WEISSENBERG AND MERRINGTON EFFECTS IN NON-NEWTONIAN FLUIDS

By (Miss) S. L. Rathna and P. L. Bhatnagar
(Department of Applied Mathematics, Indian Institute of Science, Bangalore 12)

(Received on December 1, 1962)

Abstract

Weissenberg in 1946, first demonstrated an interesting phenomenon in the hydrodynamic behaviour of certain highly viscous liquids. In the experimental set-up, the liquid is sheared in a gap between two rotating cylinders both of which move at different but with steady rotational speeds, the liquid is drawn inwards against the action of centrifugal force and upwards against the force of gravity, the whole arrangement forming a sort of " Centripetal pump ". Weissenberg attributes this effect to the elasticity of the liquid. In the present investigation, the flows of general Reiner-Rivlin fluids and Rivlin-Ericksen fluids with constant coeffieients between two coaxial rotating cylinders are studied. Explicit expressions for the velocity distribution, stress components, pressure distribution, the equation to the free surface at the inner and outer cylinders have been obtained for various kinds of non-Newtonian fluids. The effect of visco-elasticity and cross-viscocity are represented graphically. From the trend of the theoretical investigation, we conclude that either cross-viscocity alone or visco-elasticity alone or both will modify the shape of the free surface in the annulus prescribed by centrifugal force and the force of gravity.

One interesting feature to be noticed in the flow of non-Newtonian fluids through tubes, is the tendency of the fluid stream to swell at the exit section of the pipe, a phenomenon called Merrington effect after its discoverer. In the present investigation, the Merrington effect has been discussed in the case of general Reiner-Rivlin fluids and Rivlin-Ericksen fluids with constant coefficients. A method is proposed to determine the variable coefficients of viscosity and cross-viscosity by measuring the excess pressure along the length of the pipe for various flow rates per sec. per unit cross-sectional area of the tube.

In Part A of the present paper, we study the flow of (i) Newtonian fluid with the constitutive equation

$$T = -pI + \Phi_1 E, \qquad [0.1]$$

where T is the stress tensor, E is the rate of deformation tensor and Φ_1 is the coefficient of viscosity, (ii) Power-law fluid defined by

$$T = -pI + \beta_0 (\mathrm{II}_E)^{(s-1)/2} E, \qquad [0.2]$$

where β_0, s are constants and $s < 1$ corresponds to pseudo-plasticity, while $s > 1$ corresponds to dilatancy, (iii) General Reiner-Rivlin[1] fluid defined by

$$T = -pI + \Phi_1 E + \Phi_3 E^2, \qquad [0.3]$$

where Φ_1 and Φ_3, the coefficients of viscosity and cross-viscosity respectively are the functions of the invariants of E, (iv) Reiner-Rivlin fluid defined by (0.3), where Φ_1 and Φ_3 are constants, (v) Rivlin-Ericksen fiuid[2] with constant coefficients defined by

$$T = -pI + \Phi_1 E + \Phi_2 D + \Phi_3 E^2, \qquad [0.4]$$

where D is the aceleration gradient tensor defined by

$$D_{ij} = \frac{\partial a_i}{\partial x_j} + \frac{\partial a_j}{\partial x_i} + 2\frac{\partial u_\mu}{\partial x_i} \cdot \frac{\partial u_\mu}{\partial x_j}$$

and

$$a_i = \frac{\partial u_i}{\partial t} + u_\mu \frac{\partial u_i}{\partial x_\mu},$$

Φ_1, Φ_2, Φ_3, the coefficients of viscosity, visco-elasticity and cross-viscosity are constants, between two co-axial cylinders in the following cases:

(α) when the inner and outer cylinders are rotating with constant angular velocities Ω_1 and Ω_2 in the same sense,

(β) when the inner cylinder is at rest and the outer cylinder is rotating with constant angular velocity Ω_2, and

(γ) when the outer cylinder is at rest and the inner cylinder is rotating with constant angular velocity Ω_1.

The present investigation has been undertaken to study the cause of the Weissenberg effect *i.e.* the tendency of the fluid in the annulus to flow inwards against the action of centrifugal force and upwards against the force of gravity. This was necessary as there is a certain amount of uncertainty about the cause of the effect. Weissenberg[3] himself attributed it to elasticity of the fluid, whereas the investigations of Reiner[4], Rivlin[5] and Serrin[6] suggested that cross-viscosity is capable of explaining the phenomenon.

In Part B, we study the flow of the above liquids through a circular pipe under constant pressure gradient. This investigation has been taken up to study the cause of the Merrington effect *i.e.* a tendency of the fluid stream to swell at the exit section of the pipe. Merrington[7] attributed the phenomenon to elasticity of the fluid, while the investigations on the Reiner-Rivlin fluid by Reiner[4] and others[6,8,9] showed that the effect can arise due to the cross-viscosity present in the fluid.

PART A

1. *Flow of a general Reiner-Rivlin fluid between two co-axial rotating cylinders*: Consider the steady flow of a general Reiner-Rivlin fluid with variable co-efficients of viscosity Φ_1 and cross-viscosity Φ_3 between two infinite coaxial cylinders of radii a and b ($b > a$), when the inner cylinder is rotating with constant angular velocity Ω_1 and the outer cylinder with Ω_2 about the

axis of the cylinders. We shall assume that the upper free surface of the fluid is exposed to a constant atmospheric pressure p_0, and each point of the fluid moves in a plane perpendicular to the common axis of the cylinders and in a circular orbit about the axis.

If Ω is the angular velocity of the fluid distant R from the axis, then the no slip condition gives

$$\left.\begin{aligned} \Omega &= \Omega_1 \quad \text{when } R = a, \\ \Omega &= \Omega_2 \quad \text{when } R = b. \end{aligned}\right\} \qquad [1.1]$$

Taking the cylindrical polar coordinates (R, θ, Z), Z-axis of which coincides with the axis of the cylinders, the velocity components U, V, W in the directions of R, θ, Z are of the form

$$U = 0, \quad V = R\,\Omega\,(R), \quad W = 0. \qquad [1.2]$$

The constitutive equation for this class of fluids is given by [0.3]. For definiteness, we take the following expressions for Φ_1 and Φ_3 in terms of the second invariant $R^2\,\Omega'^2$:

$$\left.\begin{aligned} \Phi_1 &= \beta_0\,(R^2\,\Omega'^2)^{\lambda}, \\ \Phi_3 &= \sum_{i=0}^{\infty} \gamma_i\,(R^2\,\Omega'^2)^i, \end{aligned}\right\} \qquad [1.3]$$

where β_0, λ, γ_i $(i = 0, 1, \cdots)$ are constants.

We find it convenient to work through non-dimensional quantities by introducing the following substitutions :

$$R = hr, \quad Z = hz, \quad P = p_0\,p, \quad \Omega = \bar{\Omega}\,\omega\,(r), \qquad [1.4]$$

where r, z, p and ω are dimensionless quantities, $h = b - a$, $\bar{\Omega} = \Omega_1 + \Omega_2$. Further we introduce

$$m = \frac{\Omega_2}{\Omega_1} \text{ and } l = \frac{b}{a} \qquad [1.5]$$

for the sake of mathematical simplicity.

Thus, the momentum equations of the flow are reduced to

$$r\,\omega^2 = \frac{1}{A}\cdot\frac{\partial p}{\partial r} - \frac{\partial}{\partial r}\,[s_0\,r^2\,\omega'^2 + s_1\,(r^2\,\omega'^2)^2 + s_2\,(r^2\,\omega'^2)^3 + \cdots], \qquad [1.6]$$

$$0 = \frac{\partial}{\partial r}\,[r^3\,\omega'\,(r^2\,\omega'^2)^{\lambda}], \qquad [1.7]$$

$$0 = \frac{1}{A}\cdot\frac{\partial p}{\partial z} + B, \qquad [1.8]$$

where

$$A = \frac{\rho \bar{\Omega}^2 h^2}{p_0}, \quad B = \frac{g}{h \bar{\Omega}^2},$$

g being the force of gravity in the axial direction,

$$s_0 = \frac{\gamma_0}{\rho h^2}, \quad s_1 = \frac{\gamma_1 \bar{\Omega}^2}{\rho h^2}, \quad s_2 = \frac{\gamma_2 \bar{\Omega}^4}{\rho h^2}. \qquad [1.9]$$

The angular velocity ω from [1.7] satisfying the boundary conditions [1.1] is given by

$$\omega = \frac{c}{n} r^{-n} - d, \qquad [1.10]$$

where

$$n = \frac{2}{2\lambda + 1},$$

$$c = n \frac{1-m}{1+m} \cdot \frac{l^n}{(l^n - 1)(l-1)^n} \quad \text{and} \quad d = \frac{m l^n - 1}{(1+m)(l^n - 1)}. \qquad [1.11]$$

Thus, the solution of the equations [1.6] to [1.8] is written in the form

$$\omega = \frac{1}{(1+m)(l^n - 1)} \left[(1-m) \left(\frac{l}{l-1} \right)^n r^{-n} + m l^n - 1 \right] \qquad [1.12]$$

and

$$p = A \left[\frac{c^2 r^{2(1-n)}}{2n^2(1-n)} + \frac{d^2 r^2}{2} + \frac{2 c d r^{2-n}}{n(2-n)} - Bz + \sum_{i=1}^{\infty} s_{i-1} r^{2i} (\omega')^{2i} \right] + k_0,$$

for $n \neq 1$, $n \neq 2$*. [1.13]

The stress components for the state of motion considered are given by

$$T_{rr} = T_{\theta\theta} = Bz - \left[\frac{c^2 r^{2(1-n)}}{2n^2(1-n)} + \frac{d^2 r^2}{2} + \frac{2 c d r^{2-n}}{n(2-n)} \right] - k_0, \qquad [1.14]$$

$$T_{r\theta} = c^{2/n} r^{-2}, \quad T_{rz} = T_{z\theta} = 0, \qquad [1.15]$$

$$T_{zz} = Bz - \sum_{i=1}^{\infty} s_{i-1} (c^2 r^{-2n})^i - \left[\frac{c^2 r^{2(1-n)}}{2n^2(1-n)} + \frac{d^2 r^2}{2} + \frac{2 c d r^{2-n}}{n(2-n)} \right] - k_0, \qquad [1.16]$$

where c and d are given by [1.11].

---

* We shall consider the cases when $n=1$ and $n=2$ in the next section.

The flow is taking place between two rotating cylinders with the upper surface of the fluid exposed to the atmosphere. The equation to the free surface is

$$T_{zz} = \text{constant} = -p_0.$$

In view of this condition, the equation to the free surface is written in the form

$$\bar{z} = \frac{1}{B}\left[\frac{c^2 r^{2(1-n)}}{2n^2(1-n)} + \frac{d^2 r^2}{2} + \frac{2cd\,r^{2-n}}{n(2-n)} + \sum_{i=1}^{\infty} s_{i-1}\,(c^2 r^{-2n})^i\right], \qquad [1.17]$$

where

$$\bar{z} = z + \frac{p_0}{\rho g h} + k_0. \qquad [1.18]$$

2. *Special cases :*

Case when $n = 1$ $(\lambda = \frac{1}{2})$. In this case, the pressure distribution and the stress-components are given by the following expressions :

$$p = A\left[c_1^2 \ln r + \frac{d_1^2 r^2}{2} + 2c_1 d_1 r + \sum_{i=1}^{\infty} s_{i-1}\,(c_1^2 r^{-2})^i - Bz\right] + k_0, \qquad [2.1]$$

$$T_{rr} = T_{\theta\theta} = Bz - \left[c_1^2 \ln r + \frac{d_1^2 r^2}{2} + 2c_1 d_1 r\right] - k_0, \qquad [2.2]$$

$$T_{r\theta} = c_1^2 r^{-2}, \quad T_{\theta z} = T_{rz} = 0, \qquad [2.3]$$

$$T_{zz} = Bz - \sum_{i=1}^{\infty} s_{i-1}\,(c_1^2 r^{-2})^i - \left[c_1^2 \ln r + \frac{d_1^2 r^2}{2} + 2c_1 d_1 r\right] - k_0, \qquad [2.4]$$

and the equation to the free surface is

$$\bar{z} = \frac{1}{B}\left[c_1^2 \ln r + \frac{d_1^2 r^2}{2} + 2c_1 d_1 r + \sum_{i=1}^{\infty} s_{i-1}\,(c_1^2 r^{-2})^i\right], \qquad [2.5]$$

where

$$c_1 = \frac{1-m}{1+m}\cdot\frac{l}{(l-1)^2} \text{ and } d_1 = \frac{ml-1}{(1+m)(l-1)}. \qquad [2.6]$$

Case when $n = 2$ $(\lambda = 0)$. The particular case when $\lambda = 0$ and $s_i \neq 0$ represents the state of motion of a Reiner-Rivlin fluid with constant coefficient of viscosity β_0 and variable coefficient of cross-viscosity Φ_3.

We obtain the expressions for the pressure and stress components as

$$p = A\left[\frac{d_2^2 r^2}{2} + 2c_2 d_2 \ln r - \frac{c_2^2}{2r^2} - Bz + \sum_{i=1}^{\infty} s_{i-1}\,(4c_2^2 r^{-4})^i\right] + k_0, \qquad [2.7]$$

$$T_{rr} - T_{\theta\theta} = Bz - \left[\frac{d_2^2 r^2}{2} + 2 c_2 d_2 \ln r - \frac{c_2^2}{2 r^2}\right] - k_0, \qquad [2.8]$$

$$T_{r\theta} = c_2 r^{-2}, \; T_{\theta z} = T_{rz} = 0, \qquad [2.9]$$

$$T_{zz} = Bz - \left[\frac{d_2^2 r^2}{2} + 2 c_2 d_2 \ln r - \frac{c_2^2}{2 r^2}\right] - \sum_{i=1}^{\infty} s_{i-1} (4 c_2^2 r^{-4})^i - k_0. \qquad [2.10]$$

The equation to the free surface is

$$z = \frac{1}{B}\left[\frac{d_2^2 r^2}{2} + 2 c_2 d_2 \ln r - \frac{c_2^2}{2r^2} + \sum_{i=1}^{\infty} s_{i-1} (4 c_2^2 r^{-4})^i\right], \qquad [2.11]$$

where

$$c_2 = \frac{1-m}{1+m} \cdot \frac{l^2}{(l^2-1)(l-1)^2} \text{ and } d_2 = \frac{ml^2-1}{(m+1)(l^2-1)}. \qquad [2.12]$$

By putting $s_i = 0$ $(i = 0, 1, \cdots)$ in the above expressions, we obtain the stress components and pressure distribution corresponding to a Newtonian fluid with β_0 as the coefficient of viscosity.

3. *Motion of a Rivlin-Ericksen fluid with constant co-efficients between two rotating cylinders*: The constitutive equation describing the behaviour of a visco-elastic fluid in Rivlin-Ericksen theory with quadratic dependence is given by [0.4].

For the state of motion described by [1.2], the governing equations of motion in terms of non-dimensional quantities [1.4] and [1.5] are reduced to

$$r \omega^2 = \frac{1}{A} \cdot \frac{\partial p}{\partial r} - S\frac{\partial}{\partial r}(r^2 \omega'^2) - K(6 r \omega'^2 + 4 r^2 \omega' \omega''), \qquad [3.1]$$

$$0 = \left[\frac{\partial}{\partial r}(r \omega') + 2 \omega'\right], \qquad [3.2]$$

$$0 = \frac{1}{A} \cdot \frac{\partial p}{\partial z} + B, \qquad [3.3]$$

where

$$K = \frac{\beta}{h^2} \text{ and } S = \frac{\gamma}{h^2}. \qquad [3.4]$$

β and γ are kinematic coefficients of visco-elasticity and cross-viscosity respectively.

Thus the solution of the equations [3.1]—[3.3] satisfying the boundary conditions [1.1] are

$$\omega = \frac{1}{(1+m)(l^2-1)}\left[(1-m)\left(\frac{l}{l-1}\right)^2 r^{-2} + ml^2 - 1\right], \qquad [3.5]$$

$$p = A\left[\frac{d_2^2 r^2}{2} + 2\, c_2 d_2 \ln r - \frac{c_2^2}{2 r^2} - Bz + \frac{6\, K c_2^2}{r^4} + \frac{4\, S c_2^2}{r^4}\right] + k_0, \qquad [3.6]$$

where c_2 and d_2 are given by [2.12].

We notice that the angular velocity distribution is unaffected by viscosity, cross-viscosity and visco-elasticity.

We obtain the expressions for the stress components for this class of fluids as

$$T_{rr} = Bz - \left[\frac{d_2^2 r^2}{2} + 2\, c_2 d_2 \ln r - \frac{c_2^2}{2r^2}\right] + \frac{2\, K c_2^2}{r^4} - k_0, \qquad [3.7]$$

$$T_{\theta\theta} = Bz - \left[\frac{d_2^2 r^2}{2} + 2\, c_2 d_2 \ln r - \frac{c_2^2}{2r^2}\right] - k_0, \qquad [3.8]$$

$$T_{r\theta} = c_2 r^{-2}, \quad T_{rz} = T_{\theta z} = 0, \qquad [3.9]$$

$$T_{zz} = Bz - \left[\frac{d_2^2 r^2}{2} + 2\, c_2 d_2 \ln r - \frac{c_2^2}{2r^2}\right] - \frac{6\, K c_2^2}{r^4} - \frac{4\, S c_2^2}{r^4} - k_0. \qquad [3.10]$$

We find that $T_{rr} = T_{\theta\theta}$ for a visco-inelastic fluid; further, T_{rr} and T_{zz} are modified by the coefficient of visco-elasticity, while the effect of cross-viscosity is exhibited only in T_{zz}.

The equation to the free surface is

$$z = \frac{1}{B}\left[\frac{d_2^2 r^2}{2} + 2\, c_2 d_2 \ln r - \frac{c_2^2}{2r^2} + \frac{6\, K c_2^2}{r^4} + \frac{4\, S c_2^2}{r^4}\right]. \qquad [3.11]$$

4. *Discussion of the shape of the free surface in the annulus.* In the preceding sections, we have shown how the stress components, pressure and the equation to the free surface are modified by variable coefficients of viscosity and cross-viscosity, and constant coefficients of viscosity, cross-viscosity and the visco-elasticity. We shall now study the shape of the free surface in the annulus for the fluids (i) to (v) under the cases (α), (β) and (γ).

As we have not fixed the initial height of the liquid in the annulus before the rotation, we cannot make a definite statement about the rise and fall of the free surface of the liquid along the cylinders. We determine the initial height of the liquid z_0 by equating the volume of the liquid occupied in the annulus before and after rotation

i.e.,
$$\pi \rho \left(\frac{b^2 - a^2}{h^2} \right) z_0 = \int_{a/h}^{b/h} 2 \pi \rho r z \, dr , \qquad [4.1]$$

where z is given by the equation to the free surface after rotation. It is clear that z_0 is different in various cases we have considered.

(i) *Newtonian Fluids.* In this case, the equation to the free surfuce is

$$\bar{z} = \frac{1}{B} \left[\frac{d_2^2 r^2}{2} + 2 c_2 d_2 \ln r - \frac{c_2^2}{2r^2} \right] . \qquad [4.2]$$

We find that the slope of the free surface in the case (β) varies from zero at the inner cylinder to a positive value at the outer cylinder, while in the case (γ) it varies from a positive value at the inner cylinder to zero at the outer cylinder.

In the numerical work we have taken

$$B = 0.1 , \; m = 10 , \; l = 5. \qquad [4.3]$$

Curves (a) in Figure I, represent the shape of the free surface for the cases (α) and (β), while Figure II furnishes the shape of the free surface in the case (γ). Taking the initial height of the liquid $\bar{z}_0 = [z_0 + (p_0/\rho g h) + k_0]$ into consideration, we find that the liquid tends to climb along the outer cylinder and fall along the inner cylinder in all the cases (α), (β) and (γ). Further, the climbing of the liquid along the outer cylinder is more pronounced in the case (β) than in (α) or (γ). This type of behaviour is due to the centrifugal force produced by the rotation of the fluid. Hence, we conclude that Newtonian fluids do not show the Weissenberg effect as is well-known.

(ii) *Power-Law Fluids.* In the numerical work we have taken $n = 1$ for dilatant fluids $(n < 2)$, and $n = 4$ for pseudo-plastic fluids $(n > 2)$.

We can easily check that the slopes of the free surface at the inner and outer cylinders for a power-law fluid are the same as the slopes of the free surface at the inner and outer cylinders respectively for a Newtonian fluid, irrespective of the value of n that we consider.

Curves (b) and (c) in the Figure I, furnish the shape of the free surface in the annulus for dilatant and pseudo-plastic fluids. We find that both pseudo-plastic and dilatant fluids tend to fall slightly along the inner cylinder and tend to climb along the outer cylinder as in the case of Newtonian fluids. Thus, the power-law fluids, though belong to the class of non-Newtonian fluids, do not exhibit the Weissenberg effect.

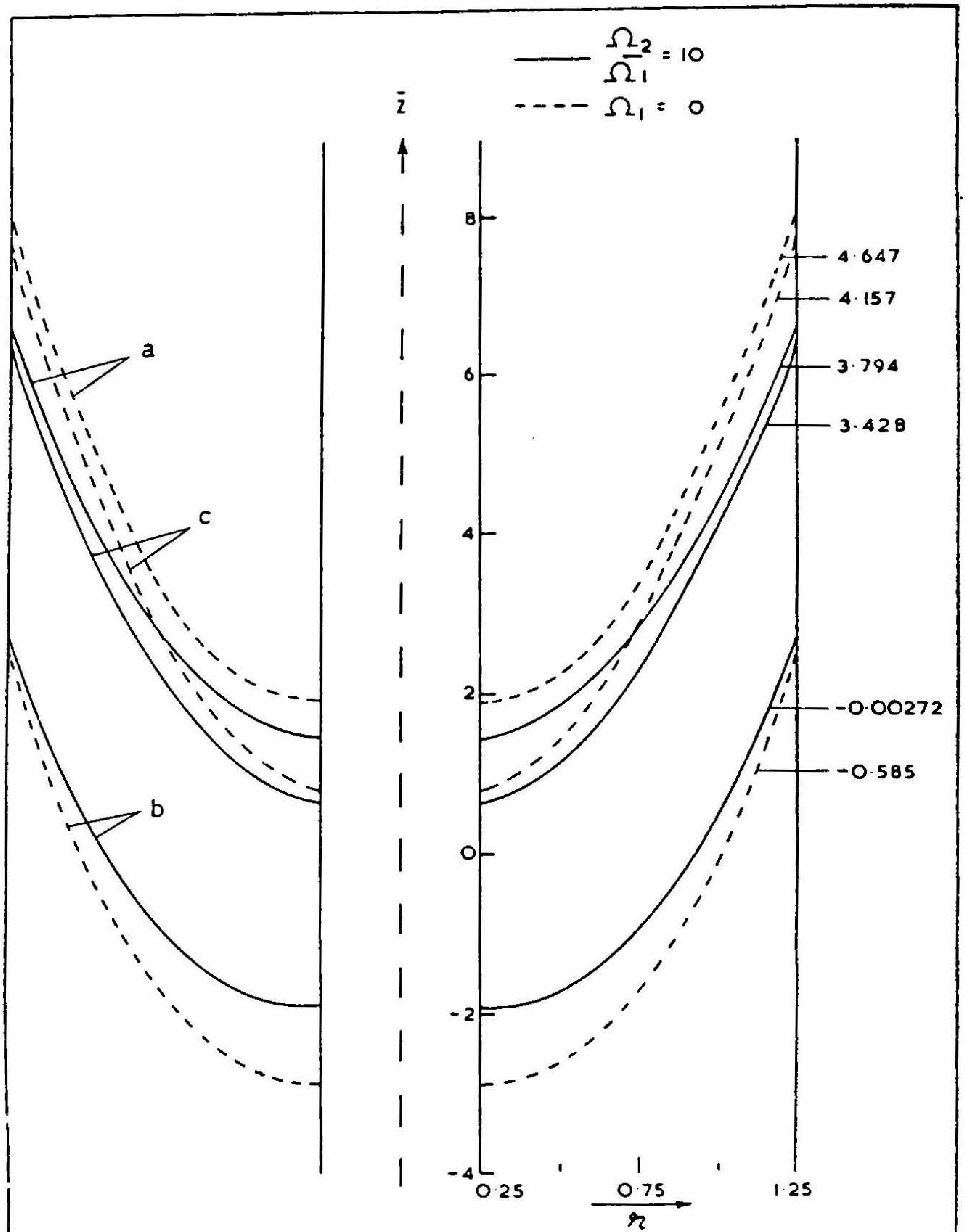

FIG I.

Shape of the free surface in the annulus for the case of (*a*) Newtonian fluid, (*b*) Dilatant fluid, (*c*) Pseudo-plastic fluid

The numbers on the right side denote $\bar{z}_0$ for the corresponding curves

(iii) *General Reiner-Rivlin Fluids.* The equation to the free surface in the annulus is given by [1.17] for $n \neq 1, 2$, [2.5] for $n = 1$, [2.11] for $n = 2$.

Figures III and IV show the shape of free surface represented by [2.5] for the cases (α) and (β) respectively, taking $\Phi_3 = \gamma_0 + \gamma_1 (I\, I_E)$. It is clear from these Figures that the fluid tends to climb along the inner cylinder and also tends to climb very slightly along the outer cylinder in the cases (α) and (β). The Figure V depicts the shape of the free surface in the case (γ). We notice that the fluid tends to climb along the inner cylinder and tends to fall along the outer cylinder, unlike the previous cases. Further, it is interesting to find that the climbing effect along the inner cylinder is more pronounced in the case (γ) than the case (α) or (β). Thus Reiner-Rivlin fluid in the annulus is drawn towards the inner cylinder against the centrifugal force showing the Weissenberg effect.

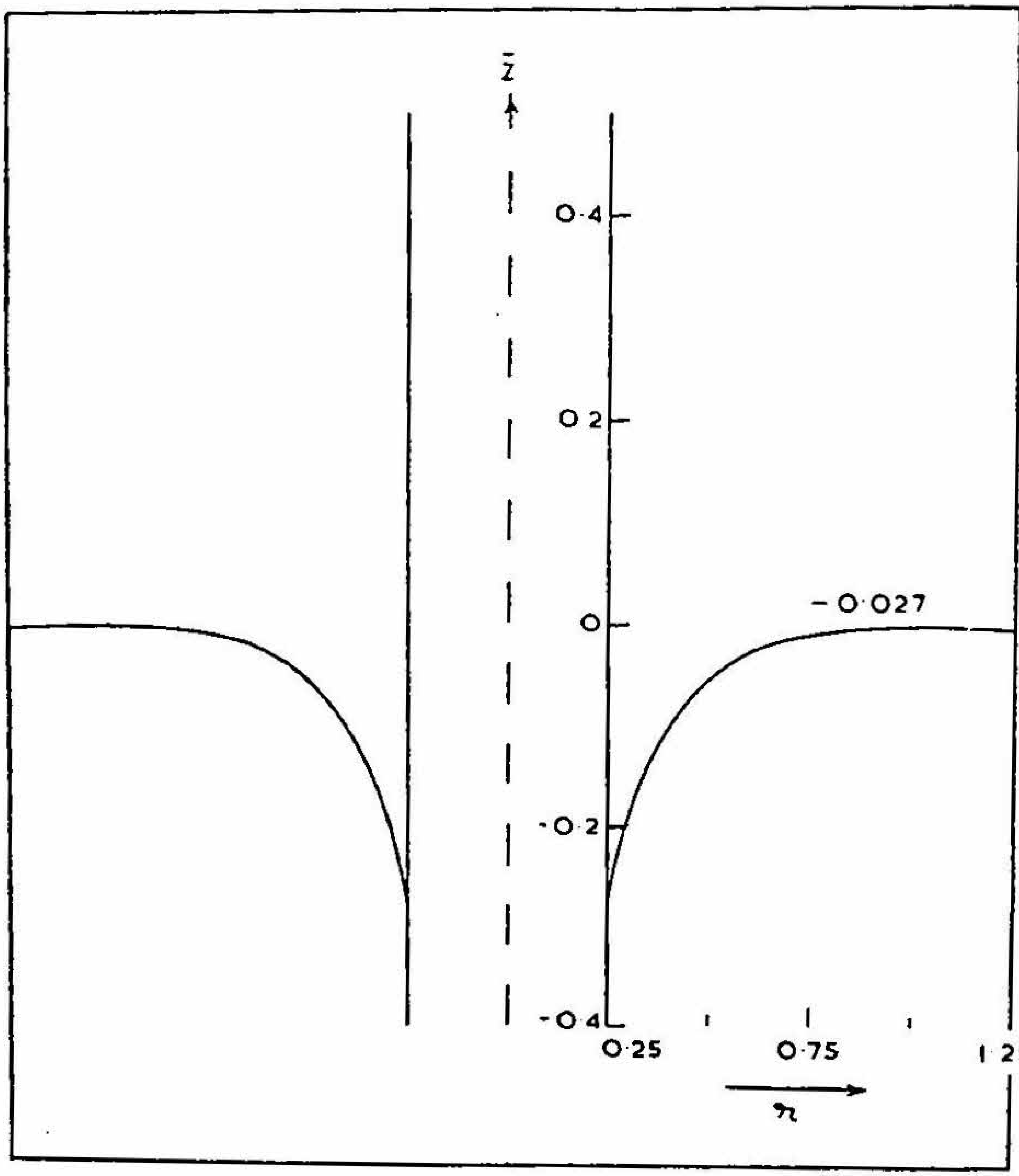

FIG. II

Shape of the free surfrce in the annulus for a Newtonian fluid (case γ)

(iv) ***Reiner-Rivlin fluids with constant coefficients of viscosity and cross-viscosity.*** The stress components and the pressure distribution for this class of fluids are obtained from [3.7] – [3.10] and [3.6] by putting $K = 0$.

The equation to the surface is

$$\bar{z} = \frac{1}{B}\left[\frac{d_2^2 r^2}{2} + 2\, c_2\, d_2 \ln r - \frac{c_2^2}{2r^2} + 4\, S\, \frac{c_2^2}{r^4}\right]. \qquad [4.4]$$

We find that the equation to the free surface for a Newtonian fluid [4.2] and for a Reiner-Rivlin fluid with constant coefficients [4.4] cannot be deduced as particular cases of the general Reiner-Rivlin fluids [1.17], but on the other hand, equation [4.2] of a Newtonian fluid can be obtained as a particular case [4,4] by putting $S = 0$ it it.

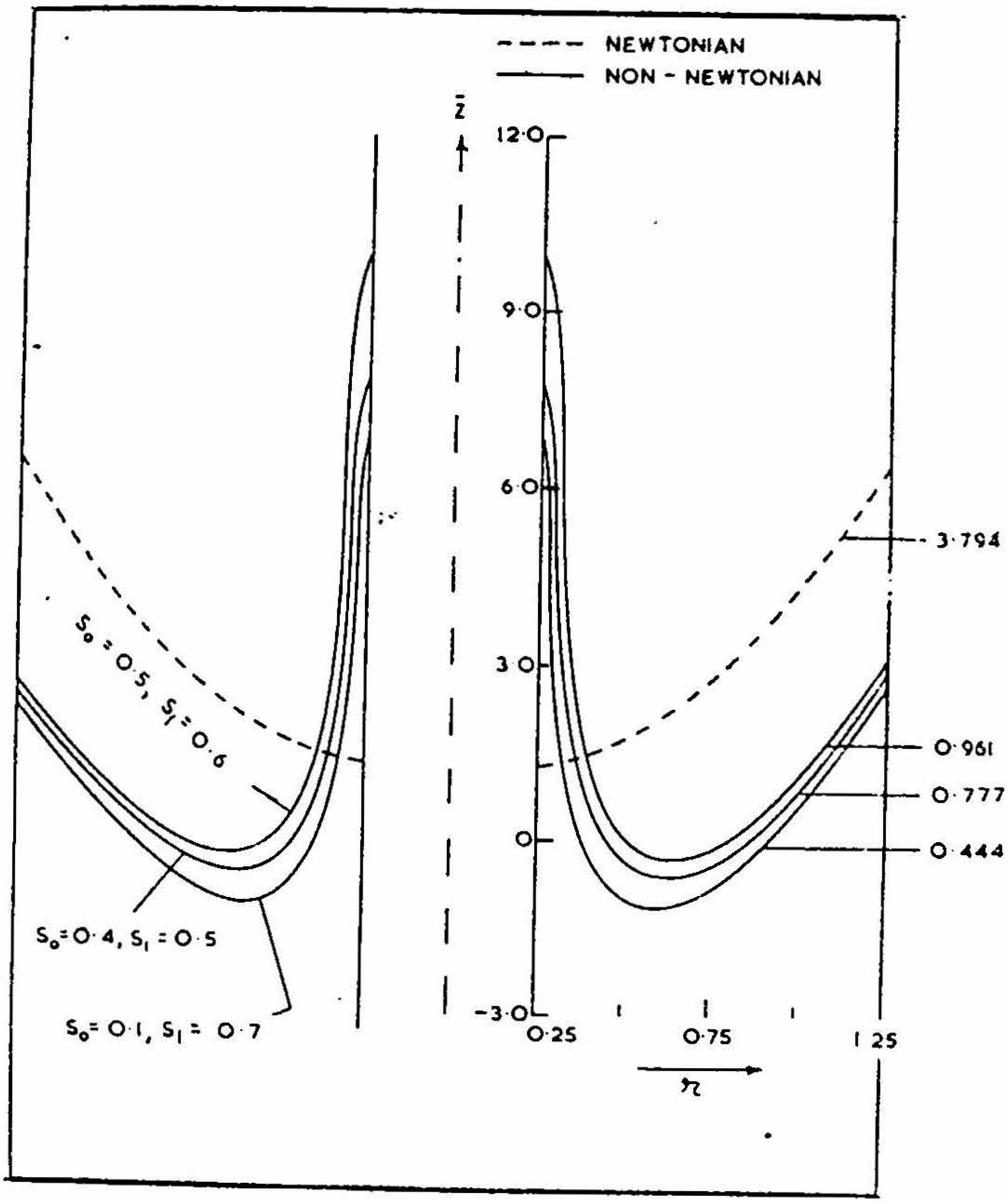

FIG. III

Shape of the free surface in the annulus for the case of a genaral Reiner–Rivlin fluid (case a)

It is interesting to find that

$$\bar{\bar{z}} \equiv (\bar{z} - \bar{z}_0)_{\text{non-New}} - (\bar{z} - \bar{z}_0)_{\text{New}} = \frac{4\,S\,c_2^2}{B}\left[\frac{1}{r^4} - \frac{(l-1)^4}{l^2}\right], \qquad [4.5]$$

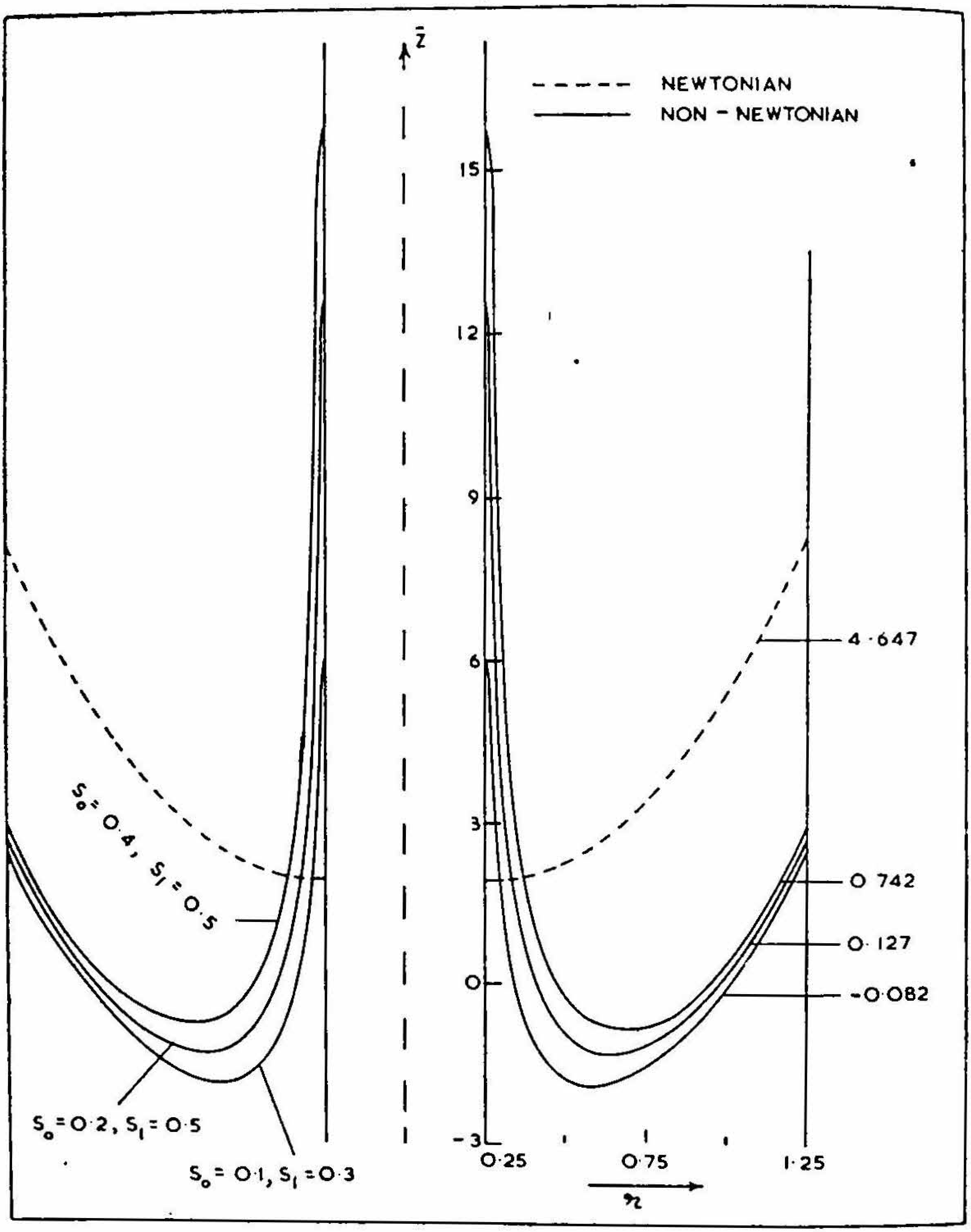

FIG. IV

Shape of the free surface in the annulus for the case of a general Reiner–Rivlin fluid (case β)

so that

$$(\bar{z})_{\text{in}} = \frac{4\,S\,(l-1)^4\,(l^2-1)\,c_2^2}{Bl^2}\ \text{(positive)}, \tag{4.6}$$

$$(\bar{z})_{\text{out}} = \frac{4\,S\,(l-1)^4\,(1-l^2)\,c_2^2}{B\,l^4}\ \text{(negative)}. \tag{4.7}$$

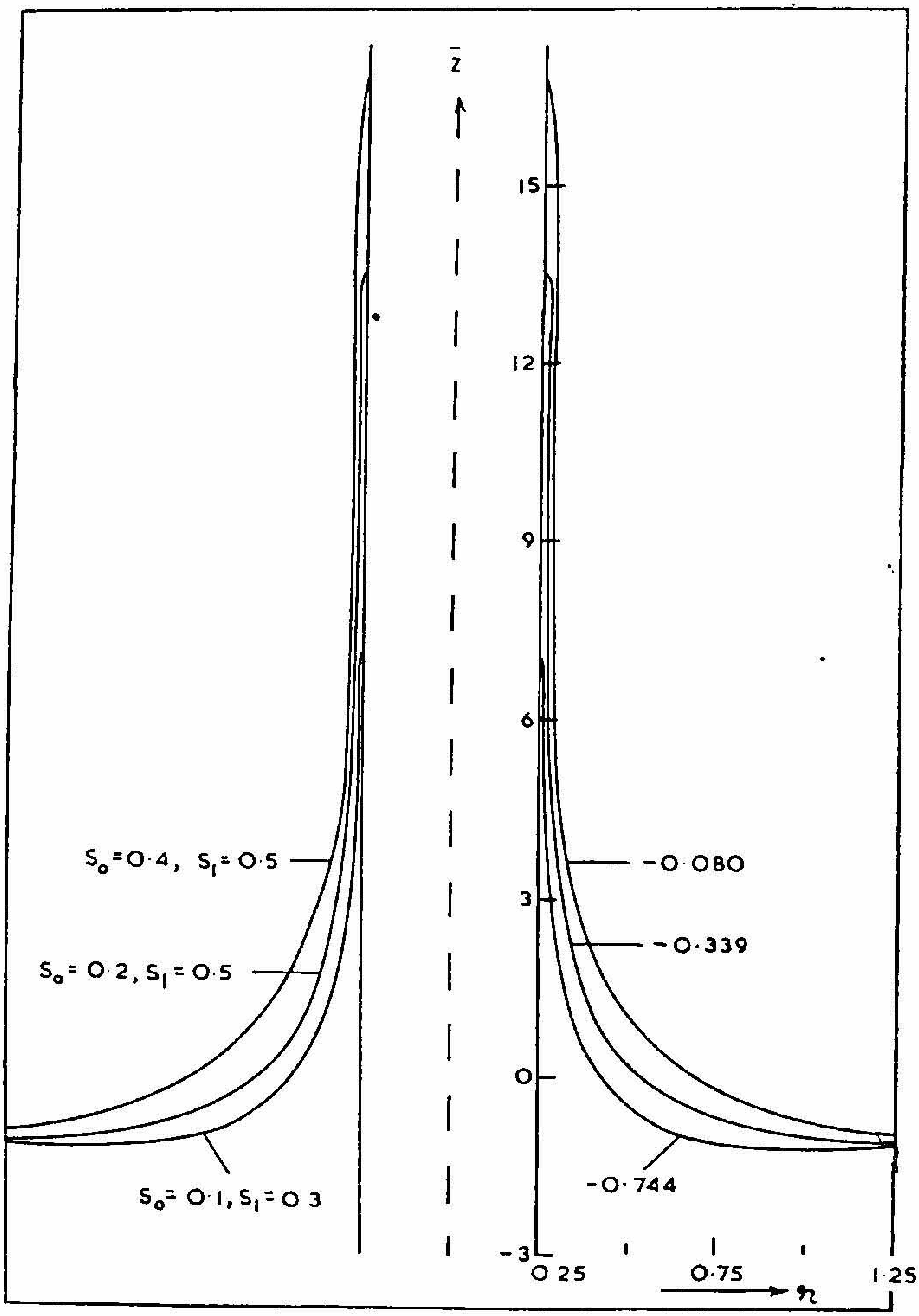

FIG. V

Shape of the free surface in the annulus for a general Reiner–Rivlin fluid (case γ)

If S can take negative values, equation [4.5] shows that Reiner-Rivlin fluid will always tend to fall along the inner cylinder showing the negative Weissenberg effect. Thus, in accordance with experiments we must take S to be positive.

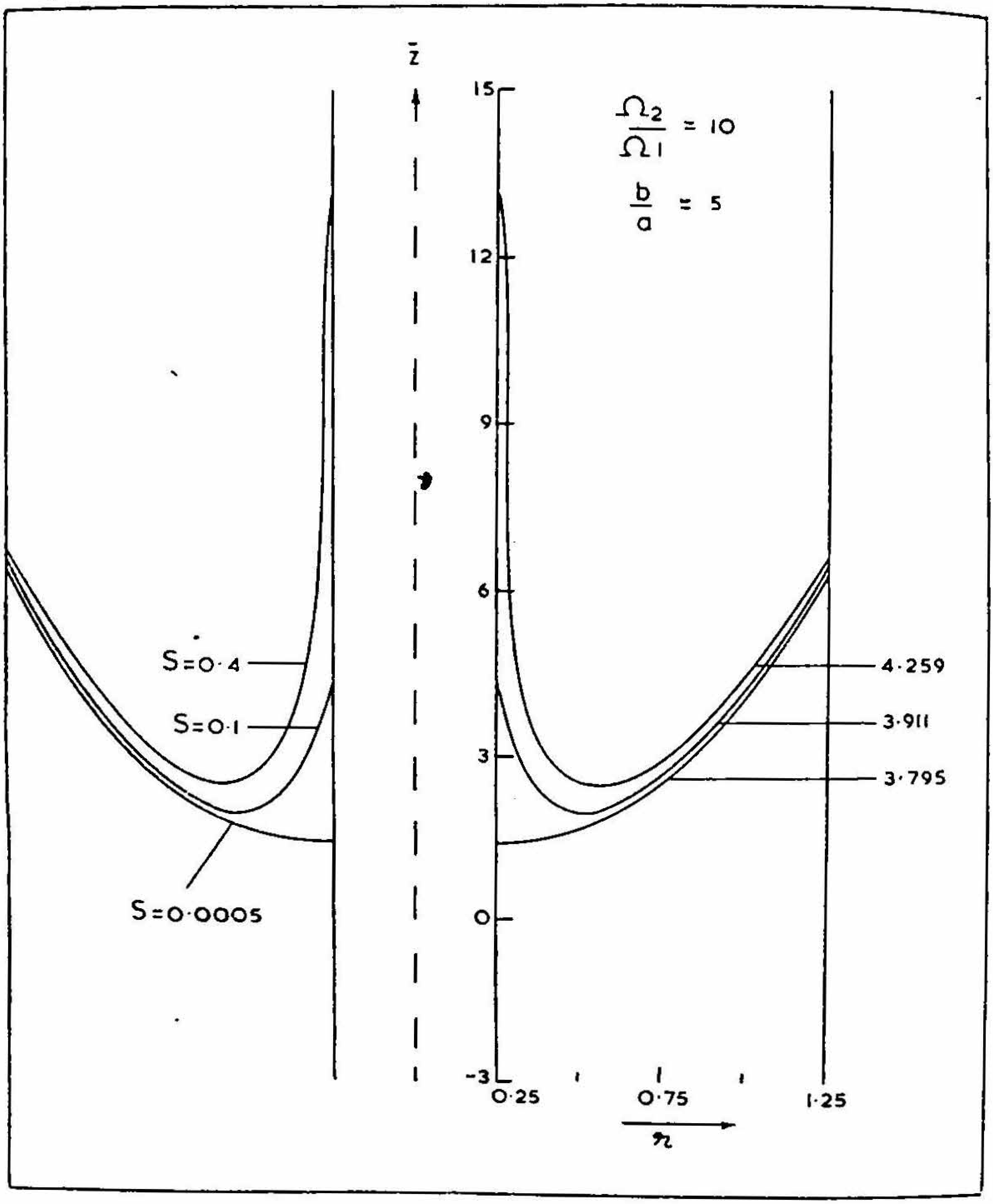

FIG. VI

Shape of the free surface in the annulus for the case of a Reiner-Rivlin fluid with constant coefficients (case α) S_c=0.0006

Figures VI and VII depict the shape of the free surface exposed to the atmosphere for the cases (α) and (β). We notice that the fluid tends to

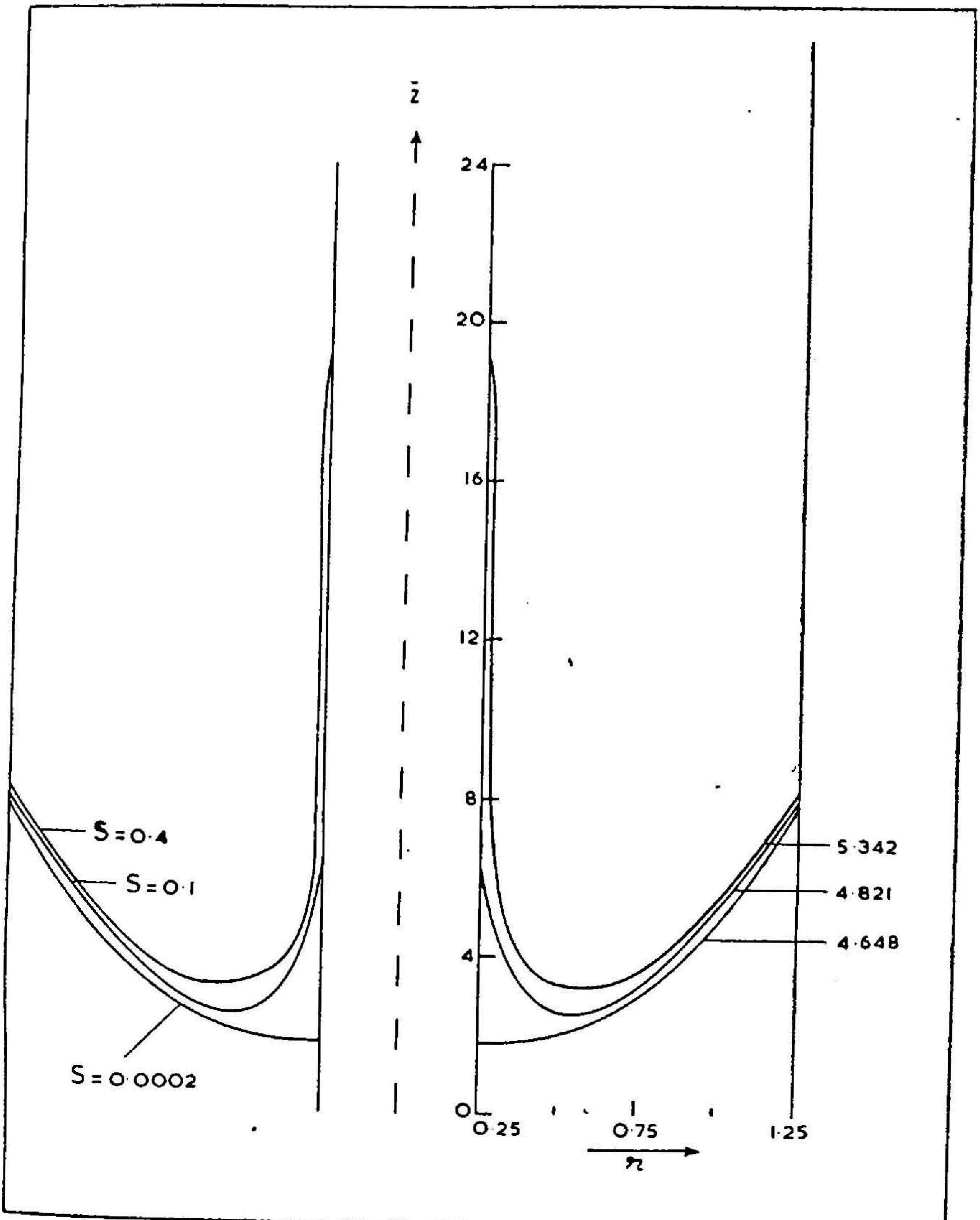

FIG. VII

Shape of the free surface in the annulus for the case of a Reiner—Rivlin fluid with constant coefficients (case β) S_c=0.0003 climb along the inner cylinder and also it climbs very slightly along the outer cylinder. A critical value of S in the case (α) is found, namely $S_c = 0.0006$, such that the shape of the free surface resembles that of the Newtonian fluid

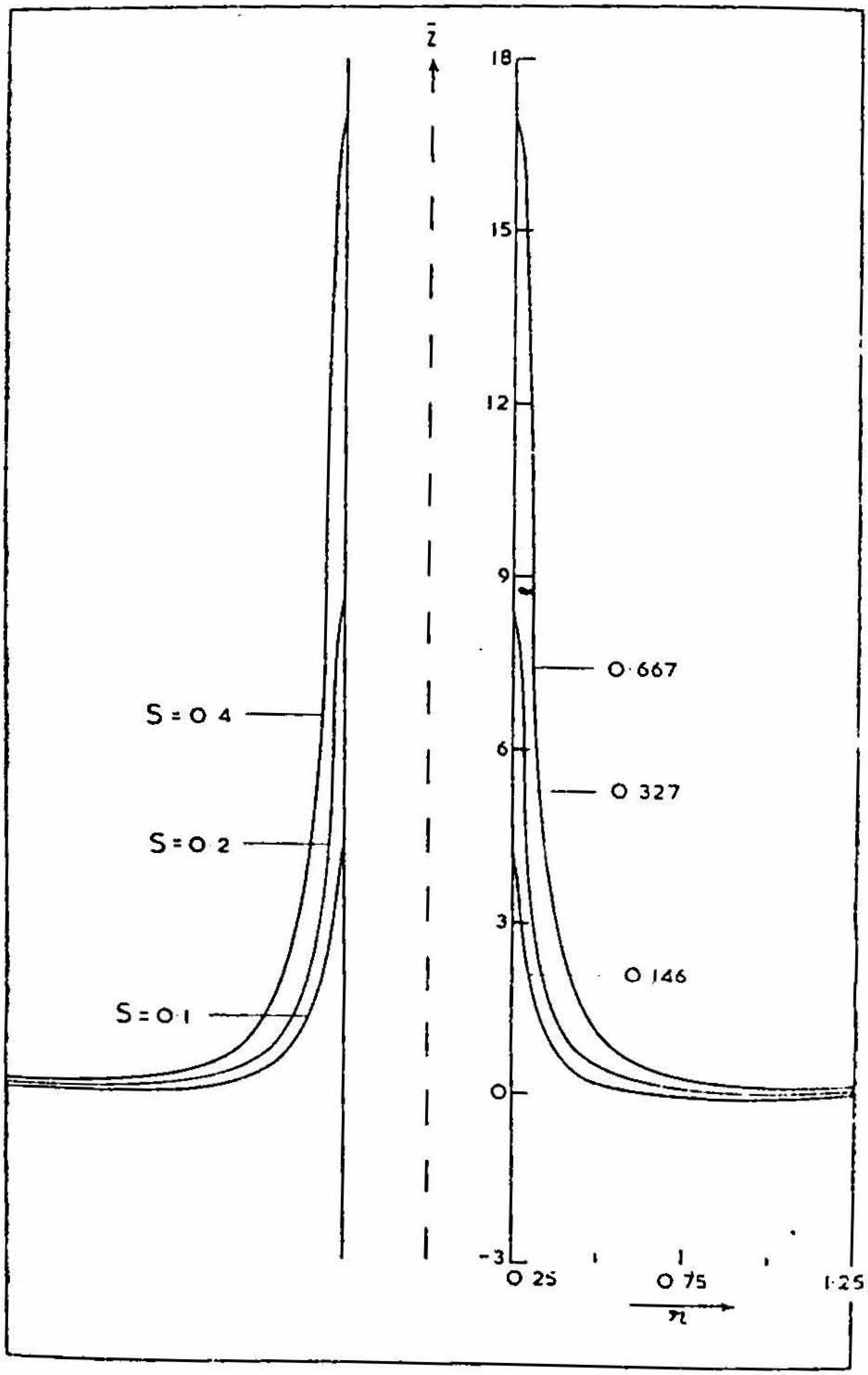

FIG. VIII

Shape of the free surface in the annulus for a Reiner–Rivlin fluid with constant coefficients (case γ) $S_c = 0.005$

when $S < S_c$ and the shape of the free surface changes gradually showing the Weissenberg effect wher $S \geqslant S_c$. Similarly, the critical value of S in case (β) is found to be 0.0003.

Figure VIII shows the shape of the free surface in the case (γ). We observe shat the fluid tends to raise along the inner cylinder and tends to fall along the outer cylinder, unlike the previous cases (α) and (β). The critical value of S in this case is 0.0005. This critical value of the non-Newtonian parameter will help us to determine the coefficient of cross-viscosity experimentally in relatiou to the shear viscosity.

(v) *Rivlin-Ericksen fluid with constant coefficients.* Figures IX and X depict the shape of the free surface in the cases (α) and (β) when $S=0$. As in Reiner-Rivlin fluids, here also we notice that the fluid tends to climb along

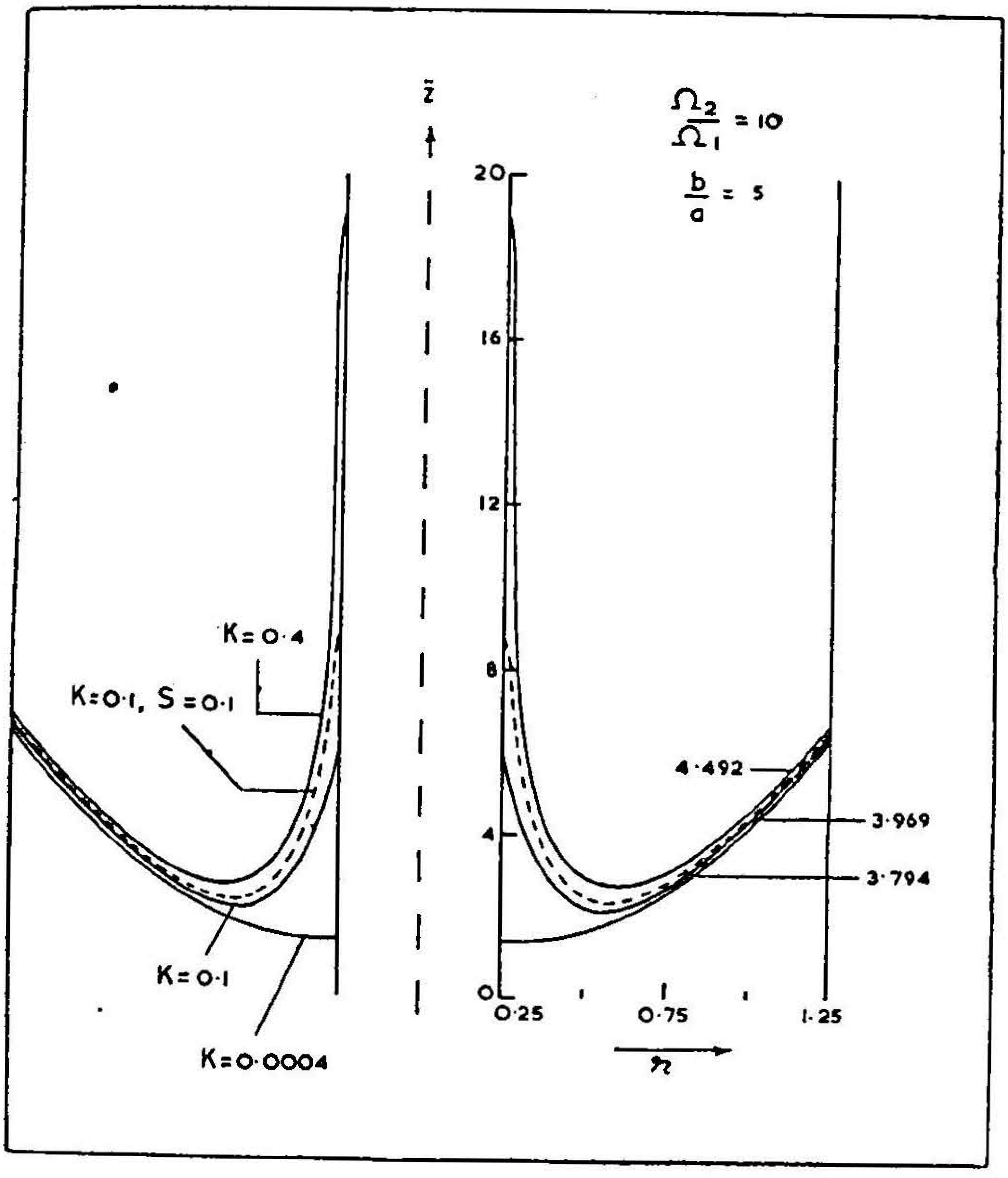

FIG. IX

Shape of the free surface in the annulus for the case of a Visco—elastic fluid (case α) $K_c=0.0005$.

the inner cylinder and it also climbs very slightly along the outer cylinder. A typical curve representing the shape of the free surface for $K = S = 0.1$ is drawn in Figure IX to take account of cross-viscosity and visco-elasticity simultaneously. The critical value of K in the cases (α) and (β) are 0.0005 and 0.0002 respectively.

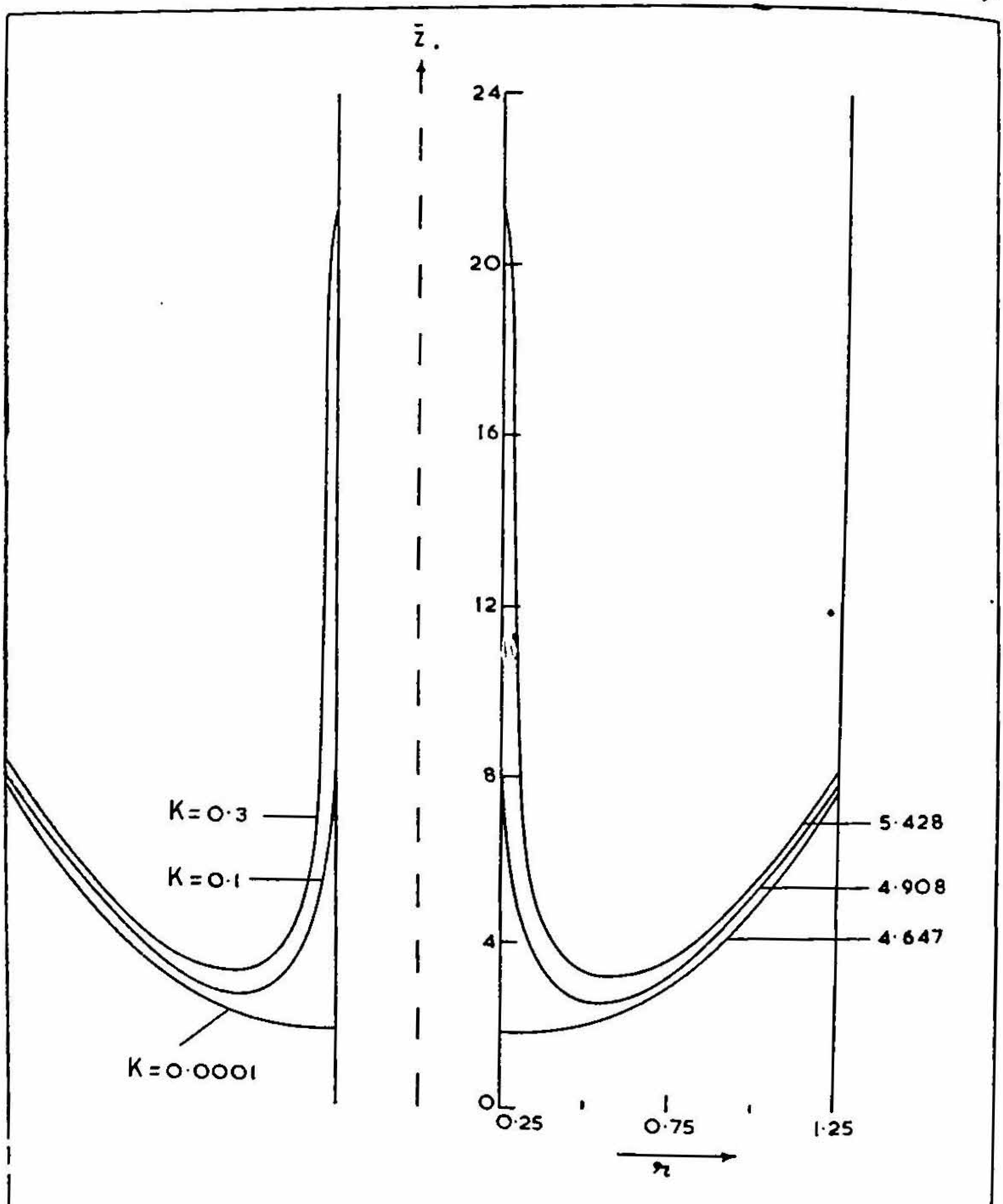

FIG. X

Shape of the free surface in the annulus for a Rivlin-Ericksen fluid (case β) $K_c = 0.0002$

Figure XI gives the shape of the free surface in the annulus for the case (γ). We observe that the fluid tends to rise along the inner cylinder and tends to fall along the outer cylinder unlike the previous cases (α) and (β). The critical value of K in the case (γ) is 0.002. As in the case of Reiner-Rivlin fluids, here also we find that the Weissenberg effect is more pronounced in the case (γ) than in (α) or (β). Thus, we conclude that cross-viscosity and visco-elasticity produce similar effects on the shape of the free surface in the annulus.

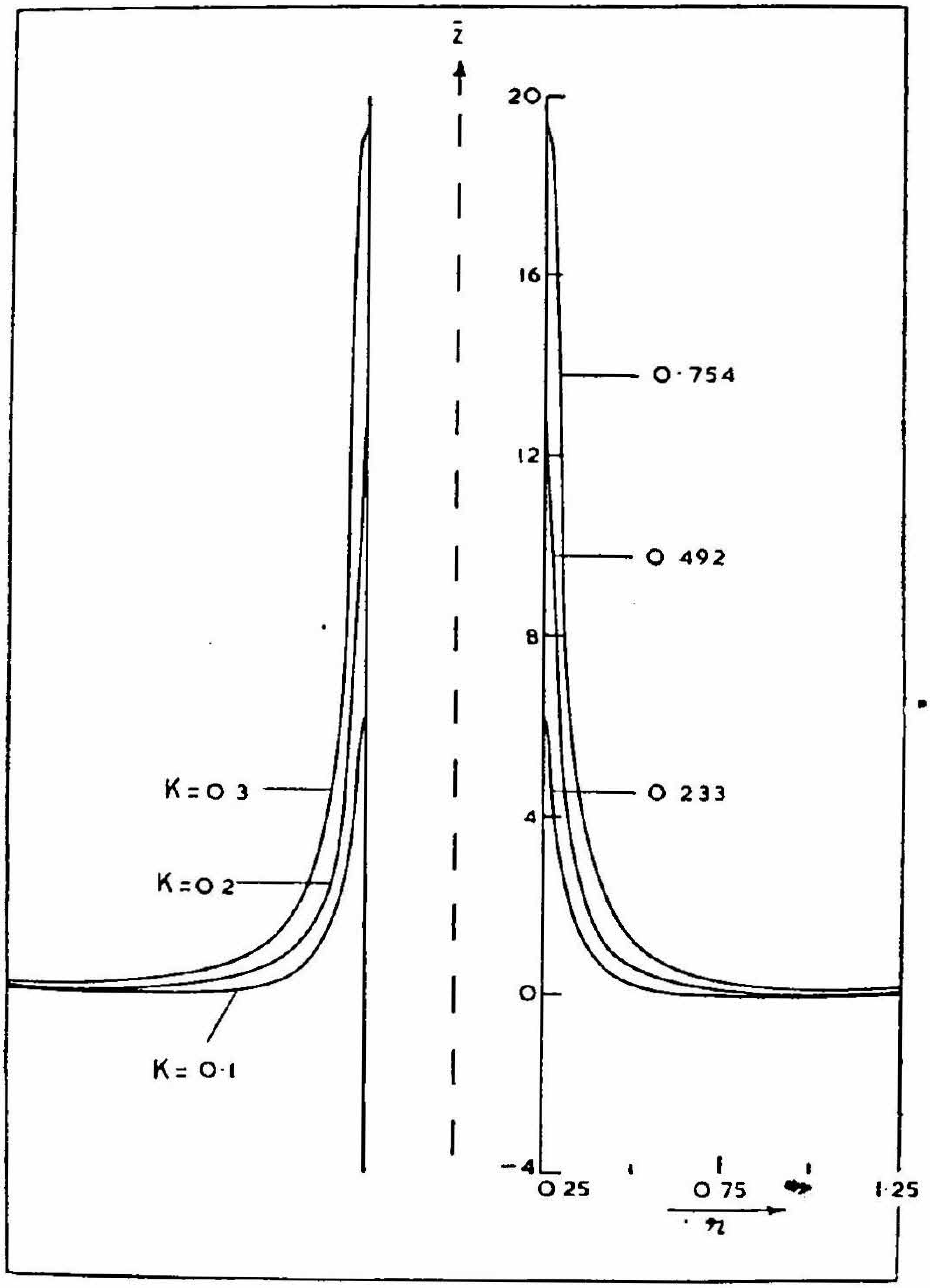

Fig. XI

Shape of the free surface in the annulus for a Rivlin–Ericksen fluid (case γ) $K_c = 0.002$

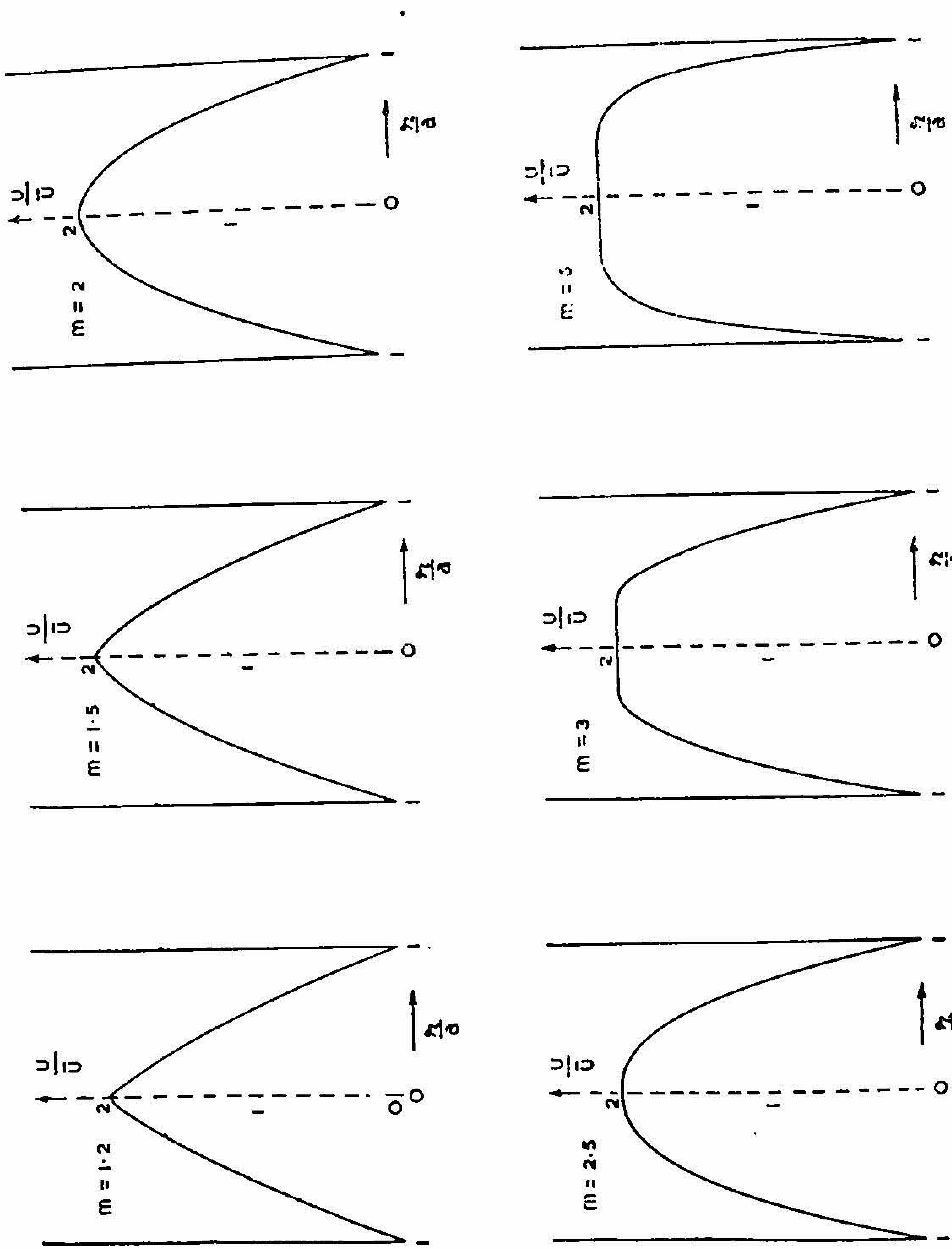

Fig. XII

Axial velocity profiles for different values of m ; m=2 corresponds to Newtonian parabolic velocity profile

Concluding remarks. Fgures III to XI show the rising of the fluid along the inner cylinder representing the Weissenberg effect in non-Newtonian fluids. Comparing the Figures VII and X corresponding to the Weissenberg experiment for visco-inelastic and visco-elastic fluids respectively, it is seen that the climbing effect along the inner cylinder is more pronounced in Figure X than in Figure VII. Further, we find that the rising of the fluid along the cylinders entirely depends on the value of the non-Newtonian parameter that we choose. Thus, we conclude that either cross-viscosity alone or visco-elasticity alone or both will modify the shape of the free surface in tha annulus prescribed by centrifugal force and force of gravity. However, the analysis based on the general Reiner-Rivlin fluids is more relevant in explaining the Weissenberg effect than the other cases.

PART B

5. *Steady flow of a general Reiner-Rivlin fluid through a pipe of circular cross-section.* Consider a steady flow of a general Reiner-Rivlin fluid through a straight pipe of circular cross-section of diameter $2a$ under the influence of a constant pressure gradient acting in the direction of the axis. We shall use the cylindrical polar coordinates (r, θ, z), where z-axis is taken along the axis of the pipe which is placed in a vertical position and r is measured from the axis of the pipe. Assuming the axial symmetry we shall take $\partial/\partial\theta \equiv 0$. Since the motion is purely axial, we have

$$u_r = 0, \quad u_\theta = 0, \quad u_z = u(r, z). \qquad [5.1]$$

In view of the equation of continuity we find that u is a function of r only.

The constitutive equation for a general Reiner-Rivlin fluid is given by [0.3], where Φ_1 and Φ_3 are functians of the second invariant u'^2, the prime denoting the differentiation with respect to r.

The Momentum equations are:

$$0 = -\frac{\partial p}{\partial r} + \frac{1}{r}\cdot\frac{\partial}{\partial r}\left(\Phi_3\, r\, u'^2\right), \qquad [5.2]$$

$$\rho g = -\frac{\partial p}{\partial z} + \frac{1}{r}\cdot\frac{\partial}{\partial r}\left(\Phi_1\, r\, u'\right). \qquad [5.3]$$

The boundary condition for the velocity profile is

$$u(r) = 0 \quad \text{when} \quad r = a. \qquad [5.4]$$

As in the Part A, here also we shall take the following expressions for the coefficients of viscosity Φ_1 and cross-viscosity Φ_3:

$$\Phi_1 = \beta_0 \left(u'^2\right)^\lambda, \qquad \Phi_3 = \sum_{i=0}^{\infty} \gamma_i \left(u'^2\right)^i, \qquad [5.5]$$

where β_0, γ_i $(i = 0, 1, \cdots)$ are constants.

The solution of [5.2] and [5.3] satisfying the boundary condition [5.4] is given by

$$u = -(A/2\,m)\,(a^m - r^m), \qquad [5.6]$$

$$p = (c - \rho\, g)\, z + \frac{A^2\, r^{2(m-1)}}{16\,(m-1)}\,[2\,\gamma_0\,(2m-1) + \gamma_1\,(4m-3)\,A^2\, r^{2(m-1)} + \cdots] + N, \qquad [5.7]$$

where

$$A = \left(\frac{c}{\beta_0}\right)^{m-1}, \qquad m \equiv \frac{2\,\lambda + 2}{2\,\lambda + 1}.$$

c corresponds to the given constant pressure gradient along the axis, and N is the constant of integration determined later.

It is readily seen from [5.6] and [5.7] that cross-viscosity does not affect the velocity profile whereas pressure is modified by the presence of cross-viscosity. By putting $m = 2$ in [5.6], we obtain the velocity profile for Reiner-Rivlin fluid with constant coefficients as given by Serrin[6].

In each cross-section the velocity attains its maximum value

$u_m = -A\, a^m/2m$ on the axis, while the average velocity is $\bar{u} = -A\, a^m/4m$.

The total mass flux M is

$$M = -\iint \rho\, ur\, dr\, d\theta$$

i.e.,

$$M = -2\,\pi\,\rho \int_0^a ur\, dr = \frac{\pi\,\rho\, A\, a^{m+2}}{2\,(m+2)}. \qquad [5.8]$$

Figure XII represents the axial velocity profile for various values of m. It is clear from the figure that the velocity profile is parabolic for $m = 2$ as in the case of Newtonian fluid, as m increases the profile gets flattened and as m decreases it gets steepened at the middle of the tube. When the cross-viscosity is zero, equation [5.6] represents velocity profile for the power-law fluids in which $m < 2$ corresponds to dilatant fluids, while $m > 2$ corresponds to pseudo-plastic fluids.

The parabolic distribution of the axial velocity is attained at some distance from the entry. The distribution of the velocity in this "inlet length" depends on the conditions at the entry. This inlet length has been determined by Bhatnagar and Rao[10] for a general Reiner-Rivlin fluid and Bogue[11] for pseudo-plastic fluids.

The stress components for the state of motion considered are:

$$T_{rr} = T_{zz} = (\rho g - c)\, z - \frac{A^2}{16\,(m-1)}\, r^{2(m-1)} \left[\gamma_0 + \gamma_1 A^2 r^{2(m-1)} + \cdots\right] - N, \qquad [5.9]$$

$$T_{\theta\theta} = (\rho g - c) z - \frac{A^2}{16(m-1)} r^{2(m-1)} \left[2\gamma_0(2m-1) + \gamma_1 A^2 (4m-3) r^{2(m-1)} + \cdots\right] - N, \qquad [5.10]$$

$$T_{r\theta} = T_{\theta z} = 0, \quad T_{rz} = \tfrac{1}{2}\, cr. \qquad [5.11]$$

We notice that the normal stresses on the pipe vary linearly along the length of the pipe.

6. *Discussion of the Ressults.* In contradistinction with Newtonian fluids, we find that the pressure distribution across a section of the pipe is not uniform for a non-Newtonian fluid.

To investigate the effect of cross-viscosity in detail, we suppose the fluid to issue from the pipe into the atmosphere at pressure p_0, the latter exerting a force equal to $\pi\, a^2 p_0$ on the output cross-section.

We use

$$\pi\, a^2 p_0 = -\int_0^a 2\,\pi\, r\, T_{zz}\, dr \qquad [6.1]$$

at the exit section $(z = 0)$ to determine the constant N in [5.7]. Thus, we have

$$T_{rr} = T_{zz} = (\rho\, g - c)\, z + \frac{A^2}{16(m-1)} \times$$

$$\times \left[2\,\gamma_0 \left\{\frac{a^{2(m-1)}}{m} - r^{2(m-1)}\right\} + \gamma_1 A^2 \left\{\frac{a^{4(m-1)}}{2\,m-1} - r^{4(m-1)}\right\} + \cdots\right] - p_0\,. \qquad [6.2]$$

Let P be the normal force per unit area which the fluid exerts on the pipe walls, then we have

$$P = -\,(T_{rr})_{\text{wall}}\,. \qquad [6.3]$$

In view of the condition (6.3), we get

$$P - p^* = cz + \left(\frac{\Gamma}{a}\right)^2 (m+2)^2 \left[\frac{\gamma_0}{8m} + \frac{\gamma_1}{8\,(2\,m-1)} (m+2)^2 \left(\frac{\Gamma}{a}\right)^2 + \cdots\right], \qquad 6.4]$$

where

$$p^* = p_0 - \rho\, g\, z \qquad [6.5]$$

and $\Gamma = M/(\pi a^2 \rho) = A\, a^m/(m+2) =$ Average volume of the flow per sec. per unit cross-sectional area of the pipe.

The excess pressure at the exit section of the pipe is

$$P - p_0 = (m+2)^2 \left(\frac{\Gamma}{a}\right)^2 \left[\frac{\gamma_0}{8m} + \frac{\gamma_1}{8} \cdot \frac{(m+2)^2}{2m-1}\left(\frac{\Gamma}{a}\right)^2 + \cdots\right]. \qquad [6.6]$$

This indicates that there is an excess pressure at the exit section of the pipe provided $\gamma_0, \gamma_1 \cdots$ are positive. This may provide in a way a theoretical explanation of the Merrington effect, namely the tendency of a fluid stream to swell at the exit section of the tube viscometer. This also supports the view point of Reiner that the Merrington effect arises from the cross-viscosity even when it is taken as constant. Further, equation [6.6] implies that the swelling at the exit section becomes more prominent when the flux is high and pipe radius small. This fact has been actually observed by Merrington [loc. cit.].

Let us write the wall pressure in the form

$$(P - p_0)\left(\frac{a}{\Gamma}\right)^2 = \left[(m+2)^2 A^{1-m} \beta_0 a^{2(1-m)} - \rho g\left(\frac{a}{\Gamma}\right)^2\right] z$$

$$+ \left[\frac{\gamma_0}{8m}(m+2)^2 + \frac{\gamma_1 (m+2)^4}{8(2m-1)}\left(\frac{\Gamma}{a}\right)^2 + \cdots\right]. \qquad [6.7]$$

For a given mean flow rate Γ, we measure P at a number of stations along the tube wall. The plot of $(P - p_0)(a/\Gamma)^2$ against z is a straight line. The slope of this line gives

$$\frac{(m+2)^2}{c^{(m-1)^2}} a^{2(1-m)} \beta_0^{m^2-2m+2} - \rho g\left(\frac{a}{\Gamma}\right)^2,$$

while the intercept δ on the $(p - p_0)(a/\Gamma)^2$ – axis gives the value of

$$\delta = (m+2)^2 \frac{\gamma_0}{8m} + \frac{\gamma_1 (m+2)^4}{8(2m-1)} \cdot \left(\frac{\Gamma}{a}\right)^2 + \cdots \qquad [6.8]$$

We thus see that the slope of the plot depends on β_0, constant pressure gradient c, radius a and the power λ of the second invariant in the expression of Φ_1.

If we now determine the slope $\tan\theta$ of the plot for the same fluid with the same pressure gradient but with different tube radii a_1 and a_2, we get

$$\left.\begin{aligned} \tan\theta_1 &= \frac{(m+2)^2}{c^{(m-1)^2}} a_1^{2(1-m)} \beta_0^{m^2-2m+2} - \rho g\left(\frac{a_1}{\Gamma}\right)^2, \\ \tan\theta_2 &= \frac{(m+2)^2}{c^{(m-1)^2}} a_2^{2(1-m)} \beta_0^{m^2-2m+2} - \rho g\left(\frac{a_2}{\Gamma}\right)^2, \end{aligned}\right\} \qquad [6.9]$$

so that

$$\frac{\tan\theta_1 + \rho g (a_1/\Gamma)^2}{\tan\theta_2 + \rho g (a_2/\Gamma)^2} = \left(\frac{a_1}{a_2}\right)^{2(1-m)}. \qquad [6.10]$$

Equation [6.10] determines m. Substituting this value of m in any one of the equations [6.9] we get the value of β_0.

If now, for a number of values of Γ, we determine the intercepts δ, we obtain the following type of equations:

$$\delta_i = \frac{(m+2)^2}{8m}\gamma_0 + \frac{(m+2)^4}{8(2m-1)}\left(\frac{\Gamma_i}{a}\right)^2 \gamma_1 + \cdots . \qquad [6.11]$$

These equations in principle determine $\gamma_0, \gamma_1 \cdots$. If on the other hand, we truncate the expression for Φ_3 to take only finite number of terms, we can determine the finite number of coefficients $\gamma_0, \gamma_1 \cdots \gamma_k$ by the method of least squares.

These ideas have been put forward entirely in the nature of suggestions. The authors have no facilities for experimentation and hence could not verify the applicability of these ideas.

7. *Merrington Effect in Rivlin-Ericksen Fluids with Constant Coefficients.*

In this case, we find that the axial velocity distribution

$$u = (c/4\Phi_1)(r^2 - a^2) \qquad [7.1]$$

is same as in the Newtonian fluids giving the parabolic profile.

The stress components and the pressure distribution are

$$T_{rr} = (\rho g - c) z - p_0 + (c^2/8\Phi_1^2)\left[(3a^2 - 2r^2)\Phi_2 + \{(a^2/2) - r^2\}\Phi_3\right], \qquad [7.2]$$

$$T_{\theta\theta} = (\rho g - c) z - p_0 + (c^2/16\Phi_1^2)\left[6\Phi_2(a^2 - r^2) + (a^2 - 3r^2)\Phi_3\right], \qquad [7.3]$$

$$T_{zz} = (\rho g - c) z - p_0 + (c^2/16\Phi_1^2)(\Phi_3 + 6\Phi_2)(a^2 - 2r^2), \qquad [7.4]$$

$$T_{r\theta} = T_{\theta z} = 0, \quad T_{rz} = \tfrac{1}{2} c r, \qquad [7.5]$$

and

$$p - p_0 = (c - \rho g) z + (3c^2/8\Phi_1^2)(2\Phi_2 + \Phi_3) r^2 - (c^2/16\Phi_1^2)(\Phi_3 + 6\Phi_2) a^2. \qquad [7.6]$$

If P is the normal force per unit area which the fluid exerts on the pipe walls, we have

$$P = -(T_{rr})_{r=a}, \qquad [7.7]$$

which gives

$$P - p^* = \left[c z + (c^2 a^2/16\Phi_1^2)(\Phi_3 - 2\Phi_2)\right]. \qquad [7.8]$$

This equation can be written in the form.

$$P - p^* = (\Gamma/a)^2 \left[(\Phi_3 - 2\,\Phi_2) + (4\,\Phi_1\, z/\Gamma) \right]. \qquad [7.9]$$

Therefore the excess pressure at the exit section of the tube is

$$P - p_0 = (\Gamma/a)^2\, (\Phi_3 - 2\,\Phi_2). \qquad [7.10]$$

When the fluid emerges from the tube the result will be a swelling of the emergent column of the fluid provided $(\Phi_3 - 2\Phi_2)$ is positive. In other words, cross-viscosity alone will show the Merrington effect when Φ_3 is positive, while visco-elasticity alone will show the Merrington effect when Φ_2 is negative. Thus we find that cross-viscosity and visco-elasticity exhibit opposite effects when they are considered separately. A fluid for which $\Phi_3 = 2\,\Phi_2$ will not show any swelling at the exit section of the pipe.

8. ACKNOWLEDGEMENT

One of us (S. L. R.) is grateful to the University Grants Commission for the award of a research fellowship.

REFERENCES

1. Reiner, M. .. .. *Amer. J. Math.*, 1945, 67, 350.
2. Rivlin, R. S. and Ericksen, J. L. *J. Rat. Mech. Anal.* 1955, 4 329.
3. Weissenberg, K. .. *Nature*, 1947, 159, 310.
4. Braun, I. and Reiner, M. .. *Quart. J. Mech. App. Math.* 1952, 5, 44.
5. Rivlin, R. S. .. .. *Proc. Roy. Soc.*, 1948, 193A, 260.
6. James Serrin .. .. ZAMM., 1959, 39, 295.
7. Merrington, A. C. .. *Nature*, 1943, 152, 663.
8. Rivlin, R. S. .. .. *Proc. Camb. Phil. Soc.*, 1949, 45, 88,
9. Rathna, S. L. .. .. *Proc. Natl. Inst. Sci.*, 1960, 26, 392.
10. Bhatnagar, P. and Rao S. K. L. *Proc. Third Cong. Theo. App. Mech.*, 1957, 29.
11. Bogue, D. C. .. .. *Indus. Engg. Chem.*, 1959, 51, 874.