# ACCRETION OF INTERSTELLAR GAS BY A STAR IN THE PRESENCE OF RADIATION

BY PARSHOTAM LAL

(*Department of Applied Mathematics, Indian Institute of Science, Bangalore-12*)

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ABSTRACT

Following the work of Bondi and McCrea, we have considered a point stationary star surrounded by an infinitely extending cloud of gas. The material in the cloud moves towards the star due to its gravitational force so that a spherically symmetrical steady motion is established in the cloud and matter continually falls in the star.

The main contribution in this paper is the inclusion of radiation effects through the adiabatic coefficient introduced by Klimshin.

## INTRODUCTION

1. The phenomenon of accretion of interstellar gas by stars is of considerable importance in astrophysics. Bondi[1] considered this problem in the following form. A point stationary star is assumed to be surrounded by an infinitely extending cloud of gas, having uniform density and pressure at infinity and at rest there. The motion of the gas is assumed to be steady and spherically symmetrical. The star acts as a sink and the gravitational force on the gas is due entirely to the star of unchanging mass. The gas is assumed to be perfect and the motion of the gas is taken to be isentropic.

Under these conditions, he considered the accretion of this inward moving gas by the star and dealt with three different cases: when the rate of accretion is less, equal to and greater than a critical value determined by the maximum rate of accretion. Only this critical value of rate of accretion gave a solution which satisfied boundary conditions at infinity, but a finite jump in acceleration would appear at a point, where Mach number is unity, which, though physically possible, was not given meaning to by Bondi.

McCrea[2] reconsidered essentially the same problem and corroborated it by introducing the existence of a stationary shock wave in the flow. This made all the solutions of Bondi for the maximum rate of accretion plausible and his type I and II solutions appeared as extreme cases of the general flow, including a standing shock wave.

The gas surrounding the star is generally at a very high temperature and the neglect of radiation effects is not valid in general. We have incorporated the effect of radiation pressure and radiation energy, but have neglected radiation flux. In the discussion we have used the adiabatic coefficient, introduced by Klimshin[3] that renders the whole treatment quite analogus to ordinary gas dynamics. Finally, we have given a numerical example which satisfies boundary conditions at infinity and which allows a standing shock wave in the physically meaningful region.

## 2. Equations of the Problem

The following equations govern the problem :

*Equation of continuity* :

$$4\pi r^2 \rho v = A, \qquad [2.1]$$

where $A$ is a constant flux across any spherical surface surrounding the origin and so is the rate of flow into the sink at 0.

Since $\rho \to \rho_\infty \neq 0$ as $r \to \infty$, it follows that

$$v = 0\,(r^{-2}) \text{ as } r \to \infty. \qquad [2.2]$$

*Bernoulli's equation* :

$$\tfrac{1}{2} v^2 + \int (dp/\rho) - (GM/r) = \text{constant}. \qquad [2.3]$$

Here $r$ is the distance from the origin, where the point star of mass $M$ is situated, $\rho$ is the density, $v$ is the radial velocity, $p$ is the sum of the gas pressure and radiation pressure, *i.e.* $p = p_g + p_R$, and $G$ is the gravitational constant.

*Equation of energy* :

$$\frac{\partial E}{\partial r} + p\frac{\partial}{\partial r}\left(\frac{1}{\rho}\right) = 0, \qquad [2.4]$$

where the total energy $E$ is given by

$$E = E_g + E_R. \qquad [2\ 5]$$

For a perfect gas,

$$E_g = p_g/(\gamma - 1)\,\rho \qquad [2.6]$$

and

$$p_g = \bar{R}\,\rho T, \qquad [2.7]$$

where $T$ is the absolute temperature, $\gamma$ the ratio of specific heats and $\bar{R}$ the gas constant.

Assuming local thermodynamic equilibrium

$$\rho E_R = 3p_R = aT^4, \qquad [2\ 8]$$

where $a$ is Stefan's constant.

If we set
$$p_g = \beta p, \quad p_R = (1-\beta)p,$$

then
$$E = \frac{1}{k-1}\,\frac{p}{\rho}, \tag{2.9}$$

where
$$k = \frac{(\gamma-1)(4-3\beta)+\beta}{\beta+3(1-\beta)(\gamma-1)} \tag{2.10}$$

This adiabatic coefficient $k$ has been introduced by Klimshin and is different from $\gamma$ and $\Gamma_1$, $\Gamma_2$, $\Gamma_3$, the three adiabatic coefficients associated with a system consisting of matter and radiation[4]. This is equal to $\gamma$ when $\beta = 1$ and 4/3 when $\beta = 0$ so that it coincides with the three adiabatic coefficients $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ in these extreme cases of pure gas and pure radiation.

Thus, the energy equation now becomes

$$\frac{1}{k-1}\cdot\frac{\partial}{\partial r}\left(\frac{p}{\rho}\right) + p\,\frac{\partial}{\partial r}\left(\frac{1}{\rho}\right) = 0 \tag{2.11}$$

or
$$p\rho^{-k} = \text{constant}, \tag{2.12}$$

and Bernoulli's equation becomes

$$\frac{v^2}{2} + \frac{k}{k-1}\cdot\frac{p}{\rho} - \frac{GM}{r} = \text{constant}. \tag{2.13}$$

## 3. SHOCK CONDITIONS

The shock conditions with the inclusion of radiation pressure and radiation energy are

$$\rho_1 v_1 = \rho_2 v_2, \tag{3.1}$$

$$p_1 + \rho_1 v_1^2 = p_2 + \rho_2 v_2^2, \tag{3.2}$$

$$\tfrac{1}{2} v_1^2 + w_1 = \tfrac{1}{2} v_2^2 + w_2, \tag{3.3}$$

where $w$ is the heat function

$$w = \frac{(4-3\beta)(\gamma-1)+\beta}{\gamma-1}\,\frac{p}{\rho} = \frac{k}{k-1}\,\frac{p}{\rho}, \tag{3.4}$$

and suffixes 1 and 2 denote the conditions just before and after the shock front.

The equation [3.3] can be written as

$$\frac{1}{2} v_1^2 + \frac{k_1}{k_1 - 1} \frac{p_1}{\rho_1} = \frac{1}{2} v_2^2 + \frac{k_2}{k_2 - 1} \frac{p_2}{\rho_2}. \qquad [3.5]$$

No doubt, we have taken $k_1$ and $k_2$ to be constants for the entire flow in front and behind the shock front, $k_1 \neq k_2$ as $k$ depends on $\beta$ which takes different values on the two sides of the shock front.

In view of the equation [3.5], the Bernoulli's equation [2.13] holds everywhere, whether the shock is present or not.

Following McCrea, we also introduce pseudo-sound velocity squared

$$c^2 = \frac{kp}{\rho}, \qquad [3\ 6]$$

in place of the actual sound velocity, given by

$$\tilde{c}^2 = \frac{\Gamma_1 p}{\rho} \qquad [3.7]$$

and the pseudo Mach number

$$u = \frac{v}{c}. \qquad [3.8]$$

The word ' pseudo ' will be dropped henceforth.

Introducing $c$ and $u$, equations [3.1], [3.2] and [3.5] become

$$\rho_1 u_1 c_1 = \rho_2 u_2 c_2, \qquad [3.9]$$

$$\frac{c_1}{k_1 u_1}(1 + k_1 u_1^2) = \frac{c_2}{k_2 u_2}(1 + k_2 u_2^2), \qquad [3.10]$$

$$c_1^2 \left(\frac{1}{2} u_1^2 + \frac{1}{k_1 - 1}\right) = c_2^2 \left(\frac{1}{2} u_2^2 + \frac{1}{k_2 - 1}\right). \qquad [3.11]$$

Combining [3.10] and [3.11], we obtain an equation giving $u_2$ in terms of $k_1$, $u_1$ and $k_2$:

$$(k_1 - 1)(k_2 - 1)(k_2^2 u_2^4 - k_1^2 u_1^4) - 2 k_1 (k_2 - 1) k_2^2 u_1^2 u_2^4 + 2 k_2 (k_1 - 1) k_1^2 u_1^4 u_2^2$$
$$+ 2 [(k_1 - 1) k_2^2 u_2^2 - (k_2 - 1) k_1^2 u_1^2] + 4 k_1 k_2 (k_1 - k_2) u_1^2 u_2^2 = 0. \qquad [3.12]$$

Since

$$p = \frac{p_R}{1 - \beta} = \frac{a T^4}{3 (1 - \beta)}, \qquad [3.13]$$

and
$$\rho = \frac{\beta a T^3}{3(1-\beta)\bar{R}}, \qquad [3.14]$$

we have
$$\left(\frac{p_2}{p_1}\right)^3 \left(\frac{\rho_1}{\rho_2}\right)^4 = \left(\frac{\beta_1}{\beta_2}\right)^4 \left(\frac{1-\beta_2}{1-\beta_1}\right). \qquad [3.15]$$

Solving [3.1], [3.2] and [3.5], we get
$$p_2/p_1 = 1 + \alpha\, F(\alpha), \qquad [3.16]$$

$$\frac{\rho_2}{\rho_1} = \frac{1}{1 - F(\alpha)}, \qquad [3.17]$$

where
$$\alpha = k_1 u_1^2 \qquad [3.18]$$

and
$$F(\alpha) = \frac{1}{k_2+1} \cdot \frac{\alpha - k_2}{\alpha} \cdot \left[1 + \sqrt{1 + \frac{2(k_2+1)(k_1-k_2)\alpha}{(k_1-1)(\alpha-k_2)^2}}\right]. \qquad [3.19]$$

The equation [3.15] now becomes
$$[1 + \alpha F(\alpha)]^3 [1 - F(\alpha)]^4 = \left(\frac{\beta_1}{\beta_2}\right)^4 \left(\frac{1-\beta_2}{1-\beta_1}\right) \qquad [3.20]$$

or
$$\phi(\alpha) = \left(\frac{\beta_1}{\beta_2}\right)^4 \left(\frac{1-\beta_2}{1-\beta_1}\right). \qquad [3.20]$$

This equation determines $\beta_2$ and $k_2$, knowing $\beta_1$, $k_1$, $u_1$ but its exact solution is difficult. We have solved it graphically (Fig. 1) for the special case corresponding to $k_1 = 1.585$, $\beta_1 = 0.86$ and $\gamma = 5/3$.

## 4. Equations Holding on the Two Sides of the Shock

Using the boundary conditions at infinity, the equation [2.13] which holds on both sides of the shock becomes

$$\frac{v^2}{2} + \frac{k}{k-1} \cdot \frac{p}{\rho} - \frac{GM}{r} = \frac{k_1}{k_1-1} \cdot \frac{p_\infty}{\rho_\infty}. \qquad [4.1]$$

If we write
$$r = x\, GM c_\infty^{-2}, \qquad [4.2]$$

where

$$c_\infty^2 = \frac{k_1 p_\infty}{\rho_\infty}, \qquad [4.3]$$

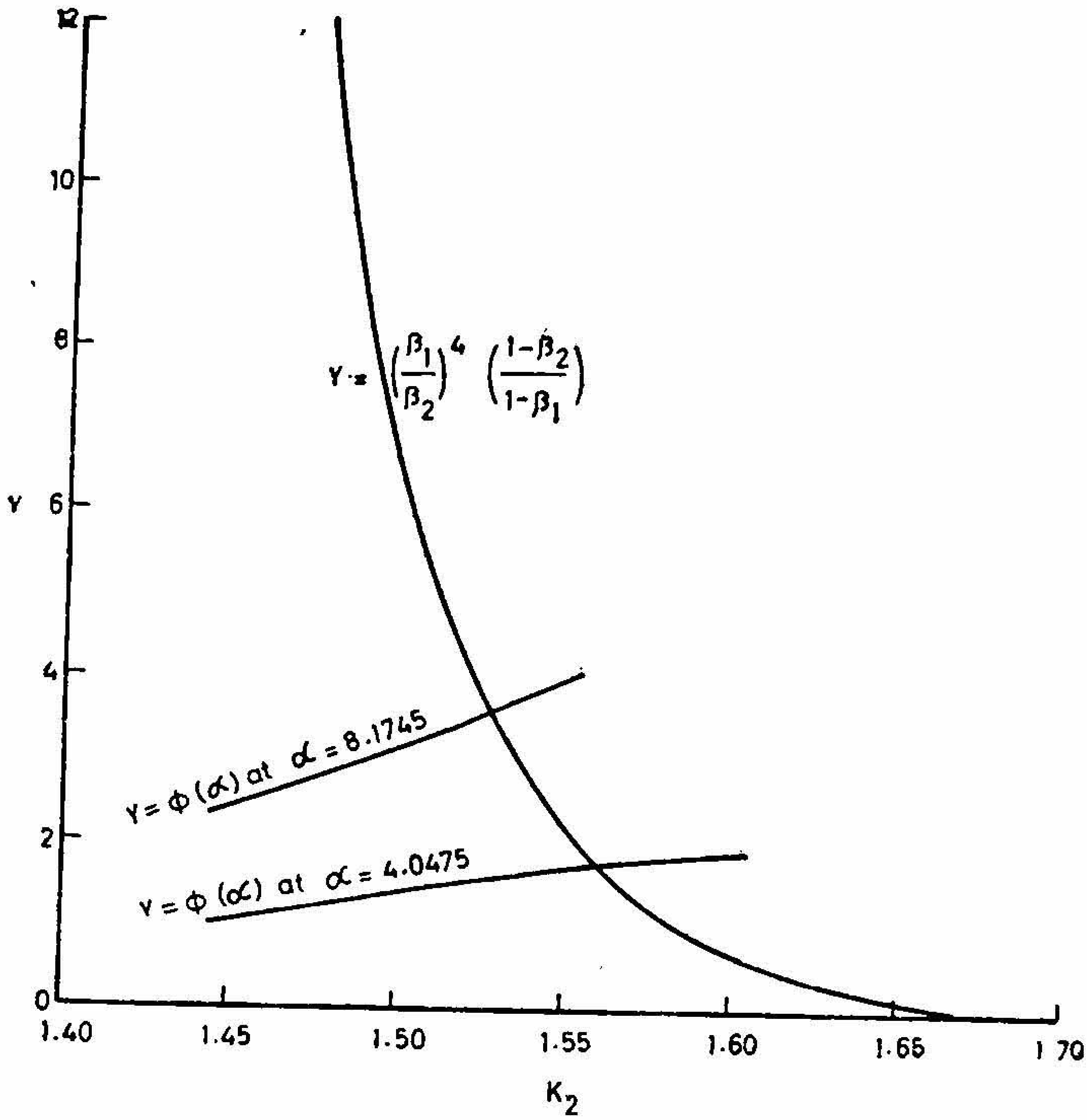


FIG. I

Solution of equation (3.20′) with $k_1=1.585$, $\beta_1=0.86$ and $\gamma=5/3$.

equation [4.1] becomes

$$c^2u^2\left(\frac{1}{2}+\frac{1}{k-1}\frac{1}{u^2}\right)=c_\infty^2\left(\frac{1}{x}+\frac{1}{k_1-1}\right). \qquad [4.4]$$

The equations [4.1] and [4.4] are true for the flow domain in front of the shock. If we assume that the shock is situated at $r=r_1$, the conditions for $r>r_1$ and $r<r_1$ can be separately discussed.

*Conditions on the front side of the Shock* $r>r_1$ :

From the energy equation

$$p/p_\infty=(\rho/\rho_\infty)^{k_1}, \qquad [4.5]$$

We can show that

$$p/p_\infty = (c/c_\infty)^{2/(k_1-1)} \qquad [4.6]$$

Making use of the equation of continuity and [4.6], we can write

$$x^2 u \left(\frac{c}{c_\infty}\right)^{(k_1+1)/(k_1-1)} = \frac{A\, c_\infty^3}{4\,\pi\, G^2 M^2 \rho_\infty} \equiv \lambda\,, \qquad [4.7]$$

where $\lambda$ is a non-dimensional quantity of matter that falls into the star per unit time.

Eliminating $(c/c_\infty)$ from [4.4] specialised for $r > r_1$ and [4.7],

$$u^{-2(k_1-1)/(k_1+1)}\left(u^2 + \frac{2}{k_1-1}\right) = \lambda^{-2(k_1-1)/(k_1+1)}\, x^{-(5-3k_1)/(k_1+1)}\left(\frac{2x}{k_1-1} + 2\right). \qquad [4.8]$$

This equation gives the Mach number at any point distant $x$ from the centre on the front side of the shock, $x$ and $r$ being connected by [4.2].

*Conditions on the back side of the Shock front, i.e. when $r < r_1$:*

The Bernoulli's equation becomes

$$\frac{v^2}{2} + \frac{k_2}{k_2-1}\,\frac{p}{\rho} - \frac{GM}{r} = \frac{k_1}{k_1-1}\,\frac{p_\infty}{\rho_\infty}. \qquad [4.9]$$

As before, it can be written as

$$c^2 u^2\left(\frac{1}{2} + \frac{1}{k_2-1}\,\frac{1}{u^2}\right) = c_\infty^2\left(\frac{1}{x} + \frac{1}{k_1-1}\right). \qquad [4.10]$$

The energy equation now becomes

$$\frac{\rho}{\rho_2} = \left(\frac{c}{c_2}\right)^{2/(k_2-1)}. \qquad [4.11]$$

In this case,

$$x^2 u\,(c/c_\infty)^{(k_2+1)/(k_2-1)} = \frac{A\, c_\infty^3}{4\,\pi\, G^2 M^2 \rho_\infty}\,\frac{\rho_\infty}{\rho}\left(\frac{c}{c_\infty}\right)^{2/(k_2-1)}$$

$$= \lambda\,\frac{\rho_\infty}{\rho_2}\,\frac{\rho_2}{\rho}\left(\frac{c}{c_2}\,\frac{c_2}{c_\infty}\right)^{2/(k_2-1)}$$

$$= \lambda\,\frac{\rho_\infty}{\rho_2}\left(\frac{c_2}{c_\infty}\right)^{2/(k_2-1)}, \qquad [4.12]$$

making use of [4.11].

We can easily show, on employing equations [3.9] and [4.6], that

$$\frac{\rho_\infty}{\rho_2}\left(\frac{c_2}{c_\infty}\right)^{2/(k_2-1)} = \frac{u_2}{u_1}\left(\frac{c_2}{c_1}\right)^{(k_2+1)/(k_2-1)} \left(\frac{c_\infty}{c_1}\right)^{2(k_2-k_1)/(k_1-1)(k_2-1)} \cdot \quad [4.13]$$

Thus, the equation [4.12] can be written as

$$x^2 u\left(\frac{c}{c_\infty}\right)^{(k_2+1)/(k_2-1)} = \lambda\frac{u_2}{u_1}\left(\frac{c_2}{c_1}\right)^{(k_2+1)/(k_2-1)} \left(\frac{c_\infty}{c_1}\right)^{2(k_2-k_1)/(k_1-1)(k_2-1)}$$

$$= \nu \text{ (say)}. \quad [4.14]$$

Eliminating $(c/c_\infty)$ from [4.10] and [4.14]

$$u^{-2(k_2-1)/(k_2+1)}\left(u^2+\frac{2}{k_2-1}\right) = \nu^{-2(k_2-1)/(k_2+1)}\, x^{-(5-3k_2)/(k_2+1)}\left(\frac{2x}{k_1-1}+2\right)\cdot [4.15]$$

This equation defines the Mach number at a point distant $x$ from the centre on the back side of the shock.

## 5. Calculation of Critical $\lambda$.

Following Bondi, the equation [4.8] giving $u$ as a function of $x$ in front of the shock can be written as

$$f(u) = \lambda^{-2(k_1-1)/(k_1+1)}\, g(x), \quad [5.1]$$

where

$$f(u) = u^{-2(k_1-1)/(k_1+1)}\left(\frac{u^2}{2}+\frac{1}{k_1-1}\right), \quad [5.2]$$

$$g(x) = x^{-(5-3k_1)/(k_1+1)}\left(\frac{x}{k_1-1}+1\right). \quad [5.3]$$

$f(u)$ and $g(x)$, each being sum of a positive and a negative power of their argument have a minimum.

$f(u)$ has its minimum at $u_{min} = 1$ and is given by

$$f_{min} = \tfrac{1}{2}\frac{k_1+1}{k_1-1}, \quad [5.4]$$

while $g(x)$ has its minimum at $x_{min} = (5-3k_1)/4$ and is given by

$$g_{min} = \tfrac{1}{4}\frac{k_1+1}{k_1-1}\left(\frac{5-3k_1}{4}\right)^{-(5-3k_1)/(k_1+1)}. \qquad [5.5]$$

Now $x$ varies between infinity and the value corresponding to the surface of the star, which will usually be very small so that $x_m$ lies in the physically significant interval.

The lowest value that the right hand side of [5.1] can take is $\lambda^{-2(k_1-1)/(k_1+1)} g_{min}$. The equation

$$f(u) = \lambda^{-2(k_1-1)/(k_1+1)} g_{min}. \qquad [5.6]$$

should hold for some value of $u$, but $f(u)$ cannot be less than $f_{min}$ so that

$$\lambda^{-2(k_1-1)/(k_1+1)} g_{min} \geqslant f_{min}$$

or

$$\lambda \leqslant \left(\frac{g_{min}}{f_{min}}\right)^{(k_1+1)/2(k_1-1)}. \qquad [5.7]$$

Thus, the maximum value of $\lambda$ is

$$\lambda_c = \left(\frac{1}{2}\right)^{(k_1+1)/2(k_1-1)}\left(\frac{5-3k_1}{4}\right)^{-(1/2)(5-3k_1)/(k_1-1)}. \qquad [5.8]$$

Hence the rate of accretion, given by [4.7] cannot exceed

$$4\pi\lambda_c G^2 M^2 c_\infty^{-3}\rho_\infty.$$

When $\lambda < \lambda_c$, one branch of the solution gives throughout supersonic velocity and the other throughout subsonic so that in the first branch boundary conditions at infinity is not satisfied and in the second, the shock wave cannot exist. The solution $\lambda > \lambda_c$ does not give any meaningful solution, showing the impossibility of a rate of accretion greater than that corresponding to $\lambda = \lambda_c$. Thus, only $\lambda = \lambda_c$ gives physically realisable solution of the present problem *i e.* only this solution satisfies the boundary condition of zero velocity at infinity with the Mach number gradually rising inwards and having a value unity at $x = x_{min}$ and greater than unity for all $x < x_m$. We can, therefore, assume the shock to be situated at any $x < x_{min}$ with the Mach number of the flow greater than unity there. Using $x = x_1$ (say), $\lambda = \lambda_c$ and $\beta_1$, $k_1$, corresponding to the temperature conditions at infinity, we find from equation [4.8] the value of $u_1$. The additional shock relation [3.20] determines $k_2$. Substitution of $u_1$, $k_1$ and $k_2$ in [3.12] gives $u_2$. Equation [4.7] gives $c_1$ and $c_2$ then is obtained from the shock condition [3.11]. Knowing $u_1$, $u_2$, $c_1$, $k_1$, $k_2$, $\lambda_c$, we can find $\nu$ from equation [4.14]. Finally, equation [4.15] gives the variation of $u$ with $x$ for $x < x_1$, that is behind the shock.

## 6. Numerical Illustration

Let us consider a hypothetical situation for a star with mass and radius given by

$$M = 40\ M_{\odot},\quad R = 15\ R_{\odot},$$

and for the following conditions at infinity :

$$T_{\infty} = (10^4)^0 \text{ Kelvin},\quad \rho_{\infty} = 10^{-20}\ gm/cm^3,$$

$$\beta_1 = 0.86,\ k_1 = 1.585.$$

$$x_{surf} = 0.00013.$$

$$\lambda_c = 0.3881,\ x_m = 0.06125,\ \nu_m = 1.$$

For $r > r_1$ :

$$u^{-0.4526}(u^2 + 3.4188) = (0.3881)^{-0.4526}\, x^{\cdot 0.0948}(3.4188\, x + 2).$$

Assuming the shock to be at $x_1 = 0.001$, we find that $u_1 = 2.271$. Behind the shock, $k_2 = 1.527$ and $u_2$ is determined by

$$19.3710\ u_2^4 - 125.0085\ u_2^2 + 34.2574 = 0,$$

admitting meaningful root $u_2 = 0.5357$, also $v = 0.9070$.

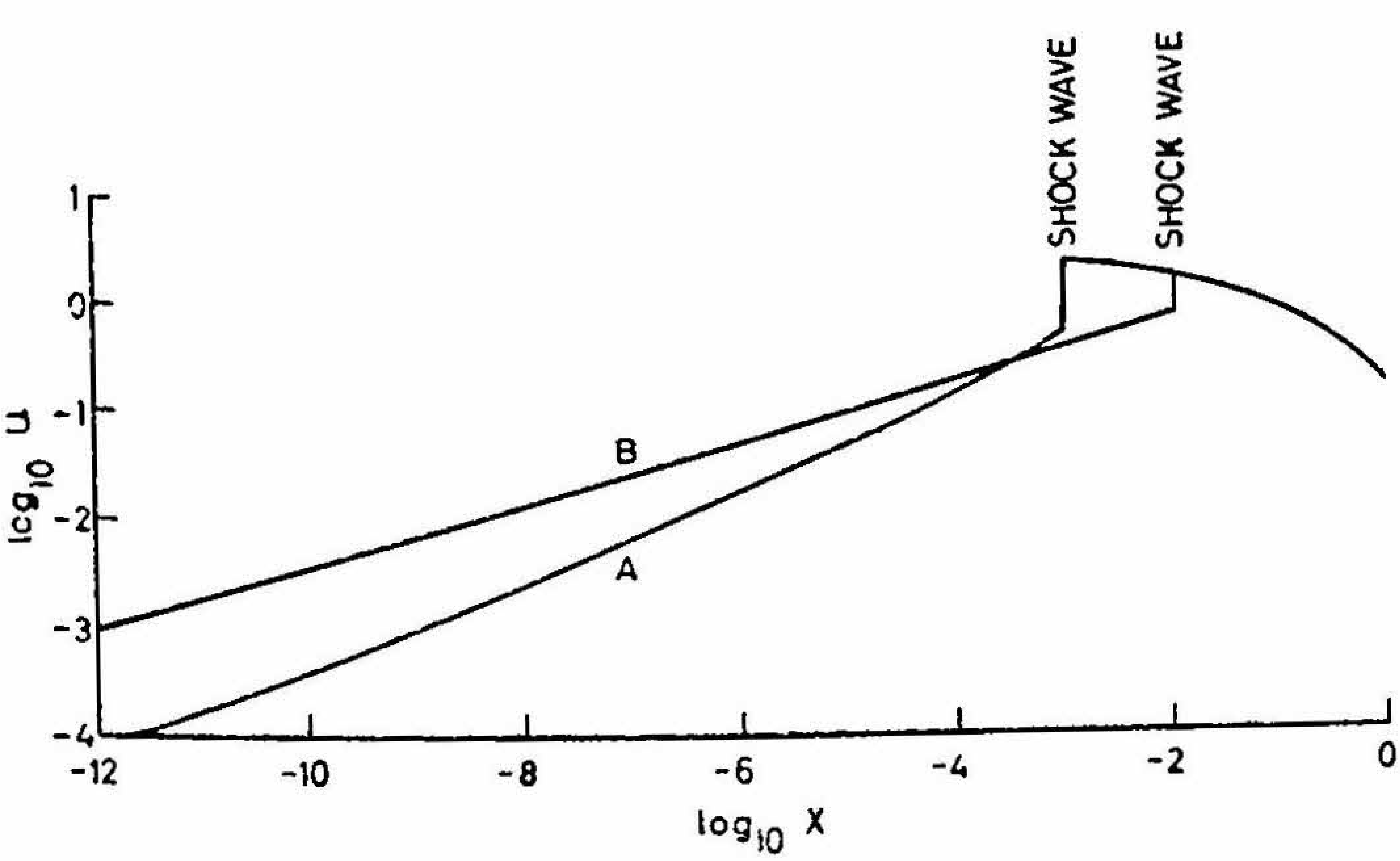


Fig. II

Accretion of interstellar gas by a star in the presence of radiation

The value of $u$ behind the sbock is determined from

$$u^{-0.4171}(u^2+3.7951)=(0.9070)^{-0.4171}x^{-0.1658}(3.4188x+2).$$

Figure II shows the graphs of $u$ against $x$ with shock at $x=10^{-3}$ (curve A) and $x=10^{-2}$ (curve B).

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