TRANSPORT PROCESSES IN A MULTI-COMPONENT ASSEMBLY ON THE BASIS OF GENERALIZED BGK COLLISION MODEL

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ABSTRACT

Bhatnagar, Gross and Krook developed a collision model for one component neutral assembly of particles in order to overcome the inherent difficulties of the Boltzmann collision integral. This has been generalized to an N-component assembly of charged and neutral particles by Bhatnagar and Devanathan. However, the transport equations obtained directly from these kinetic equations are far from simple. In this paper, simpler and elegant transport equations have been obtained by expanding the distribution functions in generalized Hermite Polynomials following Grad. From these generalized coefficients of direct electrical conductivity, diffusivity, viscosity, and heat conductivity are obtained in the presence of magnetic field. Also the relaxation times have been calculated. These naturally lead to a mechanism of the occurrence of Gross-gaps.

1. INTRODUCTION

The transport processes are essentially non-equilibrium processes. In the study of non-equilibrium processes one attempts to derive from the kinetic equations a consistent closed system of transport equations involving the macroscopic quantities associated with the system like density, velocity, temperature, stresses, heat flux, etc. In such macroscopic equations certain parameters occur. For instance, the stresses are proportional to certain space derivatives of velocity components. The corresponding coefficient of proportionality is defined as the coefficient of viscosity. Similarly, the heat flux vector is directly proportional to the temperature gradient and the coefficient of proportionality is the coefficient of heat conductivity and so on. In certain simple flow problems of an ideal gas, we can identify these coefficients exactly as the momentum transfer and heat transfer per unit area per unit time due to molecular interactions. But in general, the dependence is very complicated and we consider the former statement as the definition of transport coefficients. Thus, the main purpose of the present paper is to start with suitable kinetic equations and to deduce a closed system of transport equations in order to obtain expressions for transport coefficients such as viscosity, heat conductivity and electrical conductivity. In § 2, we shall discuss the basic kinetic equations which we have used and in § 3, the outline of the procedure for solution is explained and the closed system of transport equations are derived. In § 4, we consider three simple problems to derive the coefficients of viscosity, heat conductivity and the electrical conductivity. These simple processes are generalized and stationary non-equilibrium processes in the presence of magnetic field are considered in § 5. Finally, in § 6. we disouss the unsteady relaxation problem and attempt a plausible physical explanation of Gross gaps in frequency spectrum.

2. KINETIC EQUATIONS

Consider an assembly of N kinds of particles. Let m_s and e_s denote the mass and charge of a particle of s-type. Further, let, at time t and position $\mathbf{r}$, $\vec{\xi}_s$ be the molecular velocity and $\mathbf{F}_s$ be the external non-electromagnetic force acting on that particle. Then, the state of the system is described by the distribution functions $f_s(\vec{\xi}_s, \mathbf{r}, t)$ satisfying the Maxwell-Boltzmann equations and the self-consistent electromagnetic equations. With the usual notation of Chapman and Cowling[1], these equations are:

Maxwell-Boltzmann equations

$$\frac{\partial f_s}{\partial t} + \xi_{si}\frac{\partial f_s}{\partial x_i} + \frac{1}{m_s}\Big[F_{si} + e_s\,(E_i + [1/c]\,\epsilon_{ijk}\,\xi_{sj}\,H_k)\Big]\frac{\partial f_s}{\partial \xi_{si}}$$

$$= \sum_{j=1}^{N}\iint\Big[f_j'(\vec{\xi}_j', \mathbf{r}, t)f_s'(\vec{\xi}_s', \mathbf{r}, t) - f_j(\vec{\xi}_j, \mathbf{r}, t)\,f_s(\vec{\xi}_s, \mathbf{r}, t)\Big] \times$$

$$\times\, g_{js}\,b\,db\,d\epsilon\,d\vec{\xi}_j, \qquad s = 1, \cdots, N \qquad [2.1]$$

Maxwell equations

$$c\nabla \times \mathbf{H} = 4\pi\mathbf{J} + \partial\mathbf{E}/\partial t, \qquad [2.2]$$

$$c\nabla \times \mathbf{E} = -\partial\mathbf{H}/\partial t, \qquad [2.3]$$

$$\nabla\cdot\mathbf{H} = 0, \qquad [2.4]$$

$$\nabla\cdot\mathbf{E} = 4\pi q, \qquad [2.5]$$

where the current density $\mathbf{J}$ and the charge density q are given by

$$\mathbf{J} = \sum_{j=1}^{N} e_j \int \vec{\xi}_j f_j(\vec{\xi}_j, \mathbf{r}, t)\,d\vec{\xi}_j \qquad [2.6]$$

and

$$q = \sum_{j=1}^{N} e_j \int f_j(\vec{\xi}_j, \mathbf{r}, t)\,d\vec{\xi}_j. \qquad [2.7]$$

The above equations are highly coupled nonlinear integro-partial differential equations and in order to simplify the basic kinetic equations, Bhatnagar, Gross and Krook[2] proposed a simple tractable Collision model retaining the essential physical characteristics of the Maxwell-Boltzmann equations. This has been generalized to a multicomponent assembly by Bhatnagar and Devanathan[3]. In the collision integral, the term

$$-f_s(\vec{\xi}_s, \mathbf{r}, t) \iint f_j(\vec{\xi}_j \mathbf{r}, t)\, g_{js}\, b\, db\, d\epsilon\, d\vec{\xi}_j$$

represents the number of s-th type of particles removed from the velocity range $(\vec{\xi}_s, d\vec{\xi}_s)$ by the interaction with j-th type of particles and hence replaced by an equivalent model

$$-\frac{N_j(\mathbf{r}, t)}{\sigma_{js}} f_s(\vec{\xi}_s, \mathbf{r}, t)$$

where $N_j(\mathbf{r}, t)$ is the number density of j-th type of particles given by

$$N_j(\mathbf{r}, t) = \int f_j(\vec{\xi}_j, \mathbf{r}, t)\, d\vec{\xi}_j.$$

The nonlinear term

$$\iint f_j'(\vec{\xi}_j', \mathbf{r}, t)\, f_s'(\vec{\xi}_s', \mathbf{r}, t)\, g_{js}\, b\, db\, d\epsilon\, d\vec{\xi}_j$$

representing the number of particles brought into the velocity range concerned is replaced by the following:

(total number of collisions per unit volume per unit time) ×
× (probability that the particle goes into the concerned velocity range)

We shall denote the average total number of collisions between s-th type and j-th type by

$$\frac{N_j N_s}{\sigma_{js}} = \iiint f_j(\vec{\xi}_j, \mathbf{r}, t)\, f_s(\vec{\xi}_s, \mathbf{r}, t)\, g_{js}\, b\, db\, d\epsilon\, d\vec{\xi}_j\, d\vec{\xi}_s, \qquad [2.8]$$

and instead of taking the detailed mechanism of collision in evaluating the probability mentioned above, we take it to be locally Maxwellian given by

$$\Phi_{js}(\vec{\xi}_s, \mathbf{r}, t) = \left(\frac{m_s}{2\pi K T_{js}}\right)^{3/2} \exp\left[-\frac{m_s}{2 K T_{js}}(\vec{\xi}_s - \mathbf{u}_{js})^2\right]$$

assuming that the s-particles are scattered randomly by the j-particles. In the above expression K is the Boltzmann constant, $\mathbf{u}_{js}$ and T_{js} are mean velocity and temperature of the scattered s-th type of particles during their interaction with j-th type of particles. For these cross-velocities and temperatures we choose the phenomenalogical relations

$$\mathbf{u}_{js} = a_{jj}\,\mathbf{u}_{jj} + a_{js}\,\mathbf{u}_{ss}$$

and
$$T_{js} = b_{jj}\,T_{jj} + b_{js}\,T_{ss} + D_{js}\,\mathbf{u}_{jj}^2 + E_{js}\,\mathbf{u}_{jj}\cdot\mathbf{u}_{ss} + F_{js}\,\mathbf{u}_{ss}^2,$$

where $\mathbf{u}_{ii}$ and T_{ii} are the mean velocity and temperature of the i-th type of particles. Subjecting the collision terms to Maxwell's relaxation problem and to the instantaneous conservation laws of mass, momentum and energy, we get (for complete details refer[3])

$$a_{jj} = 1 - a_{js},$$

$$D_{js} = -\tfrac{1}{2}\,E_{js} = F_{js},$$

$$b_{jj} = 1 - b_{js},$$

$$m_s\,a_{jj} = m_j\,a_{ss} \equiv A_{js} = A_{sj},$$

$$b_{jj} = b_{ss} \equiv B_{js} = B_{sj}\,,$$

and
$$D_{js} + D_{sj} = 1/(3\,k)\,A_{js}\,(2 - a_{jj} - a_{ss}).$$

These relations determine only half of the phenomenalogical constants. But, considering the average momentum transfer and energy transfers, we find

$$A_{js} = \frac{m_j m_s}{m_j + m_s}\left[\frac{m_s\,|\vec{\xi}_s' - \vec{\xi}_s|}{[m_j m_s/(m_j + m_s)]\,g_{is}}\right]_{av}$$

and

$$6\,K\,D_{js} = \frac{m_j\,m_s}{m_j + m_s}\left[\frac{\frac{1}{2}\,m_s\,|\vec{\xi}_s'^{\,2} - \vec{\xi}_s^{\,2}|}{[\frac{1}{2}\,m_j\,m_s/(m_j + m_s)]\,g_{js}^2}\right]_{av}$$

Also B_{js} is just the coefficient of direct heat transfer between the two components. From the knowledge of the law of interaction, these constants have been determined in reference[3]. For ready reference we shall record them below:

$$A_{js} = \frac{m_j\,m_s}{m_j + m_s} \qquad \text{for elastic collisions}$$

$$\simeq 0.113\frac{m_j\,m_s}{m_j + m_s} \qquad \text{for Coulomb law}$$

$$\simeq 0.023\frac{m_j\,m_s}{m_j + m_s} \qquad \text{for Maxwellian law}$$

$$6 K D_{js} - 2 \frac{m_j^2 m_s}{m_j + m_s} \quad \text{for elastic collisions}$$

$$\simeq 0.10 \frac{m_j^2 m_s}{m_j + m_s} \quad \text{for Coulomb law}$$

$$\simeq 0.04 \frac{m_j^2 m_s}{m_j + m_s} \quad \text{for Maxwellian law.}$$

Thus, according to the above model, we replace the set of Maxwell-Boltzmann equations [2.1] by the following set of kinetic equations:

$$\frac{\partial f_s}{\partial t} + \xi_{si} \frac{\partial f_s}{\partial x_i} + \frac{1}{m_s} \left[F_{si} + e_s \left(E_i + \frac{1}{c} \epsilon_{ijk} \xi_{sj} H_k \right) \frac{\partial f_s}{\partial \xi_{si}} \right]$$

$$= \sum_{j=1}^{N} \frac{N_j}{\sigma_{js}} (- f_s + N_s \Phi_{js}),$$

$$s = 1, 2, \cdots, N. \qquad [2.9]$$

This leads to physically meaningful transport equations. However, as in the earlier transport equations of Chapman[4] or Burnett[5], this set of transport equations also does not form a closed system of equations.

3. TRANSPORT EQUATIONS FOR NON-EQUILIBRIUM PHENOMENA

We have already pointed out that the set of transport equations obtained earlier do not form a closed set. The usual procedure to obtain transport equations, followed often, is to consider the given system in a known equilibrium state specified by the distribution functions f_{so} and a small deviations f_{sl} from this equilibrium state resulting from the preassigned non-equilibrium situations like density, velocity, and temperature gradients. Then using the perturbation techniques first order transport equations are established which yield directly the respective transport coefficients.

One of the earliest of such methods is the classical solution of Chapman-Enskog-Hilbert[6,7,8]. We may note that their solution turns out to be a series solution in terms of a parameter involving the mean free path and even the first order corrections are quite complicated and higher approximations are almost prohibitive owing to enormous mathematical complexity.

Another effective method, due to Lorentz[9] and adopted successfully by Morse *et al*[10] and Margenau[11], is to expand the velocity dependence of the distribution function in spherical harmonics in velocity space. Spherical harmonics in velocity space are eigenfunctions of Boltzmann collision operator for Lorentzian gas, the corresponding eignvalues depending upon the collision frequency. Naze[12] extended these results to more general case. Thus, the mathematical advantage of the method is off-set by the fact that physically this expansion is a series expansion in terms of collision frequency and has very limited scope.

Grad[13] developed anothor method, which is also an orthogonal function expansion in the velocity space, employing the generalized Hermite Polynomials[14]. This method has decisive advantage over the earlier methods. The distribution functions are taken in the form

$$f_s = f_{so}\left[\Sigma\, a^{(n)} H^{(n)}\right],\ s = 1,\ 2, \cdots, N, \tag{3,1}$$

where the weighting factors f_{so} are exactly the equilibrium distributions. The second and the successive terms represent the deviation from the postulated equilibrium state with coefficients $a^{(n)}$ as linear combinations of the macroscopic variables of the system. Such a procedure is evidently very much suited for the solution of transport processes under consideration. Besides these expansions may be managed to be convergent by taking the deviations to be small. Further, on truncating the series at a convenient stage, we can obtain a closed system of equations for the physical variables.

Thus, the kinetic equations [2.9] and the expansions [3.1] form the basis of the present investigation.

In the subsequent working we concentrate on a three component assembly consisting of electrons, ions and neutral particles respectively denoted by the suffixes α, β and γ, as the generalization to any number of components is straightforward. Further, in order to facilitate the orthogonal function expansion, we introduce the nondimensional distribution functions g_α of the nondimensional molecular velocities $\mathbf{v}_\alpha$ defined by

$$g_\alpha = \frac{1}{N_\alpha}\left(\frac{KT_{\alpha\alpha}}{m_\alpha}\right)^{3/2} f_\alpha(\vec{\xi}_\alpha, \mathbf{r}, t), \tag{3.2}$$

$$\mathbf{v}_\alpha = \left(\frac{m_\alpha}{KT_{\alpha\alpha}}\right)^{1/2} \vec{\xi}_\alpha. \tag{3.3}$$

Then, the general expansion can be written in the form

$$g_\alpha(\mathbf{v}_\alpha, \mathbf{r}, t) = \omega(\mathbf{v}_\alpha) \sum_{n=o}^{\infty} a_\alpha^{(n)}(\mathbf{r}, t)\, H^{(n)}(\mathbf{v}_\alpha), \tag{3.4}$$

where
$$\omega(\mathbf{v}_\alpha) = \frac{1}{(2\pi)^{3/2}} \exp\left[-\frac{1}{2}\mathbf{v}_\alpha^2\right], \tag{3.5}$$

the nondimensional Maxwellian distribution function corresponding to the postulated equilibrium situation. Because of the orthogonality property of the Hermite polynomials with kernel ω_α we have

$$a_\alpha^{(n)}(\mathbf{r}, t) = \frac{1}{X_{(n)}} \int H^{(n)}(\mathbf{v}_\alpha)\, g_\alpha(\mathbf{v}_\alpha, \mathbf{r}, t)\, d\mathbf{v}_\alpha, \tag{3.6}$$

where $$X_{(n)} = \int (\omega(\mathbf{v}_\alpha) [H^{(n)}(\mathbf{v}_\alpha)]^2 d\mathbf{v}_\alpha. \qquad [3.7]$$

Since $H^{(n)}(\mathbf{v}_\alpha)$ is merely a polynomial in velocity components, the above expression clearly shows that $a_\alpha^{(n)}(\mathbf{r}, t)$ are linear combinations of the moments of the distribution function. For convenience we have given the Hermite polynomials upto the fourth degree and the corresponding coefficients in terms of the moments of g_α in Appendix 2. Since we are dealing with Hermite polynomials in three dimensional velocity space, the number of distinct types of $a^{(n)}$ of order n can be shown to be (Appendix 3),

$$\frac{1}{72}\left[47 + 36n + 6n^2 + (-1)^n\left(9 + 16\cos\frac{n\pi}{3}\right)\right] \qquad [3.8]$$

This differs from Grad's results who, from an analogy with Cartesian tensors, inferred that there are $n!$ distinct components. This leads to a slight inaccuracy in his numerical coefficients.

Accordingly, the non-dimensional distribution functions g_α satisfy the kinetic equation

$$\frac{\partial g_\alpha}{\partial t} + \left(\frac{KT_{\alpha\alpha}}{m_\alpha}\right)^{1/2} v_{\alpha l}\frac{\partial g_\alpha}{\partial x_l} + \frac{1}{m_\alpha}\left[F_{\alpha l} + e_\alpha\left(E_l + \frac{1}{c}\left(\frac{KT_{\alpha\alpha}}{m_\alpha}\right)^{1/2} \epsilon_{ijk} v_{\alpha j} H_k\right)\right]\frac{\partial g_\alpha}{\partial \xi_{\alpha l}}$$
$$+ g_\alpha\left[\frac{\partial}{\partial t}\left\{\log N_\alpha\left(\frac{m_\alpha}{KT_{\alpha\alpha}}\right)^{3/2}\right\} + \left(\frac{KT_{\alpha\alpha}}{m_\alpha}\right)^{1/2} v_{\alpha i}\frac{\partial}{\partial x_i}\left\{\log N_\alpha \left(\frac{m_\alpha}{KT_{\alpha\alpha}}\right)^{3/2}\right\}\right]$$
$$= -\tau_\alpha g_\alpha + \sum_{\delta=\alpha}^{\gamma} \frac{N_\delta}{\sigma_{\delta\alpha}}\left(\frac{T_{\alpha\alpha}}{2\pi T_{\delta\alpha}}\right)^{3/2} \exp\left[-\frac{T_{\alpha\alpha}}{2T_{\delta\alpha}}(\mathbf{v} - \mathbf{v}_{\delta\alpha})^2\right] \qquad [3.9]$$

where $$\tau_\alpha = \frac{N_\alpha}{\sigma_{\alpha\alpha}} + \frac{N_\beta}{\sigma_{\beta\alpha}} + \frac{N_\gamma}{\sigma_{\gamma\alpha}} \qquad [3.10]$$

and $\mathbf{v}_{\alpha\alpha}$, $\mathbf{v}_{\beta\alpha}$, $\mathbf{v}_{\gamma\alpha}$ are the non-dimensional mean velocities $\mathbf{u}_{\alpha\alpha}$, $\mathbf{u}_{\beta\alpha}$, and $\mathbf{u}_{\gamma\alpha}$. Similar equations hold for β and γ components of the assembly. In [3.9] and in the subsequent calculations, we shall use the suffixes $i, j, k, l, m, n, \cdots$ for dummy summation indices and the suffixes $r, s, t, u, v, \cdots$ for fixed indices.

Substituting the expansion [3.4] in the equation [3.9] and integrating with respect to $\mathbf{v}_\alpha$ after multiplying with $H^{(n)}(\mathbf{v}_\alpha)$, we get the equation for $a_\alpha^{(n)}$. Since $\mathbf{v}_\alpha$ is explicitly present in the equation [3.9], the equation for $a_\alpha^{(n)}$ will contain $a_\alpha^{(n+1)}$. Hence a suitable cut off is essential to obtain a closed set of equations for $a_\alpha^{(n)}$. To effect this cut off, we have retained only the terms up to $a_\alpha^{(4)}$. The explicit expressions (Appendix 2) for these coefficients in terms of the moments of the distribution function lend justification for the cut off at $a_\alpha^{(4)}$. For instance, $a_\alpha^{(1)}$ contains the mean velocities and consequently for an assembly consisting of charged particles the current density term. Thus, $a_\alpha^{(1)}$ takes into account the anisotropy caused in the momentum space.

Similarly, $a_\alpha^{(2)}$ contains both the material stresses and the Maxwell stresses and the Poynting flux. $a_\alpha^{(3)}$ mainly accounts for heat flux and energy flux and $a_\alpha^{(4)}$ takes account of the interaction between material stresses and Maxwell stresses. This fact has been pointed out by Burgers[15] and Bhatnagar[16]. Hence we have included the terms up to $a_\alpha^{(4)}$ and neglected $a_\alpha^{(5)}$ and subsequent terms, since the above physical quantities govern most of the natural phenomena that occur. However, if any physical situation warrants the inclusion of some particular higher order term, the formalism is general enough to include it. Consequently we consider the truncated expansion.

$$g_\alpha = \omega_\alpha \sum_{n=0}^{4} a_\alpha^{(n)} H^{(n)}(\mathbf{v}_\alpha). \qquad [3.11]$$

This process of truncation provides a natural way of expressing the fifth and higher order moments of the distribution function in terms of moments upto fourth order in contrast to the arbitrary definitions of earlier approaches[17].

The equation for $a_\alpha^{(n)}$ is given by :

$$\begin{aligned}
X_{(n)} \frac{\partial a_\alpha^{(n)}}{\partial t} &- \left[\frac{n+3}{2} X_{(n)} a_\alpha^{(n)} + X_{(n-2)} \delta_\alpha^{(2)} a_\alpha^{(n-2)}\right] \frac{\partial}{\partial t}\left(\log \frac{m_\alpha}{K T_{\alpha\alpha}}\right) \\
&+ \left(\frac{K T_{\alpha\alpha}}{m_\alpha}\right)^{1/2} \Bigg\{ X_{(n+1)} \frac{\partial a_{\alpha i}^{(n+1)}}{\partial x_i} \\
&- \frac{n+4}{2}\left[X_{(n+1)} a_{\alpha i}^{(n+1)} + X_{(n-1)} \delta_i^{(2)} a_\alpha^{(n-1)}\right] \frac{\partial}{\partial x_i}\left(\log \frac{m_\alpha}{K T_{\alpha\alpha}}\right) \\
&+ X_{(n-1)} \delta_i^{(2)} \frac{\partial a_\alpha^{(n-1)}}{\partial x_i} - X_{(n-3)} \delta_i^{(2)} \delta^{(2)} a_\alpha^{(n-3)} \frac{\partial}{\delta x_i}\left(\log \frac{m_\alpha}{K T_{\alpha\alpha}}\right)\Bigg\} \\
&- \left(\frac{m_\alpha}{K T_{\alpha\alpha}}\right)^{1/2} \left(\frac{e_\alpha}{m_\alpha} E_i + \frac{1}{m_\alpha} F_{\alpha i}\right) X_{(n-1)} \delta_i^{(2)} a_\alpha^{(n-1)} \\
&- \frac{e_\alpha}{c m_\alpha} \epsilon_{ijk} H_k \left[X_{(n)} \delta_i^{(2)} a_{\alpha j}^{(n)} + X_{(n-2)} \delta_i^{(2)} \delta_j^{(2)} a_\alpha^{(n-2)}\right] \\
&+ \frac{\partial}{\partial t}\left\{\log N_\alpha \left(\frac{m_\alpha}{K T_{\alpha\alpha}}\right)^{3/2}\right\} X_{(n)} a_\alpha^{(n)} \\
&+ \left(\frac{K T_{\alpha\alpha}}{m_\alpha}\right)^{1/2} \frac{\partial}{\partial x_i}\left\{\log N_\alpha \left(\frac{m_\alpha}{K T_{\alpha\alpha}}\right)^{3/2}\right\} \left[X_{(n+1)} a_{\alpha i}^{(n+1)} + X_{(n-1)} \delta_i^{(2)} a_\alpha^{(n-1)}\right] \\
= &- \tau_\alpha X_{(n)} a_\alpha^{(n)} + \sum_\delta \frac{N_\delta}{\sigma_{\delta\alpha}} \left(\frac{T_{\alpha\alpha}}{2\pi T_{\delta\alpha}}\right)^{3/2} A_{\delta\alpha}^{(n)},
\end{aligned} \qquad [3.12]$$

where

$$A_{\delta\alpha}^{(n)} = \int H^{(n)}(\vec{\omega}_\alpha + \mathbf{v}_{\delta\alpha}) \exp.\left[-\frac{T_{\alpha\alpha}}{2\,T_{\delta\alpha}} \vec{\omega}_\alpha^2 \right] d\vec{\omega}_\alpha. \qquad [3.13]$$

Putting $a_\alpha^{(5)}$, $a_\alpha^{(6)}$, $\cdots$, equal to zero, we get a closed system of equations for $a_\alpha^{(n)}$, $(n = 0, 1, 2, 3, 4)$. Reverting back to the physical variables, we have the following transport equations:

$$\frac{\partial N_\alpha}{\partial t} + \frac{\partial}{\partial x_i}(N_\alpha u_{\alpha\alpha i}) = 0, \qquad [3.14]$$

$$\frac{1}{N_\alpha} \cdot \frac{\partial}{\partial t}(N_\alpha u_{\alpha\alpha r}) + \tau_\alpha u_{\alpha\alpha r} - \sum_\delta \frac{N_\delta}{\sigma_{\delta\alpha}} u_{\delta\alpha r} + \frac{1}{N_\alpha} \cdot \frac{\partial}{\partial x_i}(N_\alpha P_{\alpha ir})$$

$$- \left(\frac{e_\alpha}{m_\alpha} E_r + \frac{1}{m_\alpha} F_{\alpha r} \right) - \frac{e_\alpha}{c m_\alpha} \epsilon_{rjk} u_{\alpha\alpha j} H_k = 0, \qquad [3.15]$$

$$\frac{1}{N_\alpha} \cdot \frac{\partial}{\partial t}(N_\alpha P_{\alpha rr}) + \tau_\alpha P_{\alpha rr} - \frac{2 e_\alpha}{c m_\alpha} \epsilon_{rjk} P_{\alpha rj} H_k$$

$$= -\frac{1}{N_\alpha} \cdot \frac{\partial}{\partial x_i}(N_\alpha S_{\alpha irr}) + 2\left(\frac{e_\alpha}{m_\alpha} E_r + \frac{1}{m_\alpha} F_{\alpha r} \right) u_{\alpha\alpha r} + \sum_\delta \frac{N_\delta}{\sigma_{\delta\alpha}} \left(u_{\delta\alpha r}^2 + \frac{K T_{\delta\alpha}}{m_\alpha} \right)$$

$$\equiv A_{\alpha rr}, \qquad [3.16]$$

$$\frac{1}{N_\alpha} \cdot \frac{\partial}{\partial t}(N_\alpha P_{\alpha rs}) + \tau_\alpha P_{\alpha rs} - \frac{e_\alpha}{c m_\alpha} \left(\epsilon_{rjk} P_{\alpha js} + \epsilon_{sjk} P_{\alpha jr} \right) H_k$$

$$= -\frac{1}{N_\alpha} \cdot \frac{\partial}{\partial x_i}(N_\alpha S_{\alpha irs}) + \frac{e_\alpha}{m_\alpha} (E_r u_{\alpha\alpha s} + E_s u_{\alpha\alpha r}) + \frac{1}{m_\alpha}(F_{\alpha r} u_{\alpha\alpha s} + F_{\alpha s} u_{\alpha\alpha r})$$

$$+ \sum_\delta \frac{N_\delta}{\sigma_{\delta\alpha}} u_{\delta\alpha r} u_{\delta\alpha r} \equiv B_{\alpha rs}, \qquad [3.17]$$

$$\frac{1}{N_\alpha} \cdot \frac{\partial}{\partial t}(N_\alpha S_{\alpha rrr}) + \tau_\alpha S_{\alpha rrr} - \frac{3 e_\alpha}{c m_\alpha} \epsilon_{rjk} S_{\alpha rrj} H_k$$

$$= -\frac{1}{N_\alpha} \cdot \frac{\partial}{\partial x_i}(N_\alpha Q_{\alpha irrr}) + 3\left(\frac{e_\alpha}{m_\alpha} E_r + \frac{1}{m_\alpha} F_{\alpha r} \right) P_{\alpha rr}$$

$$+ \sum_\delta \frac{N_\delta}{\sigma_{\delta\alpha}} \left[u_{\delta\alpha r}^3 + \frac{3 K T_{\delta\alpha}}{m_\alpha} u_{\delta\alpha r} \right] \equiv A_{\alpha rrr}, \qquad [3.18]$$

$$\frac{1}{N_\alpha}\cdot\frac{\partial}{\partial t}(N_\alpha S_{\alpha rrs}) + \tau_\alpha S_{\alpha rrs} - \frac{e_\alpha}{cm_\alpha}(2\,\epsilon_{rjk}\,S_{\alpha jrs} + \epsilon_{sjk}\,S_{\alpha jrr})\,H_k$$

$$= -\frac{1}{N_\alpha}\cdot\frac{\partial}{\partial x_i}(N_\alpha Q_{\alpha irrs}) + 2\left(\frac{e_\alpha}{m_\alpha}E_r + \frac{1}{m_\alpha}F_{\alpha r}\right)P_{\alpha rs} + \left(\frac{e_\alpha}{m_\alpha}E_s + \frac{1}{m_\alpha}F_{\alpha s}\right)P_{\alpha rr}$$

$$+ \sum_\delta \frac{N_\delta}{\sigma_{\delta\alpha}}\left[u_{\delta\alpha r}^2\,u_{\delta\alpha s} + \frac{KT_{\delta\alpha}}{m_\alpha}u_{\delta\alpha s}\right] \equiv B_{\alpha rrs} \qquad [3.19]$$

$$\frac{1}{N_\alpha}\cdot\frac{\partial}{\partial t}(N_\alpha S_{\alpha rst}) + \tau_\alpha S_{\alpha rst} - \frac{e_\alpha}{cm_\alpha}(\epsilon_{rjk}\,S_{\alpha jst} + \epsilon_{sjk}\,S_{\alpha jts} + \epsilon_{tjk}\,S_{\alpha jrs})\,H_k$$

$$= -\frac{1}{N_\alpha}\cdot\frac{\partial}{\partial x_i}(N_\alpha Q_{\alpha irst}) + \left(\frac{e_\alpha}{m_\alpha}E_r + \frac{1}{m_\alpha}F_{\alpha r}\right)P_{\alpha st} + \left(\frac{e_\alpha}{m_\alpha}E_s + \frac{1}{m_\alpha}F_{\alpha s}\right)P_{\alpha tr}$$

$$+ \left(\frac{e_\alpha}{m_\alpha}E_t + \frac{1}{m_\alpha}F_{\alpha t}\right)P_{\alpha rs} + \sum_\delta \frac{N_\delta}{\tau_{\delta\alpha}}u_{\delta\alpha r}\,u_{\delta\alpha r}\,u_{\delta\alpha t} \equiv C_{\alpha rst}, \qquad [3.20]$$

$$\frac{1}{N_\alpha}\cdot\frac{\partial}{\partial t}(N_\alpha Q_{\alpha rrrr}) + \tau_\alpha Q_{\alpha rrrr} - \frac{4e_\alpha}{cm_\alpha}\epsilon_{rjk}\,Q_{\alpha jrrr}\,H_k = A_{\alpha rrrr}, \qquad [3.21]$$

$$\frac{1}{N_\alpha}\cdot\frac{\partial}{\partial t}(N_\alpha Q_{\alpha rrrs}) + \tau_\alpha Q_{\alpha rrrs} - \frac{e_\alpha}{cm_\alpha}(\epsilon_{sik}\,Q_{\alpha jrrr} + 3\,\epsilon_{rjk}\,Q_{\alpha jrrs})\,H_k = B_{\alpha rrrs}, \qquad [3.22]$$

$$\frac{1}{N_\alpha}\cdot\frac{\partial}{\partial t}(N_\alpha Q_{\alpha rrst}) + \tau_\alpha Q_{\alpha rrst} - \frac{e_\alpha}{cm_\alpha}(\epsilon_{sjk}\,Q_{\alpha rrjt} + \epsilon_{tjk}\,Q_{\alpha rrjs}$$

$$+ 2\,(\epsilon_{rsk}\,Q_{\alpha ssrt} + \epsilon_{rtk}\,Q_{\alpha ttrs})]\,H_k = C_{\alpha rrst}. \qquad [3.23]$$

$$\frac{1}{N_\alpha}\cdot\frac{\partial}{\partial t}(N_\alpha Q_{\alpha rrrs}) + \tau_\alpha Q_{\alpha rrss} - \frac{2e_\alpha}{cm_\alpha}(\epsilon_{rjk}\,Q_{\alpha ssrj} + \epsilon_{sjk}\,Q_{\alpha rrsj})\,H_k = D_{\alpha rrss}. \qquad [3.24]$$

For convenience, we have recorded the lengthy expressions $A_{\alpha rrrr}$, $B_{\alpha rrrs}$, $C_{\alpha rrst}$, and $D_{\alpha rrss}$ together with the solutions of Q_α in Appendix 4.

Equations [3.14] – [3.24] along with similar transport equations for components β, and γ with $e_\gamma \equiv 0$ govern the behaviour of the assembly.

4. Transport Coefficients Deduced from Simple Flow Problems

In order to understand the significance of the transport equations [3.14] – [3.24], we shall consider three simple flow problems.

In the case of a steady plane Couette flow with no heat flux, we replace the third order moments by the equivalent moments of lower order and obtain

$$\partial S_{\alpha 221}/\partial x_2 = -\tau_\alpha P_{\alpha 12}, \tag{4.1}$$

or

$$P_{\alpha 12} = -\left(\frac{2KT_{\alpha\alpha}}{m_\alpha \tau_\alpha}\right)\left(\frac{1}{2}\frac{\partial u_{\alpha\alpha 1}}{\partial x_2}\right). \tag{4.2}$$

Hence, we conclude that in this simple problem the coefficient of viscosity is given by

$$\mu = \frac{2KT_{\alpha\alpha}}{m_\alpha \tau_\alpha}. \tag{4.3}$$

Similarly, if we consider steady one-dimensional heat flow in a fluid at rest, after replacing the fourth order moments by their equivalent lower order moments, we get

$$\frac{\partial Q_{\alpha 1111}}{\partial x_1} = -\tau_\alpha S_{\alpha 111}, \tag{4.4}$$

or

$$q_{\alpha 1} = \tfrac{1}{2} S_{\alpha 1} = -\left(\frac{5K^2 T_{\alpha\alpha}}{m_\alpha^2 \tau_\alpha}\right)\frac{\partial T_{\alpha\alpha}}{\partial x_1}. \tag{4.5}$$

This leads to the coefficient of heat conductivity :

$$k = \frac{5K^2 T_{\alpha\alpha}}{m_\alpha^2 \tau_\alpha}. \tag{4.6}$$

Finally, considering Lorentz problem of steady, homogeneous flow of a macroscopically neutral mixture of charged particles in the presence of electric field $\mathbf{E}$ $(E, 0, 0)$, the basic momentum equations reduce to

$$\frac{N_\beta A_{\beta\alpha}}{\sigma_{\beta\alpha}}(u_{\beta\beta 1} - u_{\alpha\alpha 1}) + \frac{N_\gamma A_{\gamma\alpha}}{\sigma_{\gamma\alpha}}(u_{\gamma\gamma 1} - u_{\alpha\alpha 1}) = -e_\alpha E, \tag{4.7}$$

$$\frac{N_\gamma A_{\gamma\beta}}{\sigma_{\gamma\beta}}(u_{\gamma\gamma 1} - u_{\beta\beta 1}) + \frac{N_\alpha A_{\alpha\beta}}{\sigma_{\alpha\beta}}(u_{\alpha\alpha 1} - u_{\beta\beta 1}) = -e_\beta E, \tag{4.8}$$

and

$$\frac{N_\alpha A_{\alpha\gamma}}{\sigma_{\alpha\gamma}}(u_{\alpha\alpha 1} - u_{\gamma\gamma 1}) + \frac{N_\beta A_{\beta\gamma}}{\sigma_{\beta\gamma}}(u_{\beta\beta 1} - u_{\gamma\gamma 1}) = 0. \tag{4.9}$$

Eliminating $u_{\gamma\gamma 1}$ and substituting for the current density

$$j_1 = e_\alpha N_\alpha u_{\alpha\alpha 1} + e_\beta N_\beta u_{\beta\beta 1} \qquad [4.10]$$

and making use of the neutrality condition

$$e_\alpha N_\alpha + e_\beta N_\beta = 0, \qquad [4.11]$$

we obtain

$$j_1 = \sigma E, \qquad [4.12]$$

where

$$\sigma = \frac{1}{4}\frac{(e_\beta N_\beta - e_\alpha N_\alpha)^2}{N_\alpha N_\beta} \times$$

$$\frac{N_\alpha A_{\alpha\gamma}/\sigma_{\alpha\gamma} + N_\beta A_{\beta\gamma}/\sigma_{\beta\gamma}}{N_\alpha A_{\beta\alpha} A_{\alpha\gamma}/(\sigma_{\beta\alpha}\sigma_{\alpha\gamma}) + N_\beta A_{\gamma\beta} A_{\beta\alpha}/(\sigma_{\gamma\beta}\sigma_{\beta\gamma}) + N_\gamma A_{\gamma\alpha} A_{\alpha\beta}/(\sigma_{\gamma\alpha}\sigma_{\alpha\beta})} \qquad [4.13]$$

Thus, we can interpret σ as the direct electrical conductivity of the macroscopically neutral medium. The expressions [4.3], [4.6] and [4.13] have the same structure as those given by Chapman and Cowling[1] and Grad[13].

5. Stationary Non-equilibrium Phenomena

In this section we generalize the simple results obtained in the previous section to include all stationary phenomena such as density gradient, velocity gradients, stress variations, and heat flux vector. We shall as usual consider the system to be macroscopically neutral.

We shall first consider the expression for the current density $\mathbf{J}$. In order to find the contribution of the density and temperature gradients to the current, we shall replace the second order moments by their equivalent lower order moments. Then from the momentum equations [3.15], after straightforward calculation we get the expressions $\mathbf{J}_{11}$ for the current density parallel to the magnetic field $\mathbf{H}$ and $\mathbf{J}_{\alpha\perp}$ and $\mathbf{J}_{\beta\perp}$ for perpendicular component of the current densities due to electrons and ions:

$$\mathbf{J}_{11} = \sigma \mathbf{E}_{11} + \sigma_\alpha [\nabla_{11}(K T_{\alpha\alpha}) + K T_{\alpha\alpha} \nabla_{11} \log N_\alpha - \mathbf{F}_{\alpha 11}]$$
$$+ \sigma_\beta [\nabla_{11}(K T_{\beta\beta}) + K T_{\beta\beta} \nabla_{11} \log N_\beta - \mathbf{F}_{\beta 11}], \qquad [5.1]$$

$$\mathbf{J}_{\alpha\perp} = \frac{c(e_\alpha N_\alpha - e_\beta N_\beta)}{2H^2} \mathbf{E} \times \mathbf{H}$$

$$- \frac{c N_\alpha}{H^2} [\nabla (K T_{\alpha\alpha}) + K T_{\alpha\alpha} \nabla \log N_\alpha - \mathbf{F}_\alpha] \times \mathbf{H}$$

$$- \frac{c N_\gamma}{D_2 H^2} \cdot \frac{N_\alpha A_{\alpha\gamma}}{\sigma_{\beta\gamma}} [\nabla (K T_{\gamma\gamma}) + K T_{\gamma\gamma} \nabla \log N_\gamma - \mathbf{F}_\gamma] \times \mathbf{H}$$

$$+ \frac{2 c^2 N_\alpha N_\beta D_1}{(e_\beta N_\beta - e_\alpha N_\alpha) D_2 H^2} [\nabla (N_\alpha K T_{\alpha\alpha} + N_\beta K T_{\beta\beta} + N_\gamma K T_{\gamma\gamma})$$

$$- (N_\alpha \mathbf{F}_\alpha + N_\beta \mathbf{F}_\beta + N_\gamma \mathbf{F}_\gamma)], \qquad [5.2]$$

$$\mathbf{J}_{\beta\perp} = \frac{c (e_\beta N_\beta - e_\alpha N_\alpha)}{2 H^2} \mathbf{E} \times \mathbf{H}$$

$$- \frac{c N_\beta}{H^2} [\nabla (K T_{\beta\beta}) + K T_{\beta\beta} \nabla \log N_\beta - \mathbf{F}_\beta] \times \mathbf{H}$$

$$- \frac{c N_\gamma}{D_2 H^2} \cdot \frac{N_\beta A_{\beta\gamma}}{\sigma_{\beta\gamma}} [\nabla (K T_{\gamma\gamma}) + K T_{\gamma\gamma} \nabla \log N_\gamma - \mathbf{F}_\gamma] \times \mathbf{H}$$

$$+ \frac{2 c^2 N_\alpha N_\beta D_1}{(e_\alpha N_\alpha - e_\beta N_\beta) D_2 H^2} [\nabla (N_\alpha K T_{\alpha\alpha} + N_\beta K T_{\beta\beta} + N_\gamma K T_{\gamma\gamma})$$

$$- (N_\alpha \mathbf{F}_\alpha + N_\beta \mathbf{F}_\beta + N_\gamma \mathbf{F}_\gamma)] \qquad [5.3]$$

where

$$\sigma = \frac{(e_\beta N_\beta - e_\alpha N_\alpha)^2}{4 N_\alpha N_\beta} \cdot \frac{D_2}{D_1}, \qquad [5.4]$$

$$\sigma_\alpha = \frac{(e_\beta N_\beta - e_\alpha N_\alpha)}{2 D_1} \cdot \frac{A_{\alpha\gamma}}{\sigma_{\alpha\gamma}}, \qquad [5.5]$$

$$\sigma_\beta = \frac{-(e_\alpha N_\alpha - e_\beta N_\beta)}{2 D_1} \cdot \frac{A_{\beta\gamma}}{\sigma_{\beta\gamma}} \qquad [5.6]$$

$$D_1 = \frac{N_\alpha A_{\beta\alpha} A_{\alpha\gamma}}{\sigma_{\beta\alpha}\sigma_{\alpha\gamma}} + \frac{N_\beta A_{\gamma\beta} A_{\beta\alpha}}{\sigma_{\gamma\beta}\sigma_{\beta\alpha}} + \frac{N_\gamma A_{\alpha\gamma} A_{\gamma\beta}}{\sigma_{\alpha\gamma}\sigma_{\gamma\beta}}, \qquad [5.7]$$

and

$$D_2 = \frac{N_\alpha A_{\alpha\gamma}}{\sigma_{\alpha\gamma}} + \frac{N_\beta A_{\beta\gamma}}{\sigma_{\beta\gamma}}. \qquad [5.8]$$

These expressions clearly exhibit the effect of temperature and density gradients as well as that of other external forces on current density. From [5.1] we conclude that σ is the direct electrical conductivity along the magnetic field, while σ_α and σ_β can termed as diffusion coefficients. We note that the expression for σ is the same as [4.13] However, the dependence of current density in [5.2] and [5.3] are more complicated.

In order to deal with other physical variables we shall choose the coordinate axes in such a way that H = (0, 0, H) without loss of generality.

Solving the stress equations [3.16] and [3.17] and denoting the electron and ion gyrofrequencies by ω_α and ω_β, given by

$$\omega_\alpha = \frac{e_\alpha H}{c\, m_\alpha},\ \omega_\beta = \frac{e_\beta H}{c m_\beta}, \qquad [5.9]$$

we have

$$P_{\alpha 33} = \frac{1}{\tau_\alpha} A_{\alpha 33}, \qquad [5.10]$$

$$P_{\alpha 11} = \frac{1}{\Delta_2\, \tau_\alpha}\left[2\,\omega_\alpha \tau_\alpha B_{\alpha 12} + (\tau_\alpha^2 + 2\,\omega_\alpha^2) A_{\alpha 11} + 2\,\omega_\alpha^2 A_{\alpha 22}\right] \qquad [5.11]$$

$$P_{\alpha 22} = \frac{1}{\Delta_2 \tau_\alpha}\left[-2\,\omega_\alpha \tau_\alpha B_{\alpha 12} + 2\,\omega_\alpha^2 A_{\alpha 11} + (\tau_\alpha^2 + 2\,\omega_\alpha^2) A_{\alpha 22}\right], \qquad [5.12]$$

$$P_{\alpha 12} = (1/\Delta_2)\left[\tau_\alpha B_{12} - \omega_\alpha (A_{\alpha 11} - A_{\alpha 22})\right], \qquad [5.13]$$

$$P_{\alpha 23} = (1/\Delta_1)\left[\tau_\alpha B_{\alpha 23} - \omega_\alpha B_{\alpha 31}\right], \qquad [5.14]$$

$$P_{\alpha 31} = (1/\Delta_1)\left[\omega_\alpha B_{\alpha 23} + \tau_\alpha B_{\alpha 31}\right], \qquad [5.15]$$

where

$$\Delta_1 = \tau_\alpha^2 + \omega_\alpha^2,\ \ \Delta_2 = \tau_\alpha^2 + 4\,\omega_\alpha^2. \qquad [5.16]$$

We note that $P_{\alpha 33}$ is not affected by the magnetic field and

$$P_{\alpha 11} + P_{\alpha 22} + P_{\alpha 33} = [1/\tau_\alpha]\,(A_{\alpha 11} + A_{\alpha 22} + A_{\alpha 35})$$

is also independent of the magnetic field. Thus, the stress component along the magnetic field acting on a plane perpendicular to the magnetic field and the isotropic pressure and hence the total internal energy are unaffected by the magnetic field.

Concentrating on the dependence of the stresses on the gradients of velocity components, temperature and the density, we find that

$$P_{a33} = -\mu_{a33}^{(0)} e_{a33} - \mu_{a33}^{(1)} \nabla T_{aa} - \mu_{a33}^{(2)} \nabla_3 N_a, \qquad [5.17]$$

where

$$\mu_{a33}^{(0)} = \frac{2KT_{aa}}{m_a \tau_a}, \qquad [5.18]$$

$$\mu_{a33}^{(2)} = \frac{K}{m_a \tau_a} \{ u_{aa1} \quad u_{aa2} \quad u_{aa3} \}. \qquad [5.19]$$

$$\mu_{a33}^{(2)} = \frac{u_{aa3}}{N_a} \mu_{a33}^{(0)}. \qquad [5.20]$$

We can call these μ matrices the generalized viscosity matrix. $\mu_{a33}^{(0)}$ has the same form as [4.3] and is unaffected by the magnetic field.

Similarly, we have

$$\begin{pmatrix} P_{a23} \\ P_{a31} \end{pmatrix} = -\mu_{a3}^{(0)} \begin{pmatrix} e_{a23} \\ e_{a31} \end{pmatrix} - \mu_{a3}^{(1)} \nabla T_{aa} - \mu_{a3}^{(2)} \nabla N_a. \qquad [5.21]$$

where

$$\mu_{a3}^{(0)} = \frac{2KT_{aa}}{m_a \Delta_1} \begin{pmatrix} \tau_a & -\omega_a \\ \omega_a & \tau_a \end{pmatrix}, \qquad [5.22]$$

$$\mu_{a3}^{(1)} = \frac{K}{m_a \Delta_1} \begin{pmatrix} -\omega_a u_{aa3} & \tau_a u_{aa3} & \tau_a u_{aa2} - \omega_a u_{aa1} \\ \tau_a u_{aa3} & \omega_a u_{aa3} & \omega_a u_{aa2} + \tau_a u_{aa1} \end{pmatrix}, \qquad [5.23]$$

$$\mu_{a3}^{(2)} = (T_{aa}/N_a)\, \mu_{a3}^{(1)}. \qquad [5.24]$$

Regarding the viscosity matrices associated with the stresses along the magnetic field acting on planes containing the magnetic field, we note that asymmetry is caused by the magnetic field. Moreover, the primary viscosity coefficient is reduced by the magnetic field since the diagonal terms of $\mu_{a3}^{(0)}$ can be written as

$$\frac{2KT_{aa}}{m_a \tau_a}\left(1 - \frac{\omega_a^2}{\tau_a^2 + \omega_a^2}\right). \qquad [5.25]$$

The stresses due to temperature gradient and and density gradient are proportional to each other.

Finally, writing

$$\begin{bmatrix} P_{\alpha 11} \\ P_{\alpha 12} \\ P_{\alpha 22} \end{bmatrix} = -\mu_\alpha^{(0)} \begin{bmatrix} e_{\alpha 11} \\ e_{\alpha 12} \\ e_{\alpha 22} \end{bmatrix} - \mu_\alpha^{(1)} \nabla T_{\alpha\alpha} - \mu_\alpha^{(2)} \nabla N_\alpha \qquad [5.26]$$

we have $$\mu_\alpha^{(0)} = \frac{2KT_{\alpha\alpha}}{m_\alpha \tau_\alpha \Delta_2} \begin{bmatrix} \tau_\alpha^2 + 2\omega_\alpha^2 & 2\omega_\alpha\tau_\alpha & 2\omega_\alpha^2 \\ -\omega_\alpha \tau & \tau_\alpha^2 & -\omega_\alpha\tau_\alpha \\ 2\omega_\alpha^2 & -2\omega_\alpha\tau_\alpha & \tau_\alpha^2 + 2\omega_\alpha^2 \end{bmatrix}, \qquad [5.27]$$

$$\mu_\alpha^{(1)} = \frac{K}{m_\alpha \gamma_\alpha \Delta_2} \begin{bmatrix} 2\omega_\alpha\tau_\alpha u_{\alpha\alpha 2} + (3\tau_\alpha^2 + 8\omega_\alpha^2) u_{\alpha\alpha 1} & -2\omega_\alpha\tau_\alpha u_{\alpha\alpha 1} + (\tau_\alpha^2 + 8\omega_\alpha^2) u_{\alpha\alpha 2} & \Delta_2 u_{\alpha\alpha 3} \\ \tau_\alpha^2 u_{\alpha\alpha 2} - 2\omega_\alpha\tau_\alpha u_{\alpha\alpha 1} & 2\omega_\alpha\tau_\alpha u_{\alpha\alpha 2} + \tau_\alpha^2 u_{\alpha\alpha 1} & 0 \\ +2\omega_\alpha\tau_\alpha u_{\alpha\alpha 1} + (\tau_\alpha^2 + 8\omega_\alpha^2) u_{\alpha\alpha 2} & -2\omega_\alpha\tau_\alpha u_{dd1} + (3\tau_\alpha^2 + 8\omega_\alpha^2) u_{\alpha\alpha 2} & \Delta_2 u_{\alpha\alpha 3} \end{bmatrix} \qquad [5.28]$$

$$\mu_\alpha^{(2)} = \frac{T_{\alpha\alpha}}{N} \mu_\alpha^{(1)}, \qquad [5.29]$$

Thus, considering the viscosity matrix corresponding to the stresses perpendicular to the magnetic field, we conclude that the magnetic field introduces anisotropy. Considering the diagonal terms we see that for $P_{\alpha 11}$ and $P_{\alpha 22}$ the corresponding viscosity coefficient is

$$\frac{2KT_{\alpha\alpha}}{m_\alpha \tau_\alpha}\left(1 - \frac{2\omega_\alpha^2}{\tau_\alpha^2 + 4\omega_\alpha^2}\right), \qquad [5.30]$$

while corresponding to $P_{\alpha 12}$ we have

$$\frac{2KT_{\alpha\alpha}}{m_\alpha \tau_\alpha}\left(1 - \frac{4\omega_\alpha^2}{\tau_\alpha^2 + 4\omega_\alpha^2}\right). \qquad [5.31]$$

Once again the stresses due to density gradient are proportional to stresses due to temperature gradient.

Comparing the expressions [5.18], [5.25], [5.30] and [5.31], we conclude that the magnetic field introduces intense anisotropy. Further, the coefficients of viscosity in the plane perpendicular to the magnetic field are less than the the viscosity in a plane containing the magnetic field.

From the other terms we can deduce the effect of electric field, external forces, collisional transfers, etc., on the stresses.

Solving the equations [3.18] – [3.20] for heat flux tensor, we have

$$S_{\alpha 333} = \frac{1}{\tau_\alpha} A_{\alpha 333}, \qquad [5.31]$$

$$S_{\alpha 331} = (1/\Delta_1)\left[\tau_\alpha B_{\alpha 331} + \omega_\alpha B_{\alpha 332}\right], \qquad [5.33]$$

$$S_{\alpha 332} = (1/\Delta_1)\left[\tau_\alpha B_{\alpha 331} - \omega_\alpha B_{\alpha 332}\right], \qquad [5.34]$$

$$S_{\alpha 123} = (1/\Delta_2)\left[\tau_\alpha C_{\alpha 123} + \omega_\alpha (B_{\alpha 223} - B_{\alpha 113})\right], \qquad [5.35]$$

$$S_{\alpha 113} = (1/\tau_\alpha \Delta_2)\left[2\omega_\alpha \tau_\alpha C_{\alpha 123} + (\tau_\alpha^2 + 2\omega_\alpha^2) B_{\alpha 113} + 2\omega_\alpha^2 B_{\alpha 223}\right], \qquad [5.36]$$

$$S_{\alpha 223} = (1/\tau_\alpha \Delta_2)\left[-2\omega \tau_\alpha C_{\alpha 123} + 2\omega_\alpha^2 B_{\alpha 113} + (\tau_\alpha^2 + 2\omega_\alpha^2) B_{\alpha 223}\right], \qquad [5.37]$$

$$\begin{aligned} S_{\alpha 112} = (1/\Delta_1 \Delta_3) \big[& (\tau_\alpha^2 + 3\omega_\alpha^2)(\tau_\alpha B_{\alpha 112} - \omega_\alpha A_{\alpha 111}) \\ & + 2\omega_\alpha \tau_\alpha (\tau_\alpha B_{\alpha 221} + \omega_\alpha A_{\alpha 222}) \big], \end{aligned} \qquad [5.38]$$

$$\begin{aligned} S_{\alpha 221} = (1/\Delta_1 \Delta_3) \big[& 2\omega_\alpha \tau_\alpha (\tau_\alpha A_{\alpha 111} - \omega_\alpha B_{\alpha 112}) \\ & + (\tau_\alpha^2 + 3\omega_\alpha^2)(\tau_\alpha B_{\alpha 221} + \omega_\alpha A_{\alpha 222}) \big], \end{aligned} \qquad [5.39]$$

$$\begin{aligned} S_{\alpha 111} = (1/\Delta_1 \Delta_3) \big[& 3\omega_\alpha (\tau_\alpha^2 + 3\omega_\alpha^2) B_{\alpha 112} + \tau_\alpha (\tau_\alpha^2 + 7\omega_\alpha^2) A_{\alpha 111} \\ & + 6\omega_\alpha^2 (\tau_\alpha B_{\alpha 221} + \omega_\alpha A_{\alpha 222}) \big], \end{aligned} \qquad [5.40]$$

$$\begin{aligned} S_{\alpha 222} = (1/\Delta_1 \Delta_3) \big[& 6\omega_\alpha^2 (\tau_\alpha B_{\alpha 112} - \omega_\alpha A_{111}) - 3\omega_\alpha (\tau_\alpha^2 + 3\omega_\alpha^2) B_{\alpha 221} \\ & + \tau_\alpha (\tau_\alpha^2 + 7\omega_\alpha^2) A_{\alpha 222} \big], \end{aligned} \qquad [5.41]$$

where

$$\Delta_3 = \tau_\alpha^2 + 9\omega_\alpha^2. \qquad [5.42]$$

Concentrating mainly on the temperature and density gradients of the heat flux vector

$$S_{\alpha r} = \tfrac{1}{2} S_{\alpha r i i}, \qquad [5.43]$$

we find that we can write

$$S_{\alpha 3} = -\frac{5K^2 T_{\alpha\alpha}}{m_\alpha^2 \tau_\alpha} \cdot \frac{\partial T_{\alpha\alpha}}{\partial x_3} - \frac{5K^2 T_{\alpha\alpha}^2}{2m_\alpha^2 N_\alpha \tau_\alpha} \cdot \frac{\partial N_\alpha}{\partial x_3}, \qquad [5.44]$$

and

$$\begin{pmatrix} S_{\alpha 1} \\ S_{\alpha 2} \end{pmatrix} = -K_\alpha^{(0)} \begin{Bmatrix} (\partial T_{\alpha\alpha}/\partial x_1) \\ (\partial T_{\alpha\alpha}/\partial x_2) \end{Bmatrix} - K_\alpha^{(1)} \begin{Bmatrix} (\partial N_\alpha/\partial x_1) \\ (\partial N_\alpha/\partial x_2) \end{Bmatrix}, \qquad [5.45]$$

where

$$K_\alpha^{(0)} = \left(\frac{5K^2 T_{\alpha\alpha}}{m_\alpha^2 \Delta_1}\right) \begin{Bmatrix} \tau_\alpha & \omega_\alpha \\ -\omega_\alpha & \tau_\alpha \end{Bmatrix} \qquad [5.46]$$

and

$$K_\alpha^{(1)} = \frac{T_{\alpha\alpha}}{2N_\alpha} K_\alpha^{(0)}. \qquad [5.47]$$

From [5.44] we conclude that the heat conductivity $k_{\alpha 3}^{(0)}$ coefficient in the direction of magnetic field is unaltered and is the same as the expression [4.5]. But the heat conductivity tensor transverse to magnetic field is modified due to the presence of the magnetic field. The collisional contribution to heat conductivity is given by

$$\frac{5\,K^2\,T_{\alpha\alpha}}{m_\alpha^2\,\tau_\alpha}\left(1-\frac{\omega_\alpha^2}{\omega_\alpha^2+\tau_\alpha^2}\right), \tag{5.48}$$

which decreases as the magnetic field increases. The heat conductivity tensor $k_\alpha^{(1)}$ arising out of the density gradient is directly proportional to the heat conductivity tensor $k_\alpha^{(0)}$ due to temperature gradient both along and perpendicular to the magnetic field as seen from [5.44] and [4.47].

We emphasize here that the generalized stress and thermal transport coefficients have in their denominators factors of the type $\tau_\alpha^2+\omega_\alpha^2$, $\tau_\alpha^2+4\,\omega_\alpha^2$, $\tau_\alpha^2+9\,\omega_\alpha^2$. Proceeding in a similar manner we have established the corresponding coefficients for the fourth order moments (Appendix IV). The only remark of interest about the fourth order moments is that the denominators have $\tau_\alpha^2+16\,\omega_\alpha^2$ as an additional factor Generalizing we can state as follows: For any n-th order moment, if all the suffixes are along the magnetic field direction then it contains $(1/\tau_\alpha)$ only; if $(n-1)$ suffixes are in the direction of magnetic field it has a factor $(1/\Delta_1)$ and generally if $(n-r)$ suffixes are in the direction of the magnetic field, it has a factor $(1/\Delta_r)$ where

$$\Delta_r=\tau_\alpha^2+r^2\,\omega_\alpha^2. \tag{5.49}$$

This point is of great importance while establishing the relaxation times in §6, as it leads to an explanation of Gross gaps[18].

6. Relaxation Problem

Finally, we shall consider the relaxation problem. Following Bhatnagar[19] we shall suppose that the physical quantities depend only on time and there is no external force field excepting the magnetic field. Further, for simplicity, we shall suppose that the axes are so chosen that $\mathbf{H}=(0,0,H)$. From the zeroth order continuity equations, we conclude that

$$N_\alpha=\text{constant}. \tag{6.1}$$

The relaxation times p for the velocity components are governed by the equations

$$\left(p+\frac{N_\beta\,A_{\beta\alpha}\,m_\alpha}{\sigma_{\beta\alpha}}+\frac{N_\gamma\,A_{\gamma\alpha}\,m_\alpha}{\sigma_{\gamma\alpha}}\right)u_{\alpha\alpha r}-\omega_\alpha\,\epsilon_{rj3}\,u_{\alpha\alpha j}$$
$$=\frac{N_\beta\,A_{\beta\alpha}\,m_\alpha}{\sigma_{\beta\alpha}}\,u_{\beta\beta r}+\frac{N_\gamma\,A_{\gamma\alpha}\,m_\alpha}{\sigma_{\gamma\alpha}}\,u_{\gamma\gamma r}, \tag{6.2}$$

Neglecting the square of the collisional terms, for the components along the magnetic field, we have

$$p = -\tfrac{1}{2}\left[\frac{N_\alpha A_{\alpha\beta} m_\beta}{\sigma_{\alpha\beta}} + \frac{N_\alpha A_{\alpha\gamma} m_\gamma}{\sigma_{\alpha\gamma}} + \frac{N_\beta A_{\beta\gamma} m_\gamma}{\sigma_{\beta\gamma}} + \frac{N_\beta A_{\beta\alpha} m_\alpha}{\sigma_{\beta\alpha}} + \frac{N_\gamma A_{\gamma\alpha} m_\alpha}{\sigma_{\gamma\beta}} + \frac{N_\gamma A_{\gamma\beta} m_\beta}{\sigma_{\gamma\beta}}\right] \quad [6.3]$$

which is unaffected by the magnetic field.

For the transverse components, we have three modes given by

$$p_1 = -\left(\frac{N_\alpha A_{\alpha\gamma} m_\alpha}{\sigma_{\alpha\gamma}} + \frac{N_\beta A_{\beta\gamma} m_\gamma}{\sigma_{\beta\gamma}}\right),$$

$$p_2 = \pm i\omega_\beta - \left(\frac{N_\gamma A_{\gamma\beta} m_\beta}{\sigma_{\gamma\beta}} + \frac{N_\alpha A_{\alpha\beta} m_\beta}{\sigma_{\alpha\beta}}\right)$$

$$p_3 = \pm i\omega_\alpha - \left(\frac{N_\beta A_{\beta\alpha} m_\alpha}{\sigma_{\beta\alpha}} + \frac{N_\gamma A_{\gamma\alpha} m_\alpha}{\sigma_{\gamma\alpha}}\right). \quad [6.4]$$

We note here that the self-collisions do not contribute to these relaxation times.

Procceding in the similar fashion, the relaxation time for the temperature is given by

$$p = -\tfrac{1}{2}\left(\frac{\mathrm{N}_\alpha B_{\alpha\beta}}{\sigma_{\alpha\beta}} + \frac{\mathrm{N}_\alpha B_{\alpha\gamma}}{\sigma_{\alpha\gamma}} + \frac{\mathrm{N}_\beta B_{\beta\alpha}}{\sigma_{\beta\alpha}} + \frac{\mathrm{N}_\beta \mathrm{B}_{\beta\gamma}}{\sigma_{\beta\gamma}} + \frac{\mathrm{N}_\gamma B_{\gamma\alpha}}{\sigma_{\gamma\alpha}} + \frac{\mathrm{N}_\gamma B_{\gamma\beta}}{\sigma_{\gamma\beta}}\right), \quad [6.5]$$

while the off-diagonal terms of the pressure tensor have the relaxation times

$$p_1 = -\tau_\alpha, \; p_2 = \pm i\omega_\alpha - \tau_\alpha, \; p_3 = \pm 2\, i\omega_\alpha - \tau_\alpha. \quad [6.6]$$

The relaxation times of the third order moments are determined by equations identical in form to [5.32] – [5.41] with τ_α replaced by $\tau_\alpha + p$. Hence proceeding as in the calculation of transport coefficients we obtain the following relaxation times.

$$p = -\tau_\alpha \text{ for } S_{\alpha 333},$$

$$p = \pm i\omega_\alpha - \tau_\alpha \text{ for } S_{\alpha 331} \text{ and } S_{\alpha 332},$$

$$p_1 = -\tau_\alpha,\ p_2 = \pm 2\, i\,\omega_\alpha - \tau_\alpha \text{ for } S_{\alpha 123},\ S_{\alpha 113}, \text{ and } S_{\alpha 223},$$
$$p_1 = \pm i\,\omega_\alpha - \tau_\alpha,\ p_2 = \pm 3\, i\,\omega_\alpha - \tau_\alpha \text{ for } S_{\alpha 111},\ S_{\alpha 112},\ S_{\alpha 122}, \text{ and } S_{\alpha 222}. \qquad [6.7]$$

Finally, we shall record the relaxation times for the fourth order moments.

$$p = -\tau_\alpha \text{ for } Q_{\alpha 3333},$$
$$p = \pm i\,\omega_\alpha - \tau_\alpha\ Q_{\alpha 3331} \text{ and } Q_{\alpha 3332},$$
$$p_1 = -\tau_\alpha,\ p_2 = \pm 2\, i\,\omega_\alpha - \tau_\alpha \text{ for } Q_{\alpha 3311},\ Q_{\alpha 3312}, \text{ and } Q_{\alpha 3322},$$
$$p_1 = \pm i\,\omega_\alpha - \tau_\alpha,\ p_2 = \pm 3\, i\,\omega_\alpha - \tau_\alpha \text{ for } Q_{\alpha 3111},\ Q_{\alpha 3112},\ Q_{\alpha 3122}, \text{ and } Q_{\alpha 3222},$$
$$p_1 = -\tau_\alpha,\ p_2 = \pm 2\, i\,\omega_\alpha - \tau_\alpha,\ p_3 = \pm 4\, i\,\omega_\alpha - \tau_\alpha, \text{ for } Q_{\alpha 1111},\ Q_{\alpha 1112},\ Q_{\alpha 1122},\ Q_{\alpha 1222}, \text{ and } Q_{\alpha 2222}. \qquad [6.8]$$

From the above expressions, the following plausible explanation of Gross gaps can be given. If we take the dependence of the physical quantities as e^{ipt} instead of e^{pt} then the amplitudes of the moments of the distribution function will be obtained by putting $\tau_\alpha + ip$ instead of τ_α in § 5, so that the complex frequency of oscillation will be determined by expressions of the type $p = \pm n\,\omega_\alpha + i\,\tau_\alpha$. Correspondingly, the amplitudes of nth order moments having no suffix along the magnetic field will contain in their denominators an expression of the type $(p - i\,\tau_\alpha)^2 - n^2\omega_\alpha^2$. Thus, in the absence of collisions $(\tau_\alpha = 0)$ these amplitudes will become infinity when $p = n\,\omega_\alpha$ or $n\,\omega_\beta$. It appears, therefore, that the wave is dissipated away on account of making nth and higher order moments infinity. This result was obtained by Gross for one component assembly and extended to multicomponent assembly by Bhatnagar and Devanathan on the basis of kinetic equation. Here we have obtained the physical mechanism responsible for the decay of such waves in detail.

Appendix 1.

Average number of Collision

The average number of collisions between β and α components is defined by

$$N_\beta\, N_\alpha / \sigma_{\beta\alpha} = \iiint f_\beta\, f_\alpha\, g_{\beta\alpha}\, b\, db\, dt\, d\vec{\xi}_\beta\, d\vec{\xi}_\alpha \qquad [\text{AI.1}]$$

The exact expressions can be evaluated by using the expansion for f_α and f_β. But as can be easily shown from the kinetic equations [2.9] that for small deviations from equilibrium position only the equilibrium value of $N_\beta\, N_\alpha/\sigma_{\beta\alpha}$ is necessary. Correspondingly we choose the equilibrium Maxwellian distributions

$$f_\alpha = N_\alpha \left(\frac{m_\alpha}{2\,\pi\, K\, T_{\alpha\alpha}} \right)^{3/2} \exp. \left[-\frac{m_\alpha}{2\, K\, T_{\alpha\alpha}} \vec{\xi}_\alpha^{\,2} \right], \qquad [\text{AI.2}]$$

with similar expression for f_β. Then [AI.1] reduces to

$$\frac{1}{\sigma_{\beta\alpha}} = \left(\frac{m_\alpha m_\beta}{K^2 T_{\alpha\alpha} T_{\beta\beta}}\right)^{3/2} \iiint \exp.\left[-\frac{m_\alpha}{2KT_{\alpha\alpha}}\vec{\xi}_\alpha^2 - \frac{m_\beta}{2KT_{\beta\beta}}\vec{\xi}_\beta^2\right] \times$$

$$g_{\beta\alpha}\, b\, db\, d\epsilon\, d\vec{\xi}_\alpha\, d\vec{\xi}_\beta \qquad \text{[AI.3]}$$

The integration over the impact parameter b can be caried out exactly on the same lines as Chapman and Cowling[1], for the force law $|\mathbf{F}_{\beta\alpha}| = K_{\beta\alpha}/r^s$. In order to ensure convergence we have to introduce cut-off. In terms of the non-dimensional impact parameter v_0 we have

$$\frac{1}{\sigma_{\beta\alpha}} = \frac{1}{2K}\left(\frac{m_\alpha m_\beta}{T_{\alpha\alpha} T_{\beta\beta}}\right)^{3/2}\left[\frac{(m_\alpha m_\beta)K_{\beta\alpha}}{m_\alpha m_\beta}\right]^{\frac{2}{s-1}} (v_{ou}^2 - v_{ol}^2) \times$$

$$\iint \exp.\left[-\frac{m_\alpha}{2KT_{\alpha\alpha}}\vec{\xi}_\alpha^2 - \frac{m_\beta}{2KT_{\beta\beta}}\vec{\xi}_\beta^2\right] g_{\beta\alpha}^{\frac{s-5}{s-1}}\, d\vec{\xi}_\alpha\, d\vec{\xi}_\beta \qquad \text{[AI.4]}$$

By changing the variables to

$$\mathbf{g}_{\beta\alpha} = \vec{\xi}_\beta - \vec{\xi}_\alpha \text{ and } \mathbf{V}_{\beta\alpha} = \vec{\xi}_\beta + \vec{\xi}_\alpha$$

and carrying out the elementary integrations we get finally

$$\frac{1}{\sigma_{\beta\alpha}} = \frac{1}{(2\pi)^{\frac{1}{2}}}\left[\frac{T_{\alpha\alpha}}{m_\alpha} + \frac{T_{\beta\beta}}{m_\beta}\right]^{\frac{2s-4}{s-1}}\left[\frac{(m_\alpha + m_\beta)K_{\beta\alpha}}{m_\alpha m_\beta}\right]^{\frac{2}{s-1}} (v_{ou}^2 - v_{ol}^2) \times F(s) \qquad \text{[AI.5]}$$

where $$F(s) = \Gamma\left(\frac{2s-4}{s-1}\right) \text{ if } s \neq 2 \text{ and } F(s) = \int_{x^*}^{\infty} \frac{e^{-x}}{x}\,dx \text{ if } s = 2,$$

x^* being suitable lower cut off.

APPENDIX II

The Hermite Polynomials of first four degrees and the first four moments

$$H_\alpha^{(0)} = 1,\ H_{\alpha i}^{(1)} = v_{\alpha i},\ H_{\alpha ij}^{(2)} = v_{\alpha j} v_{\alpha j} - \delta_{ij},$$

$$H_{\alpha ijk}^{(3)} = v_{\alpha l} v_{\alpha j} v_{\alpha k} - (v_{\alpha i}\delta_{jk} + v_{\alpha j}\delta_{ik} + v_{\alpha k}\delta_{ij}),$$

$$H_{\alpha ijkl}^{(4)} = v_{\alpha i} v_{\alpha j} v_{\alpha k} v_{\alpha l} - (v_{\alpha i} v_{\alpha j}\delta_{kl} + v_{\alpha i} v_{\alpha k}\delta_{jl} + v_{\alpha i} v_{\alpha l}\delta_{jk} + v_{\alpha j} v_{\alpha k}\delta_{il} + v_{\alpha j} v_{\alpha l}\delta_{ik}$$
$$+ v_{\alpha k} v_{\alpha l}\delta_{ij}) + (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

$$a^{(0)}_{\alpha} = 1,$$

$$a^{(1)}_{\alpha i} = \left(\frac{m_\alpha}{K T_{\alpha\alpha}}\right)^{1/2} u_{\alpha\alpha i},$$

$$a^{(2)}_{\alpha ii} = \frac{1}{2}\left[\left(\frac{m_\alpha}{K T_{\alpha\alpha}}\right) P_{\alpha ii} - 1\right],$$

$$a^{(2)}_{\alpha ij} = \left(\frac{m_\alpha}{K T_{\alpha\alpha}}\right) P_{\alpha ij},$$

$$a^{(3)}_{\alpha iii} = \frac{1}{6}\left[\left(\frac{m_\alpha}{K T_{\alpha\alpha}}\right)^{3/2} S_{\alpha iii} - 3\left(\frac{m_\alpha}{K T_{\alpha\alpha}}\right)^{1/2} u_{\alpha\alpha i}\right],$$

$$a^{(3)}_{\alpha iij} = \frac{1}{2}\left[\left(\frac{m_\alpha}{K T_{\alpha\alpha}}\right)^{3/2} S_{\alpha iij} - \left(\frac{m_\alpha}{K T_{\alpha\alpha}}\right)^{1/2} u_{\alpha\alpha j}\right],$$

$$a^{(3)}_{\alpha ijk} = \left(\frac{m_\alpha}{K T_{\alpha\alpha}}\right)^{3/2} S_{\alpha ijk},$$

$$a^{(4)}_{\alpha iiii} = \frac{1}{24}\left[\left(\frac{m_\alpha}{K T_{\alpha\alpha}}\right)^{2} Q_{\alpha iiii} - 6\left(\frac{m_\alpha}{K T_{\alpha\alpha}}\right) P_{\alpha ii} + 3\right],$$

$$a^{(4)}_{\alpha iiij} = \frac{1}{6}\left[\left(\frac{m_\alpha}{K T_{\alpha\alpha}}\right)^{2} Q_{\alpha iiij} - 3\left(\frac{m_\alpha}{K T_{\alpha\alpha}}\right) P_{\alpha ij}\right],$$

$$a^{(4)}_{\alpha iijk} = \frac{1}{2}\left[\left(\frac{m_\alpha}{K T_{\alpha\alpha}}\right)^{2} Q_{\alpha iijk} - \left(\frac{m_\alpha}{K T_{\alpha\alpha}}\right) P_{\alpha jk}\right],$$

$$a^{(4)}_{\alpha iijj} = \frac{1}{4}\left[\left(\frac{m_\alpha}{K T_{\alpha\alpha}}\right)^{2} Q_{\alpha iijj} - \left(\frac{m_\alpha}{K T_{\alpha\alpha}}\right)(P_{\alpha ii} + P_{\alpha jj}) + 1\right].$$

APPENDIX 3.

Number of Distinct Components of $a^{(n)}$

Since we are dealing with generalized Hermite polynomials of the velocity in three dimensional Euclidean space, the number of distinct components of $a^{(n)}$ is nothing but the number of partitions of n each partition containing terms not exceeding three in number. Denoting this by $P_3(n)$, we have from elementary number theory[20]

$$P_3(n) = \frac{1}{n!}\left\{\frac{d^n}{dx^n}\left[\frac{1}{(1-x)(1-x^2)(1-x^3)}\right]\right\}_{x=0}$$

$$= \frac{1}{72}\left[47 + 36\,n + 6\,n^2 + (-1)^n\left(9 + 16\cos\frac{n\pi}{3}\right)\right],$$

by actual differentiation.

APPENDIX 3

Solution of fourth order moments

$$\tau_\alpha\, Q_{\alpha 3333} = A_{\alpha 3333},$$

$$\Delta_1\, Q_{\alpha 3331} = \tau_\alpha\, B_{\alpha 3331} + \omega_\alpha\, B_{\alpha 3332},$$

$$\Delta_1\, Q_{\alpha 3332} = \tau_\alpha\, B_{\alpha 3332} - \omega_\alpha\, B_{\alpha 3331},$$

$$\tau_\alpha\, \Delta_2\, Q_{\alpha 3311} = (\tau_\alpha^2 + 2\,\omega_\alpha^2)\, D_{\alpha 3311} + 2\,\omega_\alpha\,(\tau_\alpha\, C_{\alpha 3312} + \omega_\alpha\, D_{\alpha 2233}),$$

$$\Delta_2\, Q_{\alpha 3312} = -\,\omega_\alpha\, D_{\alpha 3311} + \tau_\alpha\, C_{\alpha 3312} + \omega_\alpha\, D_{\alpha 2233},$$

$$\tau_\alpha\, \Delta_2\, Q_{\alpha 3322} = 2\,\omega_\alpha\,(\omega_\alpha\, D_{\alpha 3311} - \tau_\alpha\, C_{\alpha 3312}) + (\tau_\alpha^2 + 2\,\omega_\alpha^2)\, D_{\alpha 2233},$$

$$\begin{aligned}\Delta_1\,\Delta_3\, Q_{\alpha 2223} = {} & \tau_\alpha\,(\tau_\alpha^2 + 7\,\omega_\alpha^2)\, B_{\alpha 2223} - 3\,\omega_\alpha\,(\tau_\alpha^2 + 3\,\omega_\alpha^2)\, C_{\alpha 2231} \\ & + 6\,\omega_\alpha^2\,(\tau_\alpha\, C_{\alpha 1123} - \omega_\alpha\, B_{\alpha 1113}),\end{aligned}$$

$$\begin{aligned}\Delta_1\,\Delta_3\, Q_{\alpha 2213} = {} & (\tau_\alpha^2 + 3\,\omega_\alpha^2)\,(\omega_\alpha\, B_{\alpha 2223} + \tau_\alpha\, C_{\alpha 2231}) \\ & - 2\,\omega_\alpha\,\tau_\alpha\,(\tau_\alpha\, C_{\alpha 1123} - \omega_\alpha\, B_{\alpha 1113}),\end{aligned}$$

$$\begin{aligned}\Delta_1\,\Delta_3\, Q_{\alpha 2113} = {} & 2\,\omega_\alpha\,\tau_\alpha\,(\omega_\alpha\, B_{\alpha 2223} + \tau_\alpha\, C_{\alpha 2231}) \\ & + (\tau_\alpha^2 + 3\,\omega_\alpha^2)\,(\tau_\alpha\, C_{\alpha 1123} - \omega_\alpha\, B_{\alpha 1113}),\end{aligned}$$

$$\begin{aligned}\Delta_1\,\Delta_3\, Q_{\alpha 1113} = {} & 6\,\omega_\alpha^2\,(\omega_\alpha\, B_{\alpha 2223} + \tau_\alpha\, C_{\alpha 2231}) \\ & + 3\,\omega_\alpha\,(\tau_\alpha^2 + 3\,\omega_\alpha^2)\, C_{\alpha 1123} + \tau_\alpha\,(\tau_\alpha^2 + 7\,\omega_\alpha^2)\, B_{\alpha 1113},\end{aligned}$$

$$\begin{aligned}\tau_\alpha\,\Delta_2\,\Delta_4\, Q_{\alpha 1111} = {} & (\tau_\alpha^4 + 16\,\tau_\alpha^2\,\omega_\alpha^2 + 24\,\omega_\alpha^4)\, A_{\alpha 1111} \\ & + 4\,\omega_\alpha\,\tau_\alpha\,(\tau_\alpha^2 + 10\,\omega_\alpha^2)\, B_{\alpha 1112} + 12\,\omega_\alpha^2\,(\tau_\alpha^2 + 4\,\omega_\alpha^2)\, D_{\alpha 1122} \\ & + 24\,\omega_\alpha^3\,(\tau_\alpha\, B_{\alpha 2221} + \omega_\alpha\, A_{\alpha 2222}),\end{aligned}$$

$$\begin{aligned}\tau_\alpha\,\Delta_2\,\Delta_4\, Q_{\alpha 1112} = {} & \tau_\alpha\,(\tau_\alpha^2 + 10\,\omega_\alpha^2)\,(\tau_\alpha\, B_{\alpha 1112} - \omega_\alpha\, A_{\alpha 1111}) \\ & + ?\,\omega_\alpha\,\tau_\alpha\,(\tau_\alpha^2 + 4\,\omega_\alpha^2)\, D_{\alpha 1122} + 6\,\omega_\alpha^2\,\tau_\alpha\,(\tau_\alpha\, B_{\alpha 2221} + \omega_\alpha\, A_{\alpha 2222}),\end{aligned}$$

$$\begin{aligned}\tau_\alpha\,\Delta_2\,\Delta_4\, Q_{\alpha 1122} = {} & 2\,\omega_\alpha\,(\tau_\alpha^2 + \omega_\alpha^2)\,(\omega_\alpha\, A_{\alpha 1111} - \tau_\alpha\, B_{\alpha 1112}) \\ & + (\tau_\alpha^2 + 4\,\omega_\alpha^2)^2\, D_{\alpha 1122} + 2\,\omega_\alpha\,(\tau_\alpha^2 + 4\,\omega_\alpha^2)\,(\tau_\alpha\, B_{\alpha 2221} + \omega_\alpha\, A_{\alpha 2222}),\end{aligned}$$

$$\Delta_2 \Delta_4 Q_{\alpha 2221} = 6\,\omega_\alpha^2 (\tau_\alpha B_{\alpha 1112} - \omega_\alpha A_{\alpha 1111}) - 3\,\omega_\alpha (\tau_\alpha^2 + 4\,\omega_\alpha^2) D_{\alpha 1122} + \tau_\alpha (\tau_\alpha^2 + 10\,\omega_\alpha^2)(\tau_\alpha B_{\alpha 2221} + \omega_\alpha A_{\alpha 2222}),$$

$$\tau_\alpha \Delta_2 \Delta_4 Q_{\alpha 2222} = 24\,\omega_\alpha^3 (\omega_\alpha A_{\alpha 1111} - \tau_\alpha B_{\alpha 1112}) + 12\,\omega_\alpha^2 (\tau_\alpha^2 + 4\,\omega_\alpha^2) D_{\alpha 1122} - 4\,\omega_\alpha \tau_\alpha (\tau_\alpha^2 + 10\,\omega_\alpha^2) B_{\alpha 2221} + (\tau_\alpha^4 + 16\,\omega_\alpha^2 \tau_\alpha^2 + 24\,\omega_\alpha^4) A_{\alpha 2222},$$

where

$$A_{\alpha rrrr} = -\left(\frac{K T_{\alpha\alpha}}{m_\alpha}\right)\left[4\,\frac{\partial S_{\alpha rrr}}{\partial x_r} + 4\,S_{\alpha rrr}\frac{\partial}{\partial x_r}\left(\log \frac{K T_{\alpha\alpha}}{m_\alpha}\right) + 6\,S_{\alpha rrr}\frac{\partial}{\partial x_r}(\log N_\alpha) + \frac{\partial S_{\alpha irr}}{\partial x_i} + S_{\alpha irr}\,\frac{\partial}{\partial x_i}\left(\log N_\alpha \frac{K T_{\alpha\alpha}}{m_\alpha}\right)\right]$$
$$+ \left(\frac{K T_{\alpha\alpha}}{m_\alpha}\right)^2 \left[12\,\frac{\partial u_{\alpha\alpha r}}{\partial x_r} + 12\,u_{\alpha\alpha r}\,\frac{\partial}{\partial x_r}\left\{\log \mathrm{N}_\alpha \left(\frac{K T_{\alpha\alpha}}{m_\alpha}\right)^2\right\} + 6\,u_{\alpha\alpha i}\frac{\partial}{\partial x_i}\left(\log \frac{m_\alpha}{K T_{\alpha\alpha}}\right)\right]$$
$$+ 4\left(\frac{e_\alpha}{m_\alpha} E_r + \frac{1}{m_\alpha} F_{\alpha r}\right) S_{\alpha rrr}$$
$$+ \sum_\delta \frac{N_\delta}{\sigma_{\delta\alpha}}\left[u_{\delta\alpha r}^4 + \frac{6 K T_{\delta\alpha}}{m_\alpha}\,u_{\delta\alpha r}^2 + 3\left(\frac{K T_{\delta\alpha}}{m_\alpha}\right)^2\right],$$

$$B_{\alpha rrrs} = -\left(\frac{K T_{\alpha\alpha}}{m_\alpha}\right)\left[\frac{\partial S_{\alpha rrr}}{\partial x_s} + S_{\alpha rrr}\,\frac{\partial}{\partial x_s}\left(\log \mathrm{N}_\alpha \frac{K T_{\alpha\alpha}}{m_\alpha}\right) + 3\,\frac{\partial S_{\alpha rrs}}{\partial x_r} + 3\,S_{\alpha rrs}\frac{\partial}{\partial x_r}\left(\log \mathrm{N}_\alpha \frac{K T_{\alpha\alpha}}{m_\alpha}\right) + 3\,\frac{\partial S_{\alpha irs}}{\partial x_i} + 3\,S_{\alpha irs}\frac{\partial}{\partial x_i}\left(\log \mathrm{N}_\alpha \frac{K T_{\alpha\alpha}}{m_\alpha}\right)\right]$$
$$+ 3\left(\frac{K T_{\alpha\alpha}}{m_\alpha}\right)^2 \left[\frac{\partial u_{\alpha\alpha r}}{\partial x_s} + 2 u_{\alpha\alpha r}\frac{\partial}{\partial x_s}\left\{\log \mathrm{N}_\alpha \left(\frac{K T_{\alpha\alpha}}{m_\alpha}\right)^2\right\} + \frac{\partial u_{\alpha\alpha s}}{\partial x_r} + u_{\alpha\alpha s}\,\frac{\partial}{\partial x_r}\left\{\log \mathrm{N}_\alpha \left(\frac{K T_{\alpha\alpha}}{m_\alpha}\right)^2\right\}\right]$$

$$+\left(\frac{e_\alpha}{m_\alpha}E_s+\frac{1}{m_\alpha}F_{\alpha s}\right)S_{\alpha rrr}+3\left(\frac{e_\alpha}{m_\alpha}E_r+\frac{1}{m_\alpha}F_{\alpha r}\right)\dot{S}_{\alpha rrs}$$

$$+\sum_\delta\frac{N'_\delta}{\sigma_{\delta\alpha}}\left[u^3_{\delta\alpha r}\,u_{\delta\alpha s}+\frac{3KT_{\delta\alpha}}{m_\alpha}\,u_{\delta\alpha r}\,u_{\delta\alpha s}\right],$$

$$C_{\alpha rrst}=-\left(\frac{KT_{\alpha\alpha}}{m_\alpha}\right)\left[2\frac{\partial S_{\alpha rst}}{\partial x_r}+2S_{\alpha rst}\frac{\partial}{\partial x_r}\left(\log N_\alpha\frac{KT_{\alpha\alpha}}{m_\alpha}\right)\right.$$

$$+\frac{\partial S_{\alpha rrt}}{\partial x_s}+\ddot{S}_{\alpha rrt}\frac{\partial}{\partial x_s}\left(\log N_\alpha\frac{KT_{\alpha\alpha}}{m_\alpha}\right)$$

$$+\frac{\partial S_{\alpha rrs}}{\partial x_t}+S_{\alpha rrs}\frac{\partial}{\partial x_t}\left(\log N_\alpha\frac{KT_{\alpha\alpha}}{m_\alpha}\right)$$

$$\left.+\frac{\partial S_{\alpha ist}}{\partial x_i}+S_{\alpha ist}\frac{\partial}{\partial x_i}\left(\log N_\alpha\frac{KT_{\alpha\alpha}}{m_\alpha}\right)\right]$$

$$+\left(\frac{KT_{\alpha\alpha}}{m_\alpha}\right)^2\left[\frac{\partial u_{\alpha\alpha t}}{\partial x_s}+u_{\alpha\alpha t}\frac{\partial}{\partial x_s}\left\{\log N_\alpha\left(\frac{KT_{\alpha\alpha}}{m_\alpha}\right)^2\right\}\right.$$

$$\left.+\frac{\partial u_{\alpha\alpha s}}{\partial x_t}+u_{\alpha\alpha s}\frac{\partial}{\partial x_t}\left\{\log N_\alpha\left(\frac{KT_{\alpha\alpha}}{m_\alpha}\right)^2\right\}\right]$$

$$+2\left(\frac{e_\alpha}{m_\alpha}E_r+\frac{1}{m_\alpha}F_{\alpha r}\right)S_{\alpha rst}+\left(\frac{e_\alpha}{m_\alpha}E_s+\frac{1}{m_\alpha}F_{\alpha s}\right)S_{\alpha rrt}$$

$$+\left(\frac{e_\alpha}{m_\alpha}E_t+\frac{1}{m_\alpha}F_{\alpha t}\right)S_{\alpha rrt}$$

$$+\sum_\delta\frac{N_\delta}{\bar{\sigma}_{\delta\alpha}}\left[u^2_{\delta\alpha r}\,u_{\delta\alpha s}\,u_{\delta\alpha t}+\frac{KT_{\delta\alpha}}{m_\alpha}\,u_{\delta\alpha s}\,u_{\delta\alpha t}\right]$$

$$D_{\alpha rrss}=-\left(\frac{KT_{\alpha\alpha}}{m_\alpha}\right)\left[2\frac{\partial S_{\alpha ssr}}{\partial x_r}+2S_{\alpha ssr}\frac{\partial}{\partial x_r}\left(\log N_\alpha\frac{KT_{\alpha\alpha}}{m_\alpha}\right)\right.$$

$$+2\frac{\partial S_{\alpha rrs}}{\partial x_s}+2S_{\alpha rrs}\frac{\partial}{\partial x_s}\left(\log N_\alpha\frac{KT_{\alpha\alpha}}{m_\alpha}\right)$$

$$+ \frac{\partial S_{\alpha irr}}{\partial x_i} + S_{\alpha irr} \frac{\partial}{\partial x_i}\left(\log N_\alpha \frac{K T_{\alpha\alpha}}{m_\alpha} \right)$$

$$+ \frac{\partial S_{\alpha iss}}{\partial x_i} + S_{\alpha iss} \frac{\partial}{\partial x_i}\left(\log N_\alpha \frac{K T_{\alpha\alpha}}{m_\alpha} \right)\Bigg]$$

$$+ \left(\frac{K T_{\alpha\alpha}}{m_\alpha} \right)^2 \left[2 \frac{\partial u_{\alpha\alpha r}}{\partial x_r} + 2\, u_{\alpha\alpha r} \frac{\partial}{\partial x_r} \left\{ \log N_\alpha \left(\frac{K T_{\alpha\alpha}}{m_\alpha} \right)^2 \right\} \right.$$

$$+ 2 \frac{\partial u_{\alpha\alpha s}}{\partial x_s} + 2 u_{\alpha\alpha s} \frac{\partial}{\partial x_s} \left\{ \log N_\alpha \left(\frac{K T_{\alpha\alpha}}{m_\alpha} \right)^2 \right\}$$

$$\left. + 2\, u_{\alpha\alpha i} \frac{\partial}{\partial x_i} \left(\log \frac{K T_{\alpha\alpha}}{m_\alpha} \right) \right]$$

$$+ 2\left(\frac{e_\alpha}{m_\alpha} E_r + \frac{1}{m_\alpha} F_{\alpha r} \right) S_{\alpha ssr} + 2\left(\frac{e_\alpha}{m_\alpha} E_s + \frac{1}{m_\alpha} F_{\alpha s} \right) S_{\alpha rrs}$$

$$+ \sum_\delta \frac{N_\delta}{\tau_{\delta\alpha}} \left[u^2_{\delta\alpha r}\ u^2_{\delta\alpha s} + \frac{K T_{\delta\alpha}}{m_\alpha} \left(u^2_{\delta\alpha r} + u^2_{\delta\alpha s} \right) + \left(\frac{K T_{\delta\alpha}}{m_\alpha} \right)^2 \right]$$

References

1. S. Chapman and T. G. Cowling .. "The Mathematical Theory of nonuniform gases", Cambridge, (1962)
2. Bhatnagar, P. L., Gross, E. P. and Krook, M. *Phys. Rev.*, 1954, **94**, 511.
3. Gross, E. P. and Krook, M. .. *Ibid*, 1956, **102**, 593.
4. Bhatnagar, P. L. and Devanathan, C. *Proc. Nat. Inst. Sci. (India)*, 1963, **29A**, 474.
5. Chapman, S .. .. *Phil. Trans. Roy. Soc.*, 1912, **A211**, 433.
6. Burnett, D. .. .. *Proc. Lond. Math. Soc.*, 1935, **40**, 382.
7. Chapman, S. .. .. *Phil. Trans. Roy. Soc.*, 1916, **A216**, 279; (1917) **A217**, 115.
8. Ensog, D. .. .. Svensk. Vet. *Akad. Arkiv. f. Mat. Ast. Och Fys.*, 1921, **16**, 1.
9. Hilbert, D. .. .. *Math. Ann.*, 1912, **72**, 562.
10. Lorentz, H. A. .. .. *Wien. Ber.*, 1887, **95**, 115.
11. Morse, P.M., Allis, W.P., Lamar, E.S. *Phys Rev.*, 1933, **48**, 412.

12. Margenau, H. .. .. *Phys. Rev.*, 1945, **73**, 297.

13. Naze, J. .. .. "Sur 1, equation de Boltzmann", Paris, (1962).

14. Grad, H. .. .. *Comm. Pure and Appl. Math.*, 1949, **2**, 331.

15. ——— .. .. *Ibid*, 1949; **2**, 325.

16. Burgers, J.M. .. .. "Symposium on Plasma Dynamies", Addison—Wesly, (1958).

17. Bhatnagar, P. L. .. .. "Lecture on Boltzmann equation", Summer School in Theoretical Physics, Simla, (1962).

18. Morrey, C. B. .. .. *Comm. Pure Appl. Maths.*, 1955, **8**, 279.

19. Gross, E. P. .. .. *Phys. Rev.*, 1951, **82**, 232.

20. Bhatnagar, P. L. .. .. *Z. Astrophysics*, 1962, **54**, 234.

21. Hardy, G. H. and Wright, E. M. .. "An Introduction to the Theory of Numbers" Oxford, (1960).