RECTILINEAR MOTION OF A MAXWELL FLUID

By D. K. Mohan Rao

(Department of Applied Mathematics, Indian Institute of Science, Bangalore-12)

Received on September 21, 1962

Abstract

We consider the steady rectilinear motion of a Maxwell fluid in straight tubes of arbitrary cross section. This type of motion in a circular tube is possible in the absence of body forces. The velocity at any point of a cross section is less than the corresponding velocity of a Newtonian fluid, the midstream velocity being the same in both cases. We derive the conditions necessary for the maintenance of a purely rectilinear flow in a tube of arbitrary section in the absence of body forces. These conditions restrict the form of the strain energy function in its dependence on the strain invariants.

Introduction

1. Rectilinear flow of a Newtonian fluid in a cylindrical tube of arbitrary cross-section is always possible in which the streamlines are parallel to the generators of the cylinder.

In the present note we find that, in the case of a Maxwell fluid,[1] for maintaining a purely rectilinear flow of the above type in a cylindrical tube of arbitrary cross section, certain body forcess are essential, a circular tube being an exception to this statement. This situation also arises in the case of a Reiner-Rivlin fluid as was shown by Green and Rivlin[2] and Bhatnagar and Rao[3].

2. In cylindrical coordinates (r, θ, z) taking the z-axis along the axis of the circular pipe, the deformation tensor $\mathbf{a}$, for an incompressible fluid is given by

$$\mathbf{a} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ (dw/dr)\,t & 0 & 1 \end{vmatrix} \qquad [2.1]$$

where w is the velocity independent of time t. Since

$$(d\mathbf{a}/dt)\,\mathbf{a}^{-1} = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ (dw/dr) & 0 & 0 \end{vmatrix} \qquad [2.2]$$

is independent of time t, the condition for time-independence of stressess[1] is satisfied. The internal deformation tensor $\bar{\alpha}$ is given by

$$[(d\mathbf{a}/dt)\,\mathbf{a}^{-1}]\,\bar{\alpha} - \beta(\bar{\alpha} - \mathbf{I}) = 0 \qquad [2.3]$$

where β is the reciprocal of the relaxation time and $\mathbf{I}$ is the idem tensor. Thus

$$\bar{\alpha} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2\tan\theta & 0 & 1 \end{vmatrix} \qquad [2.4]$$

where

$$(1/\beta)\,(dw/dr) = 2\tan\theta. \qquad [2.5)$$

Resolving the internal deformation tensor $\bar{\alpha}$ into an orthogonal tensor $\mathbf{R}$ and a real positive symmetric tensor $\bar{\alpha}_s$, such that

$$\bar{\alpha} = \bar{\alpha}_s'\,\mathbf{R} \qquad [2.6]$$

we have

$$\bar{\alpha}_s'\,\bar{\alpha}_s' = \bar{\alpha}\,\bar{\alpha}' = \begin{vmatrix} 1 & 0 & 2\tan\theta \\ 0 & 1 & 0 \\ 2\tan\theta & 0 & 1 + 4\tan^2\theta \end{vmatrix} \qquad [2.7]$$

and

$$\bar{\alpha}_s' = \begin{vmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta + 2(\sin^2\theta/\cos\theta) \end{vmatrix} \qquad [2.8]$$

where the tensor $\bar{\alpha}_s'$ gives the pure deformation following the rotation $\mathbf{R}$.

The stress tensor $\bar{\sigma}$ is given by

$$\bar{\sigma} = 2Q\,\bar{\alpha}\,\bar{\alpha}' + R\,\bar{\alpha}_s' + P\mathbf{I} \qquad [2.9]$$

where R, Q are the partial derivatives of the strain energy function with respect to the first and second invariants I, J of the pure deformation tensor $\bar{\alpha}_s'$ given by

$$I = T_r[\bar{\alpha}_s'] \text{ and } J = T_r[\bar{\alpha}\,\bar{\alpha}'] \qquad [2.10]$$

and P is the hydrostatic pressure.

In the usual notation, the stress components are given by

$$\sigma_{rr} = R\cos\theta + 2\theta + P$$

$$\sigma_{\theta\theta} = R + 2Q + P$$

$$\sigma_{zz} = R\left[\cos\theta + 2(\sin^2\theta/\cos\theta)\right] + 2Q(1 + 4\tan^2\theta) + P$$

$$\sigma_{rz} = \sigma_{zr} = R\sin\theta + 4Q\tan\theta$$

$$\sigma_{r\theta} = \sigma_{\theta r} = \sigma_{\theta z} = \sigma_{z\theta} = 0. \qquad [2.11]$$

We consider, for the sake of simplicity, the case in which R and Q are constants in view of the absence of definite knowledge about them. We have the equation of motion along the axis of the pipe as

$$\frac{\partial(\sigma_{rz})}{\partial r} + \frac{\sigma_{rz}}{r} + \frac{\partial(\sigma_{zz})}{\partial z} = 0 \qquad [2.12]$$

where

$$\frac{\partial(\sigma_{zz})}{\partial z} = \frac{\partial P}{\partial z} = -C \qquad [2.13]$$

is the pressure gradient.

From [2.11] and [2.12]

$$\sigma_{rz} = (Cr/2) = R\sin\theta + 4Q\tan\theta \qquad [2.14]$$

so that

$$\frac{C}{4}\left(\frac{W-V}{\beta}\right) = -R(1+\cos\theta) + 2Q\tan^2\theta, \frac{\pi}{2} \leq \theta \leq \pi \qquad [2.15]$$

where V is the velocity along the axis of the pipe. Making use of the no slip condition on the pipe, we get

$$\frac{V}{2a\beta} = \frac{R(1+\cos\theta_1) - 2Q\tan^2\theta_1}{R\sin\theta_1 + 4Q\tan\theta_1} \qquad [2.16]$$

where a is the radius of the pipe and the skin-friction on the wall of the pipe is given by

$$\left(\frac{dW}{dr}\right)_{r=a} = 2\beta\tan\theta_1 \qquad [2.17]$$

If W_0 and λ are the non-dimensional parameters

$$W_0 = \frac{V}{2a\beta} \qquad [2.18]$$

and

$$\lambda = \frac{R}{4Q} \qquad [2.19]$$

we have

$$W_0 = \frac{\lambda\,(1-\cos\theta_0) - \frac{1}{2}\tan^2\theta_0}{\lambda\sin\theta_0 - \tan\theta_0} \qquad [2.20]$$

where

$$\theta_0 = \pi - \theta_1$$

$$0 \leqslant \theta_0 \leqslant (\pi/2) \qquad [2.21]$$

The rate of dissipation of energy

$$(\delta\epsilon/dt) = Tr\,[\bar{\sigma}\,\{\beta\,(\mathrm{I}-\bar{\alpha})\}\,\bar{\alpha}^{-1}] = -2\,\sigma_{rz}\,\beta\tan\theta \qquad [2.22]$$

is positive for all values of θ in the range $\pi/2 \leqslant \theta \leqslant \pi$, when $\lambda < 1$. Fig. I shows the variation of W_0 with θ_0 for $\lambda = 0.125$, -0.125. We note that W_0 is proportional to the inlet velocity and θ_0 increases with decreasing skin-friction.

3. In rectangular cartesian coordinates (x, y, z) the axis of the pipe is taken as the z-axis. W is the velocity parallel to the axis and is indepent of time.

The condition of incompressibilite gives

$$\partial w/\partial z = 0. \qquad [3.1]$$

With the same notation as in 2, we have

$$(d\mathrm{a}/dt)\,\mathrm{a}^{-1} = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ (\partial w/\partial x) & (\partial w/\partial y) & 0 \end{vmatrix} \qquad [3.2]$$

which is independent of time.

$$\bar{\alpha} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ p & q & 1 \end{vmatrix} \qquad [3.3]$$

with

$$p = (1/\beta)\,(\partial w/\partial x) \text{ and } q = (1/\beta)\,(\partial w/\partial y) \qquad [3.4]$$

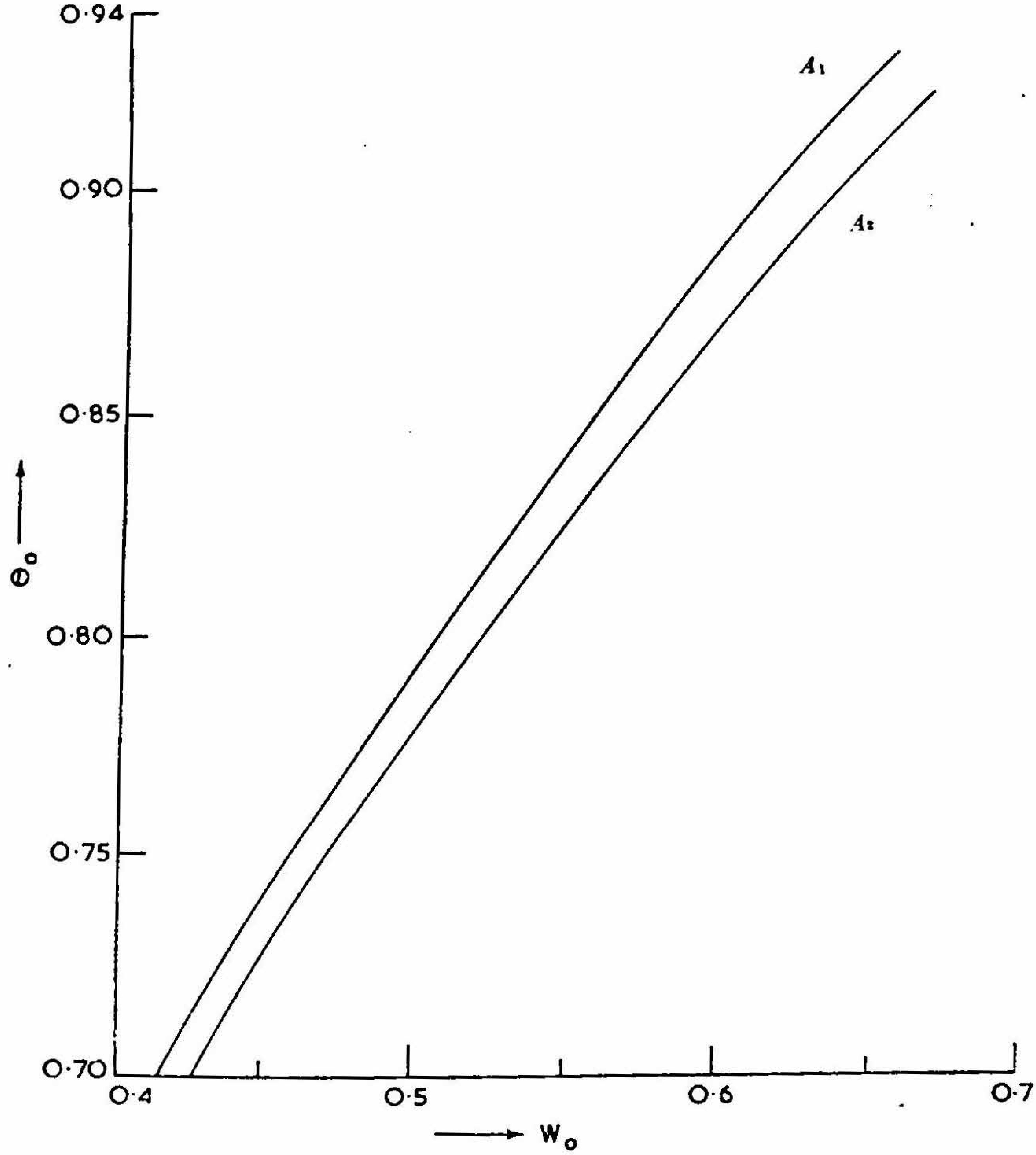

FIG. I
Effect of inlet velocity on skin friction
A_1 is = λ−0.125 and A_2 is λ =0.125

and the pure deformation $\bar{\alpha}_s'$ by

$$\sqrt{(p^2+q^2+4)}\cdot\bar{\alpha}_s' =$$

$$\begin{vmatrix} \dfrac{2p^2+q^2\sqrt{(p^2+q^2+4)}}{p^2+q^2} & \dfrac{pq\{2-\sqrt{(p^2+q^2+4)}\}}{p^2+q^2} & q \\ \dfrac{pq\{2-\sqrt{(p^2+q^2+4)}\}}{p^2+q^2} & \dfrac{2q^2+p^2\sqrt{(p^2+q^2+4)}}{p^2+q^2} & q \\ p & q & 2+p^2+q^2 \end{vmatrix} \quad [3.5]$$

The equations of motion along the x-, y- and z-axis, in the absence of body forces are

$$\frac{\partial}{\partial x}\left[\frac{R}{p^2+q^2}\cdot\frac{2p^2+q^2\sqrt{(p^2+q^2+4)}}{\sqrt{(p^2+q^2+4)}}+2Q+P\right] + \frac{\partial}{\partial y}\left[\frac{Rpq}{p^2+q^2}\cdot\frac{2-\sqrt{(p^2+q^2+4)}}{\sqrt{(p^2+q^2+4)}}\right]=0 \quad [3.6]$$

$$\frac{\partial}{\partial x}\left[\frac{Rpq}{p^2+q^2}\cdot\frac{2-\sqrt{(p^2+q^2+4)}}{\sqrt{(p^2+q^2+4)}}\right]+\frac{\partial}{\partial y}\left[\frac{R}{p^2+q^2}\cdot\frac{2q^2+p^2\sqrt{(p^2+q^2+4)}}{\sqrt{(p^2+q^2+4)}} + 2Q+P\right]=0 \quad [3.7]$$

$$\frac{\partial}{\partial x}\left[p\left\{2Q+\frac{R}{\sqrt{(p^2+q^2+4)}}\right\}\right]+\frac{\partial}{\partial y}\left[q\left\{2Q+\frac{R}{\sqrt{(p^2+q^2+4)}}\right\}\right]+\frac{\partial P}{\partial z}=0 \quad [3.8]$$

where $(\partial P/\partial z) = -C$ is the pressure gradient.

Eliminating P between [3.6] and [3.7], we have

$$\frac{\partial^2}{\partial x\,\partial y}[(p^2-q^2)F'] + \left(\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial x^2}\right)[pqF'] = 0 \quad [3.9]$$

where

$$F' = \frac{R}{p^2+q^2}\cdot\frac{2-\sqrt{(p^2+q^2+4)}}{\sqrt{(p^2+q^2+4)}} \quad [3.10]$$

Also from [3.8] we get

$$\frac{\partial^2}{\partial x\,\partial y}[(p^2-q^2)F] + \left(\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial x^2}\right)[Fpq] + D = 0 \quad [3.11]$$

where

$$D = p\frac{\partial p}{\partial x}\cdot\frac{\partial F}{\partial y} - q\frac{\partial q}{\partial y}\cdot\frac{\partial F}{\partial x} + q\frac{\partial p}{\partial y}\cdot\frac{\partial F}{\partial y} - p\cdot\frac{\partial q}{\partial x}\cdot\frac{\partial F}{\partial x}$$

vanishes in view of the fact that F is a function of $(p^2 + q^2)$, and

$$F = 2Q + R/[\sqrt{(p^2 + q^2 + 4)}] \qquad [3.12]$$

[3.9] and [3.11] are consistent if

$$(a)\ R = 0 \quad Or \quad (b)\ R/Q = 2\lambda(1 - I_1^2)/[1 + \lambda(1 + I_1)] \qquad [3.13]$$

where λ is a constant and I_1 is the first invariant of the tensor $\bar{\alpha}'_s$.

$$W = f\left[\frac{I_2 + 4I_1 + 1 + 2/\lambda}{I_1 + 1 + 1/\lambda}\right] \qquad [3.14]$$

where I_1 and I_2 are the first and second invariants of the tensor $\bar{\alpha}'_s$ and f is an arbitrary function, is an example of the strain energy function satisfying the second of conditions [3.13].

ACKNOWLEDGMENT

The author is highly indebted to Prof. P. L. Bhatnagar for suggesting this investigation and for his constant guidance and help throughout the preparation of this paper.

REFERENCES

1. Yamamoto, M. .. *J. Phys. Soc.*, Japan, 1959, **14**, 313.
2. Green, A. E. and Rivlin, R. S. .. *Quar. Appl. Maths.*, 1956, **14**, 299.
3. Bhatnagar, P. L. and Rao, S. K. L. .. *Proc. 3rd Cong. Theoretical and Applied Mechanics*, Bangalore, India, 1957.