HELICAL FLOW OF A BINGHAM PLASTIC

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ABSTRACT

The flow of a Bingham plastic between two coaxial circular cylinders rotating about their common axis with unequal angular velocities and having a relative velocity along their common axis is investigated. The position of the yield surface is determined in terms of the relative axial velocity of the cylindrical walls and the stresses acting on the outer cylinder. These results are presented graphically.

1. INTRODUCTION

In the present paper, we investigate the helical flow of a material which can support a finite stress elastically without flow and which flows with constant mobility when the stresses are sufficiently large. It is assumed[1] that the plastic flow of the isotropic material is governed by the following rheological equations of state.

$$p_{ik} = p'_{ik} - p\,\delta_{ik} \quad [1.1]$$

$$p'_{ik} = 2\,\eta\, e_{ik}, \left(p'_{ik}\, p'_{ik} \geq v^2\right) \quad [1.2]$$

$$\eta = \eta_1 + v\left(2\, e_{ik}\, e_{ik}\right)^{-1/2}, \quad [1.3]$$

where p_{ik} is the stress tensor, p'_{ik} the deviatoric stress tensor, e_{ik} the rate of strain tensor, p the hydrostatic pressure, δ_{ik} the kronecker deltas, η the effective viscosity η_1 the reciprocal mobility and v the yield stress. We further assume that the rotatory motion is induced by the relative rotation of the cylinders about their common axis, while the axial flow is induced by the relative axial motion of the cylinders in the absence of any axial pressure gradient as in the investigations of Oldroyd[2].

These equations describe the behaviour of the material only in regions where $p'_{ik}\, p'_{ik} \geq v^2$ (plastic flow regions). It may be noted that when $v = 0$, the viscosity becomes constant as for a Newtonian fluid. On the boundary surface (yield surface) separating the plastic region from the elastic region, the rate of strain components vanish, so that the frictional stress is constant and equal to the yield stress v. Thus the boundary conditions to a plastic region of the yield surface are

(i) the continuity of velocity, the elastic regions being treated as rigid and

(ii) the vanishing of the rate of strain components.

2. EQUATIONS OF THE PROBLEM AND THEIR SOLUTION

Without loss of generality (as shown in the text) we may take the inner cylinder (radius a) to be fixed and the outer cylinder (radius b) to be moving with an angular velocity Ω and an axial velocity V. We shall consider the resulting helical flow in the absence of pressure gradient along the common axis of the cylinders.

In cylindrical coordinates, we take the radial azimuthal and axial velocities to be 0, $r\,\omega\,(r)$, and $w\,(r)$.

The non-vanishing components of the rate of strain are

$$e_{r\theta} = \tfrac{1}{2} r\,(d\,\omega/d\,r), \qquad [2.1]$$

and

$$e_{rz} = \tfrac{1}{2}(d\,w/d\,r), \qquad [2.2]$$

$d\,\omega/d\,r$ and $d\,w/d\,r$ are positive except possibly on the yield surface where both of them vanish.

The stress equations of momentum give

$$dp/dr = \rho\, r\,\omega^2, \ (\rho \text{ is the density}) \qquad [2.3]$$

$$d\,(r^2\, p_{r\theta})/dr = 0, \qquad [2.4]$$

$$d\,(r\, p_{rz})/dr = 0, \ (\partial\, p/\partial z = 0) \qquad [2.5]$$

so that

$$p_{r\theta} = b_1/r^2, \qquad [2.6]$$

$$p_{rz} = b_2/r, \qquad [2.7]$$

and

$$\frac{dw}{dr} = \frac{b_2\, r^2}{b_1} \cdot \frac{d\omega}{dr}, \qquad [2.8]$$

where b_1 and b_2 are constants to be determined from the boundary conditions specified later.

From [2.8] it is evident that all the components of the rate of strain can become zero on a cylinder coaxial with the cylindrical walls.

The effective viscosity η is given by

$$\eta = \eta_1 + \frac{\upsilon\, b_1}{r\,(d\omega/dr)\,\sqrt{(b_1^2 + b_2^2\, r^2)}} = \eta_1 + \frac{\upsilon\, b_2\, r}{(dw/dr)\,\sqrt{(b_1^2 + b_2^2\, r^2)}}. \qquad [2.9]$$

On using [1.2] and [2.9], we can write the stress componets as

$$p_{rr} = p_{\theta\theta} = p_{zz} = -p = -\rho \int^r x\omega^2\,(x)\, dx, \qquad [2.10]$$

$$p_{r\theta} = \frac{b_1}{r^2} = \eta_1 r \frac{d\omega}{dr} + \frac{\upsilon b_1}{\sqrt{(b_1^2 + b_2^2 r^2)}}, \tag{2.11}$$

$$p_{rz} = \frac{b_2}{r} = \eta_1 \frac{d\omega}{dr} + \frac{\upsilon b_2 r}{\sqrt{(b_1^2 + b_2^2 r^2)}}, \tag{2.12}$$

and

$$p_{\theta z} = 0. \tag{2.13}$$

Equations [2.11] and [2 12] together with appropriate boundary conditions will determine the velocities ω and w.

It may be noted that boundary conditions to be imposed on ω and w are different in the two possible types of flow, namely

(i) there is plastic flow in the region $a \leq r \leq R$ $(R \leq b)$ and the material in the region $R \leq r \leq b$ moves as a solid plug with the velocity of the outer cylinder and

(ii) there is plastic flow in the entire region $a \leq r \leq b$.

We shall now determine the velocities in the above two cases.

Case (1): It is convenient to non-dimensionalize the various physical quantities by means of a length R (inner radius of the plug), a velocity $R\Omega$ and a stress $\eta_1 \Omega$ and use the following dimensionless quantities

$$\bar{r} = \frac{r}{R}, \quad \bar{\omega} = \frac{\omega}{\Omega}, \quad \bar{w} = \frac{w}{R\Omega},$$

$$\bar{b}_1 = \frac{b}{R^2 \eta_1 \Omega}, \quad \bar{b}_2 = \frac{b_2}{R \eta_1 \Omega},$$

$$m = \frac{V}{R\Omega}, \quad n = \frac{a}{R}$$

and

$$K = \frac{\upsilon}{\eta_1 \Omega} \qquad \text{(Bingham number)}.$$

The parameter K indicates the relative importance of plastic and viscous stresses.

The equations determining the velocities become

$$\frac{d\bar{\omega}}{d\bar{r}} = \frac{\bar{b}_1}{\bar{r}^3} - \frac{K\bar{b}_1}{\bar{r}\sqrt{(\bar{b}_1^2 + \bar{b}_2^2 \bar{r}^2)}}, \tag{2.14}$$

$$\frac{d\bar{w}}{d\bar{r}} = \frac{\bar{b}_1}{\bar{r}} - \frac{K\bar{b}_2 \bar{r}}{\sqrt{(\bar{b}_1^2 + \bar{b}_2^2 \bar{r}^2)}}, \tag{2.15}$$

with the boundary conditions

$$\bar{\omega}(1) = 1, \ \bar{w}(1) = m, \ (d\bar{w}/d\bar{r}) = 0 \text{ at } \bar{r} = 1, \qquad [2.16]$$

$$\bar{\omega}(n) = \bar{w}(n) = 0. \qquad [2.17]$$

The conditions [2.16] ensure the continuity of volocity and the vanishing of the rate of strain tensor at the yield surface and the conditions [2.17] ensure the no slip condition on the surface of the inner cylinder.

The solutions of equations [2.14] and [2.15] are

$$\bar{\omega} = -\frac{\bar{b}_1}{2\bar{r}^2} - K\log\left[\frac{\sqrt{(\bar{b}_1^2 + \bar{b}_2^2\bar{r}^2)} - \bar{b}_1}{\bar{r}}\right] + \bar{b}_3, \qquad [2.18]$$

and

$$\bar{w} = b_2\log\bar{r} - (K/\bar{b}_2)\sqrt{(\bar{b}_1^2 + \bar{b}_2^2\bar{r}^2)} + \bar{b}_4, \qquad [2.19]$$

where the arbitrary constants $\bar{b}_1$, $\bar{b}_2$, $\bar{b}_3$, $\bar{b}_4$ and the ratio n are determined from [2.16], [2.17].

The occurrence of the constants $\bar{b}_3$, $\bar{b}_4$ in the above form shows that there is no loss of generality in taking the velocity of the inner cylinder to be zero and considering only the velocities relative to the inner cylinder.

We may write $\bar{\omega}$ and $\bar{w}$ in the form

$$\bar{\omega} = \frac{\bar{b}_1}{2}\left(1 - \frac{1}{\bar{r}^2}\right) + 1 + K\log\left[\frac{(K - \bar{b}_1)\bar{r}}{\sqrt{(\bar{b}_1^2 + \bar{b}_2^2\bar{r}^2)} - \bar{b}_1}\right], \qquad [2.20]$$

and

$$\bar{w} = m + \bar{b}_2\log\bar{r} + (K/\bar{b}_2)\,[K - \sqrt{(\bar{b}_1^2 + \bar{b}_2^2\bar{r}_2)}], \qquad [2.21]$$

with

$$\bar{b}_1^2 + \bar{b}_2^2 = K^2. \qquad [2.22]$$

The condition [2.22] is equivalent to the presence of a yield surface $r = R$ on which the deviatoric stress satisfies the relation $\frac{1}{2}p'_{ik}p'_{ik} = v^2$

Setting $\bar{b}_1 = K\cos\theta$, $\bar{b}_2 = K\sin\theta$ we have the non-dimensional yield stress $\bar{v}_{ik}$ acting on the yield surface as

$$\bar{v}_{ik} = K\begin{Bmatrix} 0 & \cos\theta & \sin\theta \\ \cos\theta & 0 & 0 \\ \sin\theta & 0 & 0 \end{Bmatrix} \qquad [2.23]$$

The constants θ and n are determined by means of the following equations

$$\sqrt{(\cos^2\theta + n^2\sin^2\theta)} = (m/K)\sin\theta + \sin^2\theta\log n + 1, \qquad [2.24]$$

and

$$\frac{\cos\theta}{2n^2}(n^2-1) + \log\left[\frac{n(1-\cos\theta)}{\sqrt{(\cos^2\theta + n^2\sin^2\theta)} - \cos\theta}\right] + (1/K) = 0. \qquad [2.25]$$

We note that we may determine the various constants in terms of K and θ which describe the stresses acting on the outer cylinder or alternatively in terms of K and m giving the yield stress of the material and the axial velocity of the outer cylinder.

We have solved the above equations [2.24], [2.25] in a certain range of values of K, ϑ and m and the results are shown graphically in Fig. I, II, and III.

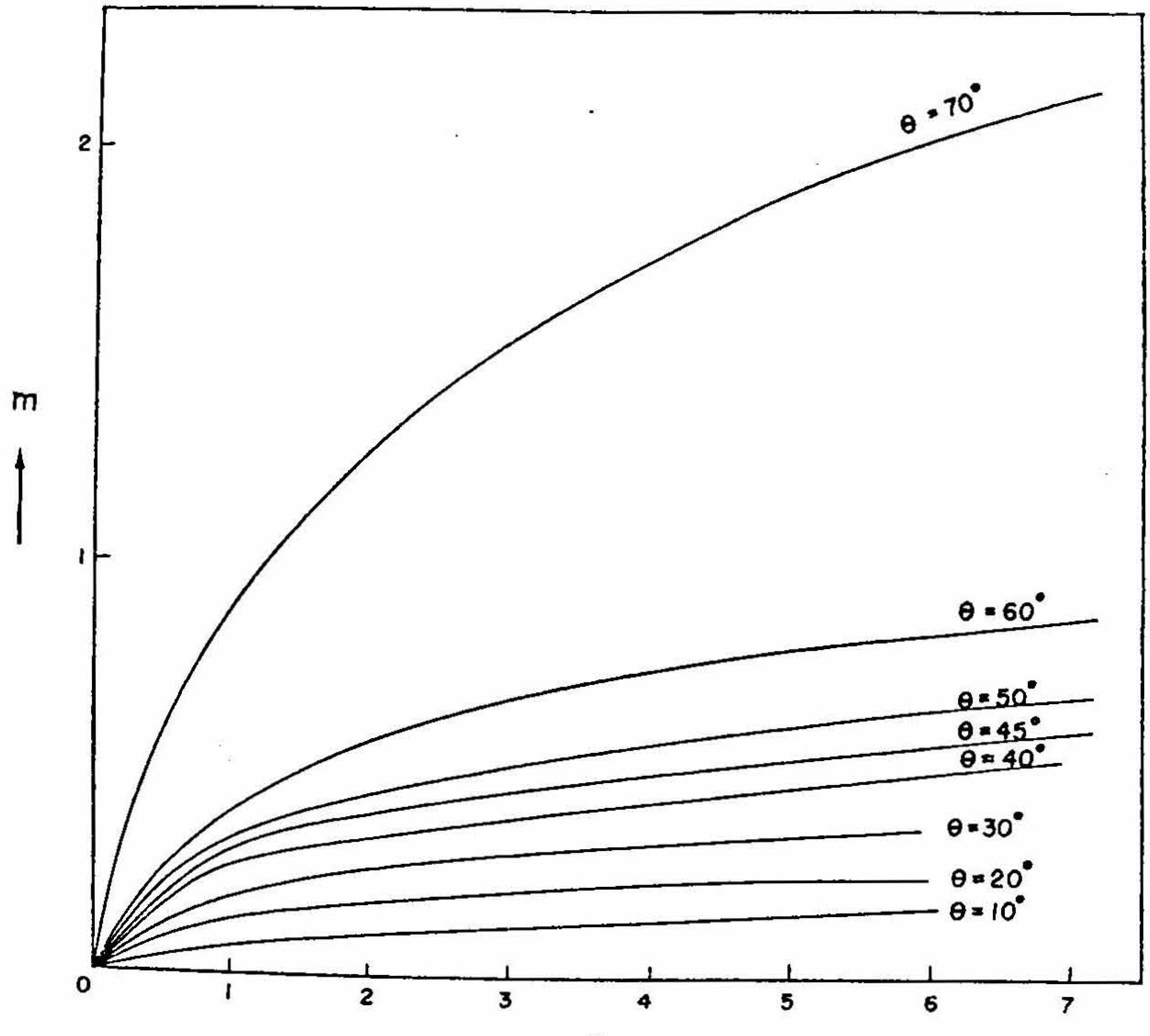

FIG. I

Variation of the Axial Velocity (Outer Cylinder) with the Stresses acting on the Outer Cylinder

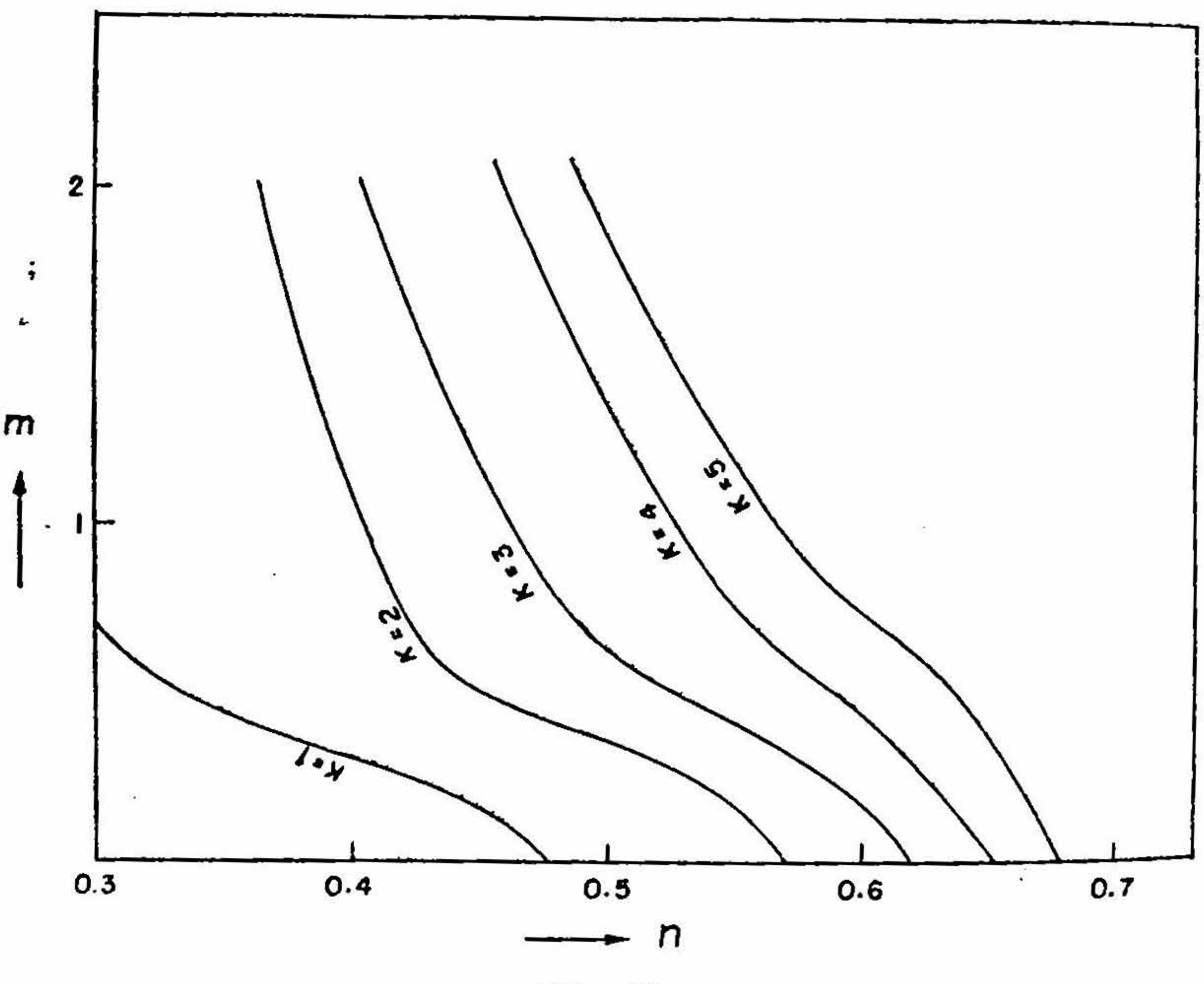

FIG. II

Variation of the Ratio *n* with the Axial Velocity of the Outer Cylinder

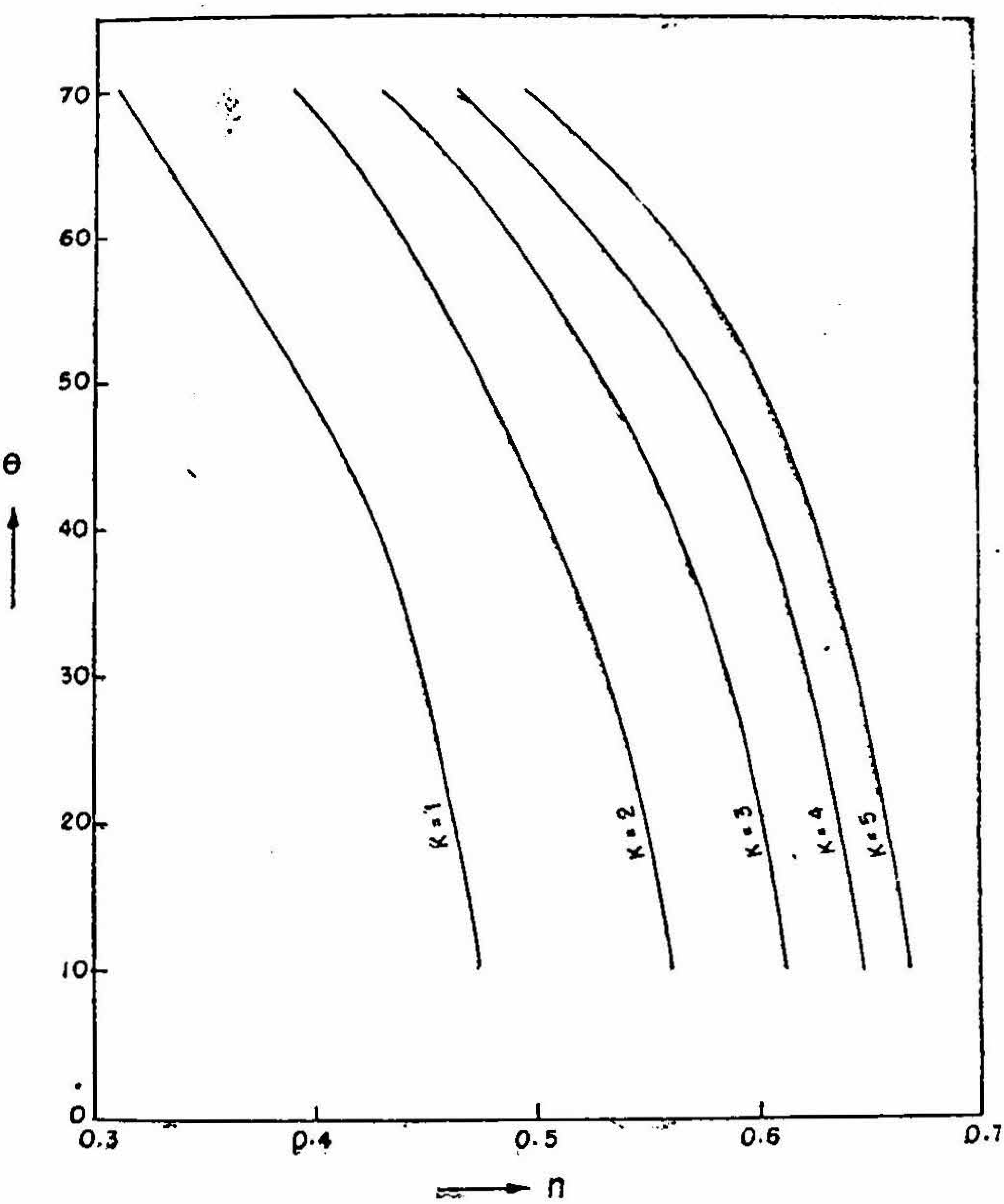

FIG. III

Variation of the Ratio n with the Stresses acting on the Outer Cylinder

Fixing m and θ in any particular flow, we may see that n increases with K and remains less t an unity for all finite values of K. Thus for large values of K we have a thin sheath of liquid surrounding the inner cylinder in a state of plastic flow. n decreases with K and the present case describes the flow for

$$n \leq (a/b) \ (=l).$$

Case (*ii*) : In this case, we use the radius a of the inner cylinder, a velocity $a\,\Omega$ and a stress $\eta_1\,\Omega$ for pruposes of non-dimensionalization.

The boundary conditions on [2.11] and [2.12] are

$$\bar{w}(1) = \bar{\omega}(1) = 0, \tag{2.26}$$

$$\bar{\omega}(l) = 1, \quad \bar{w}(l) = m. \tag{2.27}$$

Accordingly $\bar{\omega}$ and $\bar{w}$ are given by

$$\bar{\omega} = (\bar{b}_1/2)\,[1 - (1/\bar{r}^2)] - K \log\left[\frac{\sqrt{(\bar{b}_1^2 + \bar{b}_2^2 \bar{r}^2)} - \bar{b}_1}{\bar{r}\,\{\sqrt{(\bar{b}_1^2 + \bar{b}_2^2)} - \bar{b}_1\}}\right], \tag{2.28}$$

$$\bar{w} = \bar{b}_2 \log \bar{r} - \frac{K}{\bar{b}_2}\,[\sqrt{(\bar{b}_1^2 + \bar{b}_2^2 \bar{r}^2)} - \sqrt{(\bar{b}_1^2 + \bar{b}_2^2)}]. \tag{2.29}$$

where $\bar{b}_1$ and $\bar{b}_2$ are given by

$$\frac{\bar{b}_1}{2}\left(1 - \frac{1}{l^2}\right) - K \log\left[\frac{\sqrt{(\bar{b}_1^2 + \bar{b}_2^2 l^2)} - \bar{b}_1}{l\{\sqrt{(\bar{b}_1^2 + \bar{b}_2^2)} - \bar{b}_1\}}\right] = 1, \tag{2.30}$$

and

$$\bar{b}_2 \log l - \frac{K}{\bar{b}_2}\,[\sqrt{(\bar{b}_1^2 + \bar{b}_2^2 l^2)} - \sqrt{(\bar{b}_1^2 + \bar{b}_2^2)}] = m. \tag{2.31}$$

3. PARTICULAR CASES OF THE FLOW

(i) When $\Omega = 0$, we have a purely axial flow as a result of the relative axial velocity of the cylinderical walls. This has been discussed by Oldroyd[2].

(ii) When $V = 0$, we obtain Couette flow discussed by Reiner-Rivlin[3].

7. DISCUSSION OF THE RESULTS

It is interesting to note that the ratio of the shearing stresses on any cylindrical surface coaxial with the cylinder and in particular

$$\frac{(p_{rz}/p_{r\theta})\text{ inner cylinder}}{(p_{rz}/p_{r\theta})\text{ core}} = n$$

which is the same as if there were viscous flow between two cylinders of radii $\bar{r} = 1, \bar{r} = n$.

The non-dimensional axial velocity m of the outer cylinder increases with K and also with θ the rate of increase being very large for $\theta > 60°$ (Fig. 1).

For a given value of K, n decreases with m. The rate of decrease is very small for $m > 0.7$ (Fig. II). n also decreases with θ the rate of decrease being very small for $\theta < 45^\circ$ (Fig. III).

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