v) rotation does not affect the drag on the sphere, but there is a torque opposing the rotation.)

INTRODUCTION

(Recently Chester (1957, 1961) has studied under various assumptions the problem of estimating the effect of a uniform external magnetic field on the flow past a sphere or a body of revolution which at infinity, along with the magnetic field, is parallel to the axis of symmetry. Ludford and Murray (1960) have discussed the flow of an inviscid and finitely conducting liquid past a magnetized sphere for small values of the dimensionless parameter β representing the ratio of a standard magnetic pressure to the free stream dynamic pressure. Murray and Chi (1960) have considered the corresponding problem for a magnetized cylinder.

In all these problems the conductivity of the body does not come into the picture. Chakraborty (1962) has recently studied the characteristics of the flow past a conducting rotating cylinder under the assumption that the flow and the magnetic field at infinity are uniform and normal to the axis of cylinder. The conductivity of the cylinder in this case affects the flow characteristics. It is found that the Maxwell stresses produce a torque proportional to the angular velocity Ω and tends to oppose the motion.

In the present note we have studied the flow characteristics of a conducting fluid past a conducting magnetized sphere rotating about the magnetic axis under the assumption that the flow at infinity is uniform and parallel to this axis. As in the problem of Ludford and Murray (1960) we have considered only the first order effect of the magnetic field and the conductivity. We find that (i) the conductivity of the sphere affects the flow characteristics, (ii) the toroidal component of vorticity near the sphere and the poloidal component of magnetic field are not affected by rotation, (iii) toroidal components of magnetic field and velocity are induced by rotation of the sphere, (iv) in contrast with the behaviour of the toroidal component of the vorticity which becomes logarithmically infinite near the sphere, the toroidal component of liquid velocity tends to vanish as the stagnation point is approached, (v) when the magnetic field originating in the sphere has originally before rotation a component due to a dipole at the centre, the toroidal component of vorticity at a large distance from the sphere is not affected by rotation of the sphere, but when this magnetic field does not have such a component, the rotation of the sphere affects vorticity, (vi) the drag on the sphere is not affected by rotation to the present approximation, and (vii) the torque on the sphere is proportional to the angular velocity of it and the magnetic properties of the body.

THE BASIC EQUATIONS

We shall take the centre of the sphere as the origin, the axis of rotation as the $\theta = 0$ axis of the spherical polar coordinates, and use the M.K.S. system of units.

Using the radius a of the sphere, the uniform velocity U at infinity, the magnitude h of a representative magnetic field and μUh as the magnitude of a standard electric field as standard quantities, the equations of the problem reduce to the following dimensionless forms :

$$\text{curl}\,\mathbf{q} \times \mathbf{q} = -\,\text{grad}\,P + \beta\,\text{curl}\,\mathbf{H} \times \mathbf{H}, \qquad [1]$$

$$\text{div}\,\mathbf{q} = 0, \qquad [2]$$

$$\text{curl}\,\mathbf{H} = R_M[\mathbf{E} + \mathbf{q} \times \mathbf{H}]\,;\ \ \text{div}\,\mathbf{H} = 0\,;\ \ \text{curl}\,\mathbf{E} = 0, \qquad [3, 4, 5]$$

where

$$P = p + \tfrac{1}{2}\mathbf{q}^2\,;\ \ \beta = \mu h^2/\rho U^2 \text{ and } R_M = \mu a \sigma U.$$

We can conveniently choose h to be such that μh^2 is equal to the average magnetic energy per unit volume on the surface of the sphere due to the undisturbed magnetic field. When $\beta = 0$, *i.e.*, there is no magnetic field, the flow of the fluid is given by the usual potential flow

$$\mathbf{q}_0 = \{[1 - 1/r^3]\cos\theta,\ -[1 + 1/(2r^3)]\sin\theta,\ 0\}, \qquad [6]$$

where $\mathbf{q}_0$ ir curl free.

When magnetic field is weak, we expand in powers of β

$$\mathbf{q} = \mathbf{q}_0 + \beta\mathbf{q}_1 + \beta^2\mathbf{q}_2 + \cdots ; \quad p = p_0 + \beta\,\mathbf{p}_1 + \beta^2\,\mathbf{p}_2 + \cdots ,$$
$$\mathbf{H} = \mathbf{H}_0 + \beta\mathbf{H}_1 + \beta^2\mathbf{H}_2 + \cdots \qquad [7]$$

From [3], [4] and [7] we shall have

$$\text{curl}\,\mathbf{H}_0 = R_M[\mathbf{E}_0 + \mathbf{q}_0 \times \mathbf{H}_0], \qquad [8]$$

$$\text{dire}\,\mathbf{H}_0 = 0\ ; \quad \text{curl}\,\mathbf{E}_0 = 0, \qquad [9, 10]$$

$\mathbf{q}_0$ being given by [6]. Also equations [1] and [2] give

$$\text{curl}\,\mathbf{q}_1 \times \mathbf{q}_0 = -\,\text{grad}\,P_1 + \text{curl}\,\mathbf{H}_0 \times \mathbf{H}_0, \qquad [11]$$

$$\text{dire}\,\mathbf{q}_1 = 0. \qquad [12]$$

Here

$$P_1 = p_1 + \mathbf{q}_1 \cdot \mathbf{q}_0. \qquad [13]$$

We shall restrict ourselves to discussing $\mathbf{H}_0$, $\mathbf{q}_1$ and curl $\mathbf{q}_1$. The perturbation due to β is regular.

Determination of $\mathbf{H}_0$

We shall drop the subscript " 0 " in $\mathbf{H}_0$ and $\mathbf{E}_0$ and " 1 " in $\mathbf{q}_1$ and $\mathbf{p}_1$. In the axisymmetric case the equation [9] shows that we can write

$$H_r = \frac{1}{r^2\sin\theta}\cdot\frac{\partial A}{\partial\theta}, \qquad H_\theta = -\frac{1}{r\sin\theta}\cdot\frac{\partial A}{\partial r} \qquad [14, 15]$$

The equation [10] shows that

$$E_\phi = 0. \qquad [15]$$

The ϕ-component of the equation [8], in view of [14], and [15] gives

$$\frac{\partial^2 A}{\partial r^2} + \frac{\sin\theta}{r^2}\cdot\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\partial}\cdot\frac{\partial A}{\partial\theta}\right)$$

$$= R_M\left[\left(1 - \frac{1}{r^3}\right)\cos\theta\,\frac{\partial A}{\partial r} - \frac{1}{r}\left(1 + \frac{1}{2r^3}\right)\sin\theta\,\frac{\partial A}{\partial\theta}\right]. \qquad [16]$$

Noting that $\mathbf{q}_0 = (0, 0, \Omega r \sin\theta)$ for $r < 1$, we have from the ϕ-component of the equation corresponding to [8] holding inside the sphere

$$r^2 \frac{\partial^2 A^i}{\partial r^2} + \sin\theta \frac{\partial}{\partial\theta}\left\{\frac{1}{\sin\theta}\cdot\frac{\partial A^i}{\partial\theta}\right\} = 0. \qquad [17]$$

We have used superscript i for inner quantities. The equations [16] and [17], together with the boundary conditions that μH_r and H_θ are continuous at $r = 1$, determine the poloidal field completely. These equations and the boundary conditions are, however, not affected by the rotation of the sphere. Hence, the poloidal components of the magnetic field are exactly the same as in the absence of rotation, and as have been obtained by Ludford ahd Murray (1959, 1960).

When R_M is small, an appropriate solution of [16] may be obtained by a perturbation which is, however, not regular at infinity. The regular perturbation is taken by substituting

$$A = \alpha \exp.[-R_M r(1-\cos\theta)/2], \qquad [18]$$

and determining α from the equation

$$\frac{\partial^2\alpha}{\partial r^2} + \frac{\sin\theta}{r^2}\cdot\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\cdot\frac{\partial\alpha}{\partial\theta}\right)$$

$$= R_M\left[\left(1 - \frac{\cos\theta}{r^3}\right)\frac{\partial\alpha}{\partial r} - \frac{\sin\theta}{2r^4}\cdot\frac{\partial\alpha}{\partial\theta} + R_M(1-\cos\theta)(1+3\cos\theta)\frac{\alpha}{4r^3}\right] \quad [19]$$

If we represent the undisturbed field of an arbitrary magnetic distribution by

$$A = \sum_{n=1}^{\infty} \frac{A_n}{r^n}\sin^2\theta\, P_n'(\text{oos}\,\theta), \qquad [20]$$

the poloidal field outside the sphere is given by

$$A = \exp.\left[-\tfrac{1}{2}R_M r(1-\cos\theta)\right] \sum_{n=1}^{\infty} A_n\alpha_n, \qquad [21]$$

where, to the order R_M (low conductivity or slow motion of liquid and sphere), α_n is given by

$$\alpha_n = (a_n/r^n)\sin^2\theta\, P_n'(\cos\theta) + a_n R_M \sin^2\theta\left[\frac{n}{4(2n+1)}\frac{1}{r^{n+2}}P_{n-1}'(\cos\theta) + \right.$$

$$\left. + \frac{1}{2}\frac{1}{r^{n-1}}P_n'(\cos\theta) + \frac{n(n-1)}{4(n+2)(2n+1)}\frac{1}{r^{n+2}}P_{n+1}'(\cos\theta)\right] +$$

$$+ R_M \sin^2\theta \left[\lambda_{n1} \frac{1}{r^{n-1}} P'_{n-1}(\cos\theta) + \lambda_{n2} \frac{1}{r^n} P'_n(\cos\theta) + \right.$$

$$\left. + \lambda_{n3} \frac{1}{r^{n+1}} P'_{n+1}(\cos\theta) \right]. \qquad [22]$$

Also, for inside the sphere,

$$A^i = \sum_{n=1}^{\infty} A_n \left\{ [(1/r^n) + k_n r^{n+1}] \sin^2\vartheta \, P'_u(\cos\theta) + \right.$$

$$+ R_M \sin^2\theta \, [l_{n1} r^n P'_{n-1}(\cos\theta) + l_{n2} r^{n+1} P'_n(\cos\theta) +$$

$$\left. + l_{n3} r^{n+2} P'_{n+1}(\cos\theta)] \right\}. \qquad [23]$$

In the above $P_n(\text{os}\,\theta)$ is Legendre's polynomial of degree n and argument $\cos\theta$,* and $P'_n(\cos\theta)$ is its derivative.

The constants λ's and l's have been determined by Ludford and Murray (1959) by satisfying the boundary conditions on the sphere to the order R_M. These are

$$a_n = \frac{(2n+1)\mu^i}{(n+1)\mu + n\mu_i}, \qquad \lambda_{n1} = -\frac{[n(3n+2)\mu + (3n^2+2n-2)\mu^i]}{4(2n+1)[n\mu + (n-1)\mu^i]} a_n,$$

$$\lambda_{n2} = 0, \qquad \lambda_{n3} = -\frac{3n[(n+1)\mu + (n-1)\mu_i]}{4(2n+1)[(n+2)\mu + (n+1)\mu^i]} a_n,$$

and

$$k_n = \frac{n(\mu - \mu^i)}{(n+1)\mu + n\mu^i}, \qquad l_{n1} = -\frac{3n\mu}{4(2n+1)[n\mu + (n-1)\mu^i]} a_n,$$

$$l_{n2} = 0, \qquad l_{n3} = \frac{3n(n+3)\mu}{4(n+2)(2n+1)[(n+2)\mu + (n+1)\mu^i]} a_n, \qquad [23a]$$

where μ^i is the permeability of the sphere.

We shall now determine the toroidal magnetic field component.

Taking the curl of [8] and using [10], we have, from the ϕ-component of the resulting equation

* For the various recurence relations and orthogonal properties of the Legendre's polynomials and their derivatives used here see Copson (1935).

$$\frac{\partial^2\chi}{\partial r^2}+\frac{(1-t^2)}{r^2}\frac{\partial^2\chi}{\partial t^2}$$

$$=R_M\left\{\left(1-\frac{1}{r^3}\right)\cos\theta\frac{\partial\chi}{\partial r}+\frac{3\chi\cos\theta}{r^4}+\frac{1}{r}\left(1+\frac{1}{2r^3}\right)\sin^2\theta\frac{\partial\chi}{\partial t}\right\}, \qquad [24]$$

where $$t=\cos\theta \text{ and } \chi=r\sin\theta\, H_{0\phi}. \qquad [25]$$

When R_M is small an approximate solution for χ can be found out by a perturbation expansion, which, however, is not regular at infinity, the right-hand side vanishing more slowly than the left-hand side as $r\to\infty$ and χ is algebraic.

When r is large and the small terms on the right-hand side are neglected, [24] approximates to

$$\frac{\partial^2\chi}{\partial r^2}+\frac{(1-t^2)}{r^2}\cdot\frac{\partial^2\chi}{\partial t^2}=R_M\left\{\cos\theta\frac{\partial\chi}{\partial r}+\frac{(1-t^2)}{r}\cdot\frac{\partial\chi}{\partial t}\right\}. \qquad [26]$$

Proceeding as in the case of the equation [16], we can show here that for large r,

$$\chi\sim\sqrt{[\pi/R_M\cdot]}\ \sin^2\theta\ P_n'(\cos\theta)\exp.\left[-\tfrac{1}{2}R_M r(1-\cos\theta)\right]. \qquad [27]$$

Thus, the ϕ-component of the magnetic field is swept into a wake behind the sphere whose boundary is the paraboloid of revolution $r(1-\cos\theta)=b/R_M$, where b is a suitably chosen constant

Ludford and Murray (1959) have found a similar behaviour of the poloidal component of the magnetic field at a large distance from the sphere. The above discussion suggests that when R_M is small, we should assume the toroidal field in the form:

$$\chi=\gamma\exp.\left[-\tfrac{1}{2}R_M r(1-\cos\theta)\right] \qquad [28]$$

in [24] and solve for γ from the equation obtained by substituting [28] in [24], namely

$$\frac{\partial^2\gamma}{\partial r^2}+\frac{\sin\theta}{r^2}\cdot\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\cdot\frac{\partial\gamma}{\partial\theta}\right)$$

$$=R_M\left[\frac{3\cos\theta\,\gamma}{r^4}+\left(1-\frac{\cos\theta}{r^3}\right)\frac{\partial\gamma}{\partial r}-\frac{\sin\theta}{2r^4}\cdot\frac{\partial\gamma}{\partial\theta}+\right.$$

$$+R_M\gamma\left\{-\frac{(1-\cos\theta)^2}{4}-\frac{\cos\theta(1-\cos\theta)}{2}\left(1-\frac{1}{r^3}\right)+\right.$$

$$\left.\left.+\frac{r\sin^2\theta}{2}\left(\frac{1}{2r}+\frac{1}{2r^4}\right)\right\}\right]. \qquad [29]$$

by perturbation in R_M. Such a perturbation is regular for the function χ.

From [28] and [29] we find that the solution for χ correct upto first order in R_M can be written as

$$\chi = \exp.\left[-\tfrac{1}{2} R_M r(1-\cos\theta)\right] \times$$

$$\times \left[\frac{B_n}{r^n}\sin^2\theta\, P_n'(\cos\theta) + B_n R_M(1-\cos^2\theta)\left\{\frac{n(n+5)\,P_{n+1}'(\cos\theta)}{4(n+2)(2n+1)\,r^{n+2}}\right.\right.$$

$$\left. + \frac{P_n'(\cos\theta)}{2r^{n-1}} + \frac{(n+2)\,P_{n-1}'(\cos\theta)}{4(2n+1)\,r^{n+2}}\right\} + R_M(1-\cos^2\theta)\left\{\frac{\nu_{n1} P_{n-1}'(\cos\theta)}{r^{n-1}}\right.$$

$$\left.\left. + \frac{\nu_{n2} P_n'(\cos\theta)}{r^n} + \frac{\nu_{n3} P_{n+1}'\cos\theta)}{r^{n+1}}\right\}\right]. \qquad [30]$$

where $\dot{B}_n$, ν_{n1}, ν_{n2} and ν_{n3} are constants to be suitably determined. Noting that $\mathbf{q} = (0, 0, r\,\Omega\sin\theta)$ for $r < 1$, and taking the curl of [8] we have

$$\text{curl curl } \mathbf{H} = 0, \qquad [31]$$

The ϕ-component of [31] gives

$$\frac{\partial^2\chi}{\partial r^2} + \frac{\sin\theta}{r^2}\cdot\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\cdot\frac{\partial\chi}{\partial\theta}\right) = 0. \qquad [32]$$

The solution of [32] can be written as

$$\chi = C_n(1-\cos^2\theta)\,P_n'(\cos\theta)\,r^{n+1} + R_M\left[D_{n+1}(1-\cos^2\theta)\,P_{n+1}'(\cos\theta)\,r^{n+2} + \right.$$

$$\left. + D_n(1-\cos^2\theta)\,P_n'(\cos\theta)\,r^{n+1} + D_{n-1}(1-\cos^2\theta)\,P_{n-1}'(\cos\theta)\,r^n\right]. \qquad [33]$$

To determine the constants B_n, ν's, D's and C's in [30] and [33] we apply the following boundary conditions at $r = 1$, correct to order R_M,

(i) continuity of the normal component of the volume current,
(ii) continuity of χ (or H_ϕ) to avoid surface currents,
(iii) continuity of the θ-component of the electric field. (E_ϕ is zero both inside and outside).

The conditions (i) and (it) are equivalent. For the normal component of the volume current on the sphere is $(\text{curl}\,\mathbf{H})_r = [1/(r^2\sin\theta)]\cdot\partial(r\sin\theta\, H_\phi)/\partial\theta$ evaluated at $r = 1$. Hence, if H_ϕ is continuous on the surface of the sphere $r = 1$, so is $(\text{curl}\,\mathbf{H})_r$ and *vice-versa*. (ii) gives

$$B_n = C_n, \qquad [34]$$

$$\frac{3B_n n(n+3)}{4(2n+1)(n+2)} + \nu_{n3} = D_{n+1}, \qquad [35]$$

$$\nu_{n2} = D_n\,; \quad \frac{B_n(3n+4)}{4(2n+1)} + \nu_{n1} = D_{n-1}, \qquad [36, 37]$$

The condition (iii) gives, in view of [8], and the similar equation for inside

$$[\mathrm{curl}\,\mathbf{H} - R_M \mathbf{q} \times \mathbf{H}]_\theta = \frac{\mu^i}{\mu}\left[\frac{R_M}{R_M^i}\,\mathrm{curl}\,\mathbf{H}_i - R_M \mathbf{q}_i \times \mathbf{H}_i\right]_\theta, \qquad [38]$$

at $r = 1$, where $R_M^i = \mu^i a \sigma_i U$ is the magnetic Reynolds number for the sphere and σ_i is its conductivity.

In [38], $\mathbf{H}$ and $\mathbf{q}$ on the left-hand side stand for the magnetic field and the velocity outside the sphere and $\mathbf{H}_i$ and $\mathbf{q}_i$ on the right hand side stand for the corresponding quantities inside the sphere in dimentionless forms.

We satisfy the boundary condition [38] to the order R_M. Three cases are to be considered.

(*a*) $(R_M/R_M^i) = 0\,(1)$, (*b*) $R_M^i \sim 1$,

(*c*) $(1/R_M^i) \sim R_M$ or higher powers of R_M.

Case (*a*): $$n B_n = -(\mu^i/\mu)\,(R_M/R_M^i)\,C_n\,(n+1), \qquad [39]$$

obtnined by equating the coefficients of $P_n'(\cos\theta)$ that are of the order 1. [34] and [39] implies that

$$B_n = C_n = 0. \qquad [40]$$

Similarly by equating the coefficients of $R_M P_n'(\cos\theta)$ we have

$$n\,\nu_{n2} = -\frac{\mu^i R_M D_n (n+1)}{\mu R_M^i}. \qquad [41]$$

[36] and [41] give

$$\nu_{n2} = D_n = 0. \qquad [42]$$

The other constants can be easily evaluated similarly. We give the final results.

$$\nu_{n1} = D_{n-1} = \frac{A_n \Omega'(1+k_n)\,n(n+1)}{(2n+1)[(n-1) + (\mu^i n R_M)/(\mu R_M^i)]}, \qquad [43]$$

$$\nu_{n3} = D_{n+1} = -\frac{A_n \Omega' \{(1+k_n)\,n(n+1)\}}{(2n+1)\,[(n+1) + (\mu^i R_M)/(\mu R_M^i)(n+2)]}. \qquad [44]$$

where $$\Omega' = (\mu^i \Omega)/\mu.$$

Case (*b*): Proceeding as is case (*a*) we have

$$B_n = \nu_{n2} = C_n = D_n = 0, \qquad [45]$$

$$\nu_{n1} = D_{n-1} = \frac{A_n \Omega' (1+k_n) n(n+1)}{(2n+1)(n-1)}, \qquad [46]$$

$$\nu_{n3} = D_{n+1} = -A_n \Omega' \frac{(1+k_n) n}{(2n+1)}. \qquad [47]$$

Case (c) : The values of the constants are the same as in case (b).

BEHAVIOUR OF VORTICITY AND TOROIDAL COMPONENT OF VELOCITY

Let curl $\mathbf{q} = (\Omega_r, \ \Omega_\theta, \ \omega)$ [48]

and $$(\text{curl}\,\mathbf{H}) \times \mathbf{H} = (F_r, \ F_\theta, \ F_\phi). \qquad [49]$$

We introduce the potential function $\phi_0 = r\,(1 + 1/2r^3)\cos\theta$, and the stream function $\psi_0 = r\sqrt{\{[1 - 1/r^3]\}}\sin\theta$ of the potential flow [6] as the new independent variables in place of r and θ. We take the curl of [11] and the ϕ-component of the resulting equation gives

$$r\left[\left(1 - \frac{1}{r^3}\right)^2 \cos^2\theta + \left(1 + \frac{1}{2r^3}\right)^2 \sin^2\theta\right]\frac{\partial\tilde{\omega}}{\partial\phi_0} = f(r, \ \theta; \ R_M) \qquad [50]$$

where $$\tilde{\omega} = \omega/r\sin\theta$$

and $$f(r, \ \theta; \ R_M) = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial r}(rF_\theta) - \frac{\partial}{\partial\theta}(F_r)\right]. \qquad [51]$$

The equation [50] in terms of the new variables can be written as

$$\tilde{\omega} = \int_{-\infty}^{\phi_p} \frac{r^5 f(r, \ \theta; \ R_M)}{(r^3-1)^2\cos^2\theta + (r^3+\frac{1}{2})^2\sin^2\theta}\, d\phi_0. \qquad [52]$$

the integral is taken along a streamline $\psi_0 =$ constant and the lower limit is so chosen that when $r \to \infty$ upstream the quantity $\tilde{\omega}$ tends to zero. Also, taking the ϕ-component of [11], we find similarly that $\xi = r\sin\theta\, q_\phi$ given by

$$\left(r - \frac{1}{r^2}\right)\cos\theta\frac{\partial\xi}{\partial r} - \left(1 + \frac{1}{2r^3}\right)\sin\theta\frac{\partial\xi}{\partial\theta} = r^2\sin\theta\, F_\phi \qquad [53]$$

i.e., (in terms of ϕ_0 and ψ_0).

$$\left[\left(1 - \frac{1}{r^3}\right)^2\cos^2\theta + \left(1 + \frac{1}{2r^3}\right)^2\sin^2\theta\right]\frac{\partial\xi}{\partial\phi_0} = r\sin\theta\, F_\phi. \qquad [53a]$$

Integrating [53] as in the case of [50], we have

$$\xi = \int_{-\infty}^{\phi_P} \frac{r^7 \sin\theta \; F_\phi}{[(r^3-1)^2\cos^2\theta + (r^3+\frac{1}{2})^2\sin^2\theta]}\, d\phi_0 \qquad [54]$$

where we have taken $\xi \rightarrow 0$ as $r \rightarrow \infty$ upstream.

Now, in the expressions for F_r and F_ϕ the contribution of rotation is $0\,(R_M^2)$ hence, to our approximation, when r is small, we can neglect the effect of rotation in the r and θ components of Lorentz force and in $f(r,\ \theta,\ R_M)$. Hence, as shown in the absence of rotation by Ludford and Murray (1959, 1960) the ϕ component of vorticity is logarithmically infinite on the sphere.

From [53], [21], [28] and [49] we find that the ϕ-component of velocity is swept into a wake behind the sphere whose boundary is a paraboloid of revolution of half the size of that for the magnetic field. For in the right hand side of [53] the argument of the exponential, exp. $[-R_M r(1-\cos\theta)]$, is twice that of the magnetic field.

Noting that in [54] we integrate along a fixed stream line we can write

$$\xi = \int_{r=\infty}^{r(P)} \frac{r^4 \sin\theta \; F_\phi}{(r^3-1)\cos\theta}\, dr. \qquad [55]$$

Using [55] we shall obtain an expression for ξ near the front stagnation point. The integrand in [55] is

$$\frac{r^{4+1/2}\,\psi_0\, F_\phi}{\cos\theta(r^3-1)^{3/2}}. \qquad [56]$$

The denominator $\rightarrow 0$ as $r \rightarrow 1$. Hence when $r_p \sim 1$ and r_1 is sufficiently larger than 1, the leading part in [55] is

$$\int_{r=r_1}^{r(P)} \frac{r^{9/2}\,\psi_0\, F_\phi\, dr}{\cos\theta\,(r^3-1)^{3/2}}, \qquad [57]$$

so that the contribution of

$$\int_{r=\infty}^{r_1} \frac{r^{9/2}\,\psi_0\, F_\phi\, dr}{\cos\theta\,(r^3-1)^{3/2}}$$

being negligible compared with this leading part. (The details are given in Appendix A). Also when r is not very large, from [49], [30], [21] and [22] we have to the order R_M,

$$F_\phi = R_M \sin\theta \Bigg[\sum_{m,\, n=1}^{\infty} \frac{A_m\, m\, a_m\, P'_m(\cos\theta)}{r^{m+2}} \left(\frac{\nu_{n1}\, n(n-1)\, P_{n-1}(\cos\theta)}{r^{n+1}} + \frac{\nu_{n3}(n+1)(n+2)\, P_{n+1}(\cos\theta)}{r^{n+3}} \right) - \sum_{m,\, n=1}^{\infty} \frac{A_m\, m(m+1)\, a_m\, P_m(\cos\theta)}{r^{m+3}} \times \left(\frac{(n-1)\,\nu_{n1}\, P'_{n-1}(\cos\theta)}{r^{n}} + \frac{(n+1)\,\nu_{n3}\, P'_{n+1}(\cos\theta)}{r^{n+2}} \right) \Bigg]$$

$$= R_M \sin\theta\, F(r, \theta).$$

Hence [57] can be written as

$$\xi = \psi_0^2 \int_{r=r_1}^{r(P)} \frac{R_M\, F_1(r, \theta)\, dr}{(r-1)^2} \qquad [58]$$

where

$$F_1(r, \theta) = \frac{r^5\, F(r, \theta)}{\cos(r^2+r+1)^2}. \qquad [59]$$

Now, by the Mean-Value Theorem of the Differential Calculus for two variables we have

$$F_1(r, \theta) = F_1(1, \pi) + (r-1)\, \partial F_1[1+\eta(r-1),\ \pi+\eta(\theta-\pi)]/\partial r + (\theta-\pi)\, \partial F_1[1+\eta(r-1),\ \pi+\eta(\theta-\pi)]/\partial\theta,$$

where $0 < \eta < 1$. Hence

$$\int_{r=r_1}^{r(P)} \frac{R_M F_1(r, \theta)\, dr}{(r-1)^2} = R_M \Bigg[F_1(1, \pi) \left(-\frac{1}{[r(P)-1]} + \frac{1}{(r_1-1)} \right) + \int_{r=r_1}^{r(P)} \frac{\partial F_1[1+\eta(r-1),\ \pi+\eta(\theta-\pi)]/\partial r}{(r-1)}\, dr + \int_{r=r_1}^{r(P)} \frac{(\theta-\pi)\, \partial F_1[1+\eta(r-1),\ \pi+\eta(\theta-\pi)]/\partial\theta}{(r-1)^2}\, dr \Bigg]. \qquad [60]$$

We can find two numbers A and B such that

$$|\partial F_1[1+\eta(r-1),\ \pi+\eta(\theta-\pi)]/\partial r| \leq A,$$

and

$$|\partial F_1[1+\eta(r-1),\ \pi+\eta(\theta-\pi)]/\partial\theta| \leq B.$$

Thus, the mod. of the first and the second integrals on the right hand side are respectively less than or equal to

$$A\left[\log(r_1-1)-\log(r(P)-1)\right]$$

and
$$(-\theta_p+\pi)\,B\left[\frac{1}{[r(P)-1]}-\frac{1}{(r_1-1)}\right]$$

in absolute value. Hence, when $\pi-\theta_p$ and r_p-1 are sufficiently small, it follows from [60] that

$$\int_{r=r_1}^{r(P)}\frac{R_M F_1(r,\theta)}{(r-1)^2}\,dr\simeq-R_M\,F_1(1,\pi)\left(\frac{1}{r(P)-1}\right). \qquad [61]$$

From [58] and (59) we have

$$q_\phi=-R_M F_1(1,\ \pi)\,(r_P^2+r_P+1)\sin\theta_P/r_P^2. \qquad [62]$$

In the case of a dipole of strength A_1 situated at the centre of the sphere,

$$A=A_1\sin^2\theta/r,$$

$$q_\phi=\frac{3\,A_1\,\bar{u}_1\,\nu_{13}\,R_M\sin\theta_P\,(\cos^2\theta_P+1)}{r_P^4\,(r_P^2+r_P+1)\cos\theta_P}. \qquad [63]$$

We shall now find expressions for ω and q_ϕ taking $\xi_P=R_M\,r_P\,(1-\cos\theta_P)$ to be very large. This implies that r is large at every upstream point on the streamline through the point P. We put

$$u=r(1-\cos\theta)/[r_P(1-\cos\theta_P)]-1,$$

and [52] reduces to

$$\tilde{\omega}=\exp.\left[-\xi_P\right]\int_0^\infty\frac{\exp.\left[-\xi_P u\right]f'\,r_P(1-\cos\theta_P)}{r(1-\cos\theta)}\,du \qquad [64]$$

where
$$f=\exp.\left[-R_M r\,(1-\cos\theta)\right]f'.$$

By Watson's lemma* (Cospon, 1935) for fixed θ_p, we have from [64]

$$\omega\sim\frac{\exp.\left[-\xi_P\right]f'\sin\theta_P}{R_M(1-\cos\theta_P)}. \qquad [65]$$

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* The function $[f'\,r_p(1-\cos\theta_P)]/[r(1-\cos\theta)]$ occurring in [64] is a rational function of u with poles at $u=-1\pm i\cot(^1/_2\,\theta_P)$.

where f' is evaluated at (r_p, θ_p). When, in the undisturbed magnetic field of the sphere, there is component due to a dipole situated at the centre, it can be shown that, for large r,

$$f' \sim -R_M^3 \nu_{n1}^2 \sin^2\theta \, \{P'_{n-1}(\cos\theta)\}^2/[r^{2n+1}(1+\cos\theta)]. \qquad [66]$$

(The details are given in the Appendix B).

Thus, from [65], it follows that

$$\omega \sim -\exp.[-\xi_P]\, R_M^2 \nu_{n1}^2 \sin\theta_P \{P'_{n-1}(\cos\theta_P)\}^2/r_P^{2\,n+1}. \qquad [67]$$

Similarly from the equation [54] we have, for large r_P,

$$\xi \sim -\exp.[-\xi_P] \times \frac{A_n R_M^3 a_n \nu_{n1} n \sin^2\theta_P (1-\cos\theta_P)}{4 r_P{}^n R_M (1-\cos\theta_P)} \times$$

$$\times \{(n-1)\, P_{n-1}(\cos\theta_P)\, P'_n(\cos\theta_P) - (n+1)\, P_n(\cos\theta_P)\, P'_{n-1}(\cos\theta_P)\}.$$

Both the expressions [67] and [68] are to be modified in the case $n = 1$, *i.e.*, when the undisturbed magnetic field in the sphere has a component due to a magnetic dipole situated at the centre. In this case

$$\omega \sim \exp.[-\xi_P] \times \frac{A_1^2 R_M^5 a_1^2 \sin\theta_P [-3 + 4\cos\theta_P - \cos^2\vartheta_P + \sin^2\theta_P \cos\theta_P]}{16 R_M r_P^3 (1-\cos\theta_P)}. \qquad [69]$$

and

$$\xi \sim \exp.[-\xi_P] A_1 R_M^2 a_1 \nu_{21} \sin^4\theta_P / [4 r_P^3 (1-\cos\theta_P)]. \qquad [70]$$

From [67] and [68], and [43] or [46] it can be seen that ω and ξ are proportional to Ω^2 and Ω respectively and it follows from [69] that ω is independent of rotation.

Drag on the Torque on the Sphere: The force on the sphere due to the pressure is a drag along the downstream given by $\rho_0 U^2 a^2 D_p$ where

$$D_p = \pi\beta \int_1^\infty \frac{dr}{r} \int_0^\pi (F_\theta \sin\theta + 2 F_r \cos\theta) \sin\theta \, d\vartheta. \qquad [71]$$

The contribution of rotation to the terms F_r and F_θ above is $0(R_M^2)$ and hence to the order of our approximation it does not affect the drag. The drag coefficient D_P is given by Ludford and Murray (1959).

The force on the sphere also arises due the Maxwell stresses,

$$\mu H_i H_j - \tfrac{1}{2}\mu H^2 \delta_{ij}.$$

On a surface element this becomes

$$\mu h^2 [\tfrac{1}{2}(H_r^2 - H_\theta^2 - H_\phi^2), \quad H_r H_\theta, \quad H_r H_\phi]. \qquad [72]$$

Hence the drag coefficient due to the Maxwell stress is given by D_M, where

$$D_M = 2\pi\beta \int_0^\pi [\tfrac{1}{2}(H_r^2 - H_\theta^2 - H_\phi^2)\cos\theta - H_r H_\theta \sin\theta] \sin\theta \, d\theta \qquad [73]$$

and the total contribution to the force is $\rho U^2 a^2 D_M$. Contribution due to rotation is $0(R_M^2)$, hence, to our approximation D_M is that given by Ludford and Murray (1959). The torque due to the Maxwell stresses is given by $\rho a^3 U^2 \beta T_m$, where

$$T_m = 2\pi \int_{\theta=0}^{\pi} H_r H_\phi \sin^2\theta \, d\theta$$

$$= 2\pi R_M \sum_{n=1}^{\infty} \frac{A_n a_n n(n+1)}{(2n+3)} \left\{ \frac{2\nu_{n,3}(n+1)(n+2)}{(2n+3)} \right.$$

$$\left. + \nu_{n+2,1} \frac{2(n+1)(n+2)}{(2n+3)} - \frac{\nu_{n,1}\, 2n(n-1)}{(2n-1)} - \nu_{n-2,3} \frac{2n(n-1)}{(2n-1)} \right\} \qquad [74]$$

The case for a dipole can be easily deduced from the above by putting $n = 1$.

$$T_m = (16\pi R_M A_1 a_1 \nu_{13})/5 \qquad [75]$$

where A_1 is the strength of the dipole and a_1 and ν_{13} are given by [23a] and [44] or [47]. Thus

$$T_m = -\frac{96\pi \mu \mu^i R_M \Omega' A_1^2}{5(2\mu + \mu^i)^2 [2 + (3 R_M \mu^i)/(\mu R_M^i)]}$$

when $\qquad R_M/R_M^i = 0(1),$

$$T_m = -\frac{48\pi \mu \mu^i \Omega' A_1^2 R_M}{5(2\mu + \mu^i)^2}.$$

when R_M/R_M^i is $0(R_M)$ or smaller. It is evident that the torque opposes the rotation of the sphere and its magnitude increases to a limiting value when the conductivity of the sphere becomes large compared with that of the liquid outside.

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APPENDIX (A)

We shall show that the contribution of

$$I_1 = \int_{r=\infty}^{r_1} \frac{r^{9/2}\,\psi_0\,F_\phi}{\cos\theta\,(r^3-1)^{3/2}}\,dr$$

is negligible compared with

$$I_2 = \int_{r=r_1}^{r(P)} \frac{r^{9/2}\,\psi_0\,F_\phi}{\cos\theta\,(r^3-1)^{3/2}}\,dr\,,$$

where r_p is sufficiently near 1 and r_1 is sufficiently greater than 1.

By the first Mean Value Theorem of Integral Calculus, we have

$$I_1 = \int_{r=\infty}^{r=r_1} \frac{r^{9/2}\,\psi_0\,F_\phi}{\cos\theta\,(r^3-1)^{3/2}}dr = -\frac{2\,\psi_0}{(r_1-1)^{1/2}}\left[\frac{r^{9/2}\,F_\phi}{\cos\theta\,(r^2+r+1)^{3/2}}\right]_{r=r_2},$$

where $r_1 < r_2 < \infty$. Now, let

$$\left[\frac{r^{9/2}\, F\phi}{\cos\theta\, (r^2+r+1)^{3/2}}\right]_{r=r_2} = k,$$

a finite number, then

$$I_2 : I_1 \sim -\frac{R_M\, F_1(1, \pi)\, \psi_0\, (r_1-1)^{1/2}}{2k\,(r_P-1)}$$

which is large when (r_1-1) is large and $r_P - 1$ is small.

APPENDIX (B)

In view of the very complex nature of the expression for f and F_ϕ and of the approximation for large r, we shall discuss briefly this asymptotic nature below. We write

$$A = \exp.\left[-\tfrac{1}{2} R_M r (1-\cos\theta)\right] A',$$

$$\mathbf{H} = \exp.\left[-\tfrac{1}{2} R_M r (1-\cos\theta)\right] \mathbf{H}',$$

$$\chi = \exp.\left[-\tfrac{1}{2} R_M r (1-\cos\theta)\right] \chi',$$

$$\mathbf{F} = \exp.\left[-R_M r (1-\cos\theta)\right] \mathbf{F}',$$

and

$$f = \exp.\left[-R_M r (1-\cos\theta)\right] f'.$$

Let in the sequence $A_1, A_2, A_3, \cdots, A_n, A_{n+1}, \cdots$

A_n be the first non-zero number. We have

$$f(r, \theta;\, R_M)$$

$$= \frac{\exp.\left[-R_M r(1-\cos\theta)\right]}{r\sin\theta}\Big[R_M^2\{\tfrac{1}{2}(1-\cos\theta)^2\, r\, H'_\theta H'_r$$

$$-\tfrac{1}{2}\sin\theta\,(1-\cos\theta)\, r\, H_r'^2 + \tfrac{1}{2}(1-\cos\theta)\sin\theta\, r\, H_\theta'^2 - \tfrac{1}{2} H'_r H'_\theta\, r \sin^2\theta\}$$

$$+ R_M\{-(1-\cos\theta)\, H'_\theta H'_r - (1-\cos\theta)\, r\, H'_r(\partial H'_\theta/\partial r)$$

$$+ (1-\cos\theta)\, H'_r\,(\partial H'_r/\partial\theta) - H_\theta'^2 \sin\theta - r\sin\theta\, H'_\theta\,(\partial H'_\theta/\partial r)$$

$$+ \sin\theta\, H'_\theta\,(\partial H'_r/\partial\theta) - \tfrac{1}{2}(1-\cos\theta)\, H'_\theta H'_r + \tfrac{1}{2}\sin\theta\, H_r'^2$$

$$- r\,\partial\{\tfrac{1}{2}(1-\cos\theta)\, H'_\theta H'_r - \tfrac{1}{2}\sin\theta\, H_r'^2\}/\partial r - \partial\{\tfrac{1}{2}(1-\cos\theta)\, H_\theta'^2$$

$$- \tfrac{1}{2}\sin\theta\, H'_r H'_\theta\}/\partial\theta + \frac{(1-\cos\theta)}{r^2\sin^2\theta}\chi' \cdot \frac{\partial\chi'}{\partial\theta} - \frac{\chi'}{r\sin\theta}\cdot\frac{\partial\chi'}{\partial r}$$

$$+\frac{\chi'^2}{2r^2\sin\theta}+r\frac{\partial}{\partial r}\cdot\left(\frac{\chi'^2}{2r^2\sin\theta}\right)-\frac{\partial}{\partial\theta}\cdot\left(\frac{\chi'^2(1-\cos\theta)}{2r^2\sin^2\theta}\right)\Bigg\}$$

$$+\left\{\frac{H_r'H_\theta'}{r}+H_r'\frac{\partial H_\theta'}{\partial r}-\frac{H_r'}{r}\cdot\frac{\partial H_r'}{\partial\theta}\right\}+r\frac{\partial}{\partial r}\left\{\frac{H_r'H_\theta'}{r}+H_r'\frac{\partial H_\theta'}{\partial r}-\frac{H_r'}{r}\cdot\frac{\partial H_r'}{\partial\theta}\right\}$$

$$+\frac{\partial}{\partial\theta}\left\{\frac{H_\theta'^2}{r}+H_\theta'\frac{\partial H_\theta'}{\partial r}-\frac{H_\theta'}{r}\cdot\frac{\partial}{\partial\theta}H_r'\right\}$$

$$-\frac{\chi'}{r^3\sin^2\theta}\cdot\frac{\partial\chi'}{\partial\theta}-r\frac{\partial}{\partial r}\left\{\frac{\chi'}{r^3\sin^2\theta}\cdot\frac{\partial\chi'}{\partial\theta}\right\}+\frac{\partial}{\partial\theta}\left\{\frac{\chi'}{r^2\sin^2\theta}\cdot\frac{\partial\chi'}{\partial r}\right\}\Bigg]. \qquad [A_{2,1}]$$

The coefficient of R_M^2 in the above expression can be written as

$$\tfrac{1}{2}R_M r\left\{A'\frac{\partial A'}{\partial(\cos\theta)}\cdot\frac{\cos\theta(1-\cos\theta)^2}{r^3\sin\theta}-\frac{\cos\theta(1-\cos\theta)}{r^2\sin\theta}A'\frac{\partial A'}{\partial r}\right.$$

$$\left.-\frac{\sin\theta(1-\cos\theta)}{r^3}A'\frac{\partial A'}{\partial(\cos\theta)}-\frac{\sin\theta(1-\cos\theta)^2}{r^2\sin^2\theta}A'\frac{\partial A'}{\partial r}\right\}$$

$$-\frac{\cos\theta(1-\cos\theta)}{r^2\sin\theta}\cdot\frac{\partial A'}{\partial r}\cdot\frac{\partial' A'}{\partial(\cos\theta)}-\frac{\sin\theta(1-\cos\theta)}{2r^3}\left(\frac{\partial A'}{\partial(\cos\theta)}\right)^2$$

$$+\frac{\sin\theta(1-\cos\theta)}{2r\sin^2\theta}\left(\frac{\partial A'}{\partial r}\right)^2. \qquad [A_{2,2}]$$

For large r,

$$A'\sim\frac{A_n R_M\sin^2\theta}{r^{n-1}}\left(\frac{a_n}{2}P_n'(\cos\theta)+\lambda_{n1}P_{n-1}'(\cos\theta)\right)$$

and

$$\chi'\sim R_M\sin^2\theta\,\nu_{n1}P_{n-1}'(\cos\theta)/r^{n-1}.$$

From [22] it is clear that $[A_{2,2}]$ is $0(R_M^3/r^{2n})$. Hence in $[A_{2,1}]$, the contribution of $R_M^2\times$ (its coefficient) is $0(R_M^5/r^{2n})$. Similarly, it can be shown that in $[A_{2,1}]$ the contribution of $R_M\times$ (its coefficient) is $0(R_M^3/r^{2n})$. This is

$$R_M\left\{\frac{(1-\cos\theta)}{r^2\sin^2\theta}\chi'\frac{\partial\chi'}{\partial\theta}-\frac{\chi'}{r\sin\theta}\cdot\frac{\partial\chi'}{\partial r}+\frac{\chi'^2}{2r^2\sin\theta}\right.$$

$$\left.+r\frac{\partial}{\partial r}\left(\frac{\chi'^2}{2r^2\sin\theta}\right)-\frac{\partial}{\partial\theta}\left(\frac{\chi'^2(1-\cos\theta)}{2r^2\sin^2\theta}\right)\right\}$$

$$=-\frac{R_M^3\nu_{n1}^2\sin^3\theta\,[P_{n-1}'(\cos\theta)]^2}{r^{2n}(1+\cos\theta)}.$$

In the coefficient for R_M the terms contributed by H_r' and H_θ' and their derivatives are $0\left(R_M^4/r^{2n}\right)$.

Hence they are neglected when compared with the terms involving χ' and its derivatives. The rest of the terms in $[A_2, 1]$ is $0\left(1/r^{2n+1}\right)$. Similarly the expressions in [68], [69] and [70] are derived. When the original (undisturbed) magnetic field in the sphere contains a component due to a dipole situated at the centre, we have for large r

$$A \sim A_1 \alpha_1 = \frac{A_1 a_1 R_M \sin^2\theta}{2}$$

and

$$\chi' \sim \frac{R_M \nu_{21} \sin^2\theta}{r}.$$

These values are used in deriving the expressions [69] and [70].