STEADY LAMINAR FLOW OF VISCO-ELASTIC FLUID THROUGH A PIPE AND THROUGH AN ANNULUS WITH SUCTION OR INJECTION AT THE WALLS

BY R. K. BHATNAGAR

(Department of Applied Mathematics, Indian Institute of Science, Bangalore-12)

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ABSTRACT

The problem of steady laminar flow of visco-elastic fluids through a pipe and through a cylindrical annulus in the presence of suction and injection have been considered in Part A and Part B of the paper respectively. The rate of suction at one wall has been taken to be equal to the rate of injection at the other wall in the case of annulus. For the pipe flow the effect of the visco-elasticity of the fluid on the axial velocity, axial pressure and the shearing stress have been depicted graphically. For the annulus, the perturbation method has been used for the axial velocity regarding $K R$ as the perturbation parameter. For rapid convergence of the series solution, $K R$ has taken to be equal to 0.1. It has been found that for Reynolds number, $R=1$, the axial velocity profile is not symmetrical regardless of the radius ratio of the two boundaries of the annulus. This is in contrast to the Newtonian fluids where the profile is symmetrical for $R=1$. It has also been found that the points of inflexion in the axial velocity profiles occur only after a critical value of Reynolds number, *viz.*, $R_c=1.7$. The radial pressure variation depends on the direction of cross flow. In this respect, visco-elastic fluids have been found to differ from Newtonian fluids.

INTRODUCTION

1. The problem of diffusion of gases through porous pipes was first considered by Olson[1], Later Berman[2, 3, 4] made a considerable contribution to the Newtonian viscous fluids by introducing a stream function. This work was further extended by Yuan[5], Sellars[6] and Donoughe[7], who obtained lhe perturbation solutions under the limiting conditions of suction and injection.

Narasimhan[8, 9] has studied the problem of laminar flow of visco-inelastic non-Newtonian fluids in a pipe and in an annulus between two co-axial cylinders taking constant coefficient of cross viscosity into consideration, He has obtained the exact solutions of the equations assuming that the fluid is being withdrawn or injected through the wall in the case of pipe flow. In the case of co-axial cylinders, the fluid injection rate at one wall is taken equal to the fluid suction rate at the other wall.

The purpose of the present paper is to study the problem of steady laminar flow of visco-elastic fluids through a pipe [Part A] and through an annulus [Part B] with suction or injection.

The constitutive equation for a Rivlin-Ericksen visco-elastic fluid is

$$T = -pI + \phi_1 E + \phi_2 D \qquad [1.1]$$

where

$$E_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \qquad [1.2]$$

is the rate of deformation tensor,

$$D_{ij} = \frac{\partial a_i}{\partial x_j} + \frac{\partial a_j}{\partial x_i} + 2\,\frac{\partial u_m}{\partial x_i}\cdot\frac{\partial u_m}{\partial x_j} \qquad [1.3]$$

is the acceleration gradient tensor, p the pressure, and ϕ_1, ϕ_2 the kinematic coefficients of viscosity and viscoelasticity respectively.

Then the equations of continuity and momentum in the steady state and in the absence of external forces are

$$\nabla \cdot \mathbf{q} = 0 \qquad [1.4]$$

$$\rho\, \mathbf{q} \cdot \nabla \mathbf{q} = \nabla \cdot T \qquad [1.5]$$

where $\mathbf{q}$ is the velocity and ρ the density of the fluid.

PART A

Reduction of the Flow Equations

2. We shall work through cylindrical polar co-ordinate system $[r, \theta, x]$ where r is measured from the axis of the cylinder, x along the axis of the cylinder and θ from some convenient meridian plane. The θ co-ordinate will not appear in our discussion due to the axial symmetry. We have further assumed that a constant normal velocity of suction or injection is applied at the wall.

Defining a non-dimensional parameter

$$\lambda = [r/a], \quad [0 \leqslant \lambda \leqslant 1] \qquad [2.1]$$

where a is the radius of the pipe, the equations of continuity and momentum reduce to :

$$\frac{\partial}{\partial x}[\lambda u] + \frac{1}{a}\cdot\frac{\partial}{\partial \lambda}[\lambda v] = 0 \qquad [2.2]$$

$$
\begin{aligned}
u\frac{\partial u}{\partial x} + \frac{v}{a}\cdot\frac{\partial u}{\partial \lambda} = & -\frac{1}{\rho}\cdot\frac{\partial p}{\partial x} + \alpha\left(\frac{\partial^2 u}{\partial x^2} + \frac{1}{a^2}\cdot\frac{\partial^2 u}{\partial \lambda^2} + \frac{1}{a^2\lambda}\cdot\frac{\partial u}{\partial \lambda}\right) \\
& + \beta\left[\frac{3}{a^2}\cdot\frac{\partial u}{\partial x}\cdot\frac{\partial^2 u}{\partial \lambda^2} + \frac{4}{a^9}\cdot\frac{\partial u}{\lambda}\cdot\frac{\partial^2 u}{\partial \lambda\,\partial x} + \frac{u}{a^2}\cdot\frac{\partial^3 u}{\partial \lambda^2\,\partial x}\right. \\
& + \frac{1}{a^3}\cdot\frac{\partial u}{\partial \lambda}\cdot\frac{\partial^2 v}{\partial \lambda^2} + \frac{2}{a^3}\cdot\frac{\partial v}{\partial \lambda}\cdot\frac{\partial^2 u}{\partial \lambda^2} + \frac{v}{a^3}\cdot\frac{\partial^3 u}{\partial \lambda^3} + 10\frac{\partial u}{\partial x}\cdot\frac{\partial^2 u}{\partial x^2} + 2u\frac{\partial^3 u}{\partial x^3} \\
& + \frac{3}{a}\cdot\frac{\partial u}{\partial \lambda}\cdot\frac{\partial^2 v}{\partial x^2} + \frac{4}{a}\cdot\frac{\partial v}{\partial x}\cdot\frac{\partial^2 u}{\partial \lambda\,\partial x} + \frac{1}{a}\cdot\frac{\partial v}{\partial x}\cdot\frac{\partial^2 u}{\partial \lambda\,\partial x} + \frac{2v}{a}\cdot\frac{\partial^3 u}{\partial \lambda\ x^2} \\
& + \frac{3}{a^2\lambda}\cdot\frac{\partial u}{\partial \lambda}\cdot\frac{\partial u}{\partial x} + \frac{u}{a^2\lambda}\cdot\frac{\partial^2 u}{\partial \lambda\,\partial x}\cdot + \frac{1}{a^3\lambda}\cdot\frac{\partial u}{\partial \lambda}\cdot\frac{\partial v}{\partial \lambda} + \frac{v}{a^3\lambda}\cdot\frac{\partial^2 u}{\partial \lambda^2} \\
& + \frac{1}{a}\cdot\frac{\partial u}{\partial x}\cdot\frac{\partial^2 v}{\partial \lambda\,\partial x} + \frac{u}{a}\cdot\frac{\partial^3 v}{\partial x^2\ \lambda} + \frac{4}{a^2}\cdot\frac{\partial v}{\partial \lambda}\cdot\frac{\partial^2 v}{\partial \lambda\ x} + \frac{3}{a^2}\cdot\frac{\partial v}{\partial x}\cdot\frac{\partial^2 v}{\partial \lambda^2} \\
& + \frac{v}{a^2}\cdot\frac{\partial^3 v}{\partial \lambda^2\partial x} + \frac{1}{a\lambda}\cdot\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x} + \frac{u}{a\lambda}\cdot\frac{\partial^2 v}{\partial x^2} + \frac{3}{a^2\lambda}\cdot\frac{\partial v}{\partial x}\cdot\frac{\partial v}{\partial \lambda} \\
& \left. + \frac{v}{a^2\lambda}\cdot\frac{\partial^2 v}{\partial \lambda\,\partial x} + 4\frac{\partial v}{\partial x}\cdot\frac{\partial^2 v}{\partial x^2}\right]
\end{aligned}
\qquad [2.3]
$$

$$
\begin{aligned}
u\frac{\partial v}{\partial x} + \frac{v}{a}\cdot\frac{\partial v}{\partial \lambda} = & -\frac{1}{a\rho}\cdot\frac{\partial p}{\partial \lambda} + \sigma\left[\frac{\partial^2 v}{\partial x^2} + \frac{1}{a^2}\cdot\frac{\partial}{\partial \lambda}\left\{\frac{1}{\lambda}\cdot\frac{\partial}{\partial \lambda}[\lambda v]\right\}\right] \\
& + \beta\left[\frac{4}{a^2}\cdot\frac{\partial u}{\partial \lambda}\cdot\frac{\partial^2 v}{\partial x\,\partial \lambda} + \frac{2}{a^2}\cdot\frac{\partial v}{\partial x}\cdot\frac{\partial^2 u}{\partial \lambda^2} + \frac{2u}{a^2}\cdot\frac{\partial^3 v}{\partial x\ \lambda^2} + \frac{10}{a^3}\cdot\frac{\partial v}{\partial \lambda}\cdot\frac{\partial^2 v}{\partial \lambda^2}\right. \\
& + \frac{2v}{a^3}\cdot\frac{\partial^3 v}{\partial \lambda^3} + 2\frac{\partial u}{\partial x}\cdot\frac{\partial^2 v}{\partial x^2} + u\frac{\partial^3 v}{\partial x^3} + \frac{\partial v}{\partial x}\cdot\frac{\partial^2 u}{\partial x^2} + \frac{4}{a}\cdot\frac{\partial v}{\partial x}\cdot\frac{\partial^2 v}{\partial x\,\partial \lambda} \\
& + \frac{3}{a}\cdot\frac{\partial v}{\partial \lambda}\cdot\frac{\partial^2 v}{\partial x^2} + \frac{v}{a}\cdot\frac{\partial^3 v}{\partial x^2\partial \lambda} + \frac{2}{a^2\lambda}\cdot\frac{\partial u}{\partial \lambda}\cdot\frac{\partial v}{\partial x} + \frac{2u}{a^2\lambda}\cdot\frac{\partial^2 v}{\partial \lambda\,\partial x} \\
& + \frac{4}{a^3\lambda}\left(\frac{\partial v}{\partial \lambda}\right)^2 + \frac{2v}{a^3\lambda}\cdot\frac{\partial^2 v}{\partial \lambda^2}\cdot - \frac{2u}{a^2\lambda^2}\cdot\frac{\partial v}{\partial x} - \frac{2v}{a^3\lambda^2}\cdot\frac{\partial v}{\partial \lambda} \\
& + \frac{3}{a}\cdot\frac{\partial u}{\partial \lambda}\cdot\frac{\partial^2 u}{\partial x^2} + \frac{4}{a}\cdot\frac{\partial u}{\partial x}\cdot\frac{\partial^2 u}{\partial \lambda\,\partial x} + \frac{u}{a}\cdot\frac{\partial^3 u}{\partial \lambda\partial x^2} + \frac{1}{a^2}\cdot\frac{\partial v}{\partial \lambda}\cdot\frac{\partial^2 u}{\partial \lambda\,\partial x} \\
& + \frac{1}{a^2}\cdot\frac{\partial u}{\partial \lambda}\cdot\frac{\partial^2 v}{\partial \lambda\,\partial x} + \frac{1}{a^2}\cdot\frac{\partial v}{\partial x}\cdot\frac{\partial^2 u}{\partial \lambda^2} + \frac{v}{a^2}\cdot\frac{\partial^3 u}{\partial \lambda^2\partial x} - \frac{2}{a^3\lambda^3}v^2 \\
& \left. + \frac{4}{a^3}\cdot\frac{\partial u}{\partial \lambda}\cdot\frac{\partial^2 u}{\partial \lambda^2} + \frac{2}{a^3\lambda}\left(\frac{\partial u}{\partial \lambda}\right)^2\right]
\end{aligned}
\qquad [2.4]
$$

where u, v are the axial and radial components of velocity, α, β are the kinematic coefficients of viscocity and visco-elasticity respectively.

The boundary conditions of the problem are

$$u[x, 1] = 0 \quad [2.5]$$

$$[\partial u/\partial\lambda]_{\lambda=0} = 0 \quad [2.6]$$

$$v[x, 0] = 0 \quad [2.7]$$

$$v[x, 1] = v_w = \text{constant.} \quad [2.8]$$

We now introduce a stream function ψ defined by

$$u[x, \lambda] = \frac{1}{a^2\lambda} \cdot \frac{\partial\psi}{\partial\lambda} \quad [2,9]$$

$$v[x, \lambda] = -\frac{1}{a\lambda} \cdot \frac{\partial\psi}{\partial x} \quad [2.10]$$

This ψ satisfies the equation of continuity identically.

Following Berman[2] we write the streamfunction in the form

$$\psi[x, \lambda] = g[x]\ \phi[\lambda]. \quad [2.11]$$

Then the velocity components are given by

$$u = \frac{1}{a^2\lambda}\ g[x]\ \phi'[\lambda] \quad [2.12]$$

$$v = -\frac{1}{a\lambda}\ g'[x]\ \phi[\lambda]. \quad [2.13]$$

The boundary condition [2 8] together with the inlet conditions to the pipe give the expression for $g[x]$ as

$$g[x] = \frac{1}{\phi[1]} \left[\frac{a^2\,\bar{u}_0}{2} - a\,v_\omega\,x\right] \quad [2.14]$$

where $\bar{u}_0$ is the average axial velocity at $x = o$ given by

$$\bar{u}_0 = 2\int_0^1 \lambda\, u[o, \lambda]\, d\lambda \quad [2.15]$$

We now write

$$\omega[\lambda] = \frac{\phi'[\lambda]}{\lambda\phi[1]} \quad [2.16]$$

and

$$W[\lambda] = \frac{\phi[\lambda]}{\lambda\phi[1]} = \frac{1}{\lambda}\int_0^\lambda t\,\omega[t]\,dt \qquad [2.17]$$

Then the velocity components and the stream function assume the following forms :

$$u[x,\lambda] = \left(\frac{\bar{u}_0}{2} - \frac{x}{a}v_\omega\right)\omega[\lambda] \qquad [2.18]$$

$$v[x,\lambda] = v_\omega W[\lambda] \qquad [2.19]$$

and

$$\psi[x,\lambda] = \left(\frac{a^2\bar{u}_0}{2} - a\,v_\omega\,x\right)\lambda W[\lambda] \qquad [2.20]$$

In the above equations, $w[\lambda]$ is some function of the distance parameter λ, yet to be determined. We also note that since the suction velocity v_w is taken to be constant, the radial component of velocity becomes a function of λ only.

Substituting for u, v from [2.18] and [2.19] into the equations of motion [2.3] and [2.4], we get

$$-\frac{1}{\rho}\cdot\frac{\partial p}{\partial x} = \left(\frac{\bar{u}_0}{2} - \frac{x}{a}v_\omega\right)\left[-\frac{v_\omega}{a}[\omega^2 - W\omega'] - \frac{\alpha}{a^2}\left(\omega'' + \frac{\omega'}{\lambda}\right)\right.$$
$$+\beta\frac{v_\omega}{a^3}\left(4\omega\omega'' + 4\omega'^2 + \frac{4}{\lambda}\omega\,\omega' - W\omega''' - \omega' W'' - \frac{\omega' W'}{\lambda}\right.$$
$$\left.\left. - 2\,\omega'' W' - \frac{1}{\lambda}W\omega''\right)\right] \qquad [2.21]$$

$$-\frac{1}{\rho}\cdot\frac{\partial p}{\partial\lambda} = v_\omega^2\,WW' - \frac{\alpha}{a}v_\omega\left[W'' + \frac{1}{\lambda^2}[\lambda W' - W]\right]$$
$$-\beta\frac{v_\omega^2}{a^2}\left[10\,W' W'' + 2\,W W''' + \frac{4}{\lambda}W'^2 + \frac{2}{\lambda}WW''\right.$$
$$-\frac{2}{\lambda^2}WW' + 4\,\omega\,\omega' - W'\omega' - W\omega'' - \frac{2}{\lambda^3}W^2$$
$$\left. + \frac{4}{v_\omega^2}\left\{\frac{\bar{u}_0}{2} - \frac{x}{a}v_\omega\right\}^2\omega'\omega'' + \frac{2}{\lambda v_\omega^2}\left\{\frac{\bar{u}_0}{2} - \frac{x}{a}v_\omega\right\}^2\omega'^2\right] \qquad [2.22]$$

where the dashes denote differentiation with respect to λ.

Eliminating p between [2.21] and [2.22] we have

$$\left(\frac{\bar{u}_0}{2}-\frac{x}{a}v_\omega\right)\left[\frac{v_\omega}{a}[2\,\omega\,\omega'-W\omega''-W'\omega']+\frac{\alpha}{a^2}\left(\omega'''+\frac{1}{\lambda}\omega''-\frac{1}{\lambda^2}\omega'\right)\right.$$

$$-\beta\frac{v_\omega}{a^3}\left\{4\,\omega'\omega''-3\,W'\omega'''-3\,W''\omega''-\frac{2}{\lambda}W'\omega''+4\,\omega\,\omega'''\right.$$

$$-W\omega^{iv}+\frac{4}{\lambda}\omega\,\omega''-\frac{4}{\lambda^2}\omega\,\omega'-\omega'\left(W'''+\frac{1}{\lambda}W''-\frac{1}{\lambda^2}W'\right)$$

$$\left.\left.-\frac{1}{\lambda}W\omega'''+\frac{1}{\lambda^2}W\omega''\right\}\right]=0 \qquad [2.23]$$

If [2.23] is to be satisfied for all x, then

$$R\,[2\,\omega\,\omega'-W\omega''-W'\omega']+\left(\omega'''+\frac{1}{\lambda}\omega''-\frac{1}{\lambda^2}\omega'\right)$$

$$-\frac{\beta}{a^2}R\left\{4\,\omega'\omega''-3\,W'\omega'''-3\,W''\omega''-\frac{2}{\lambda}W'\omega''\right.$$

$$+4\,\omega\,\omega'''-W\omega^{iv}+\frac{4}{\lambda}\omega\,\omega''-\frac{4}{\lambda^2}\omega\,\omega'$$

$$\left.-\omega'\left(W'''+\frac{1}{\lambda}W''-\frac{1}{\lambda^2}W'\right)-\frac{1}{\lambda}W\omega'''+\frac{1}{\lambda^2}W\omega''\right\}=0 \qquad [2.24]$$

where $R=av_\omega/\alpha$ is the cross flow Reynolds number. [2.25]

The problem thus reduces to the solution of a fourth order non-linear ordinary differential equation. If the visco-elastic coefficient β vanishes, we find that [2.24] reduces to Berman's[4] equation for the Newtonian fluids.

From [2.8], [2.18] and [2.19], the boundary conditions on the functions $w[\lambda]$, $W[\lambda]$ are

$$\omega\,[1]=0=\omega'\,[0],\quad W[0]=0,\quad W[1]=1 \qquad [2.26]$$

If we consider the limiting form of [2.24] as R tends to zero, we have a simple third order equation,

$$\omega'''+[1/\lambda]\,\omega''-[1/\lambda^2]\,\omega'=0$$

the solution of which subject to [2.26] leads to a function $\omega\,[\lambda]$ describing the well known Poiseuille flow in a cylindrical pipe. Thus we can study the deviations from the Poiseuille flow by a perturbation method where the cross-flow Reynolds number R is used as a perturbation parameter. The solution thus obtained will be valid for sufficiently small values of R.

PERTURBATION SOLUTION

3. We expand the functions $w[\lambda]$ and $W[\lambda]$ in the form:

$$\omega[\lambda] = \omega_0[\lambda] + R\,\omega_1[\lambda] + R^2\,\omega_2[\lambda] + \cdots + R^n\,\omega_n[\lambda] + \cdots \qquad [3.1]$$

and
$$W[\lambda] = W_0[\lambda] + R\,W_1[\lambda] + R^2\,W_2[\lambda] + \cdots + R^n\,W_n[\lambda] + \cdots \qquad [3.2]$$

where
$$W_n[\lambda] = \frac{1}{\lambda}\int_0^{\lambda} t\,\omega_n[t]\,dt \qquad [3.3]$$

and w_n, W_n are taken to be independent of R. Substituting [3.1] and [3.2] in [2.24] and equating the coefficients of various powers of R to zero, we get the following set of equations:

Zeroth order approximation

$$\omega_0''' + \frac{1}{\lambda}\,\omega_0'' - \frac{1}{\lambda^2}\,\omega_0' = 0 \qquad [3.4]$$

First order approximation

$$\begin{aligned}
\omega_1''' + \frac{1}{\lambda}\,\omega_1'' - \frac{1}{\lambda^2}\,\omega_1' &= W_0\,\omega_0'' + W_0'\,\omega_0' - 2\omega_0\,\omega_0' \\
&+ \frac{\beta}{a^2}\Big\{4\,\omega_0'\,\omega_0'' - 3\,W_0'\,\omega_0''' - 3\,W_0''\,\omega_0'' - \frac{2}{\lambda}\,W_0'\,\omega_0'' + 4\omega_0\,\omega_0''' \\
&- W_0\,\omega_0^{iv} + \frac{4}{\lambda}\,\omega_0\,\omega_0'' - \frac{4}{\lambda^2}\,\omega_0\,\omega_0' - \omega_0'\left(W_0''' + \frac{1}{\lambda}W_0'' - \frac{1}{\lambda^2}\,W_0'\right) \\
&- \frac{1}{\lambda}W_0\,\omega_0''' + \frac{1}{\lambda^2}W_0\,\omega_0''\Big\} = 0
\end{aligned} \qquad [3.5]$$

Second order approximation

$$\begin{aligned}
\omega_2''' + \frac{1}{\lambda}\,\omega_2'' - \frac{1}{\lambda^2}\,\omega_2' &= W_0\,\omega_1'' + W_1\,\omega_0'' + W_1'\,\omega_0' + \omega_1'\,W_0' \\
&- 2\,\omega_0\,\omega_1' - 2\,\omega_1\,\omega_0' + \frac{\beta}{a^2}\Big\{4\,[\omega_0'\,\omega_1'' + \omega_0''\,\omega_1'] - 3\,[\omega_0'''\,W_1' + \omega_1'''\,W_0']
\end{aligned}$$

$$- 3[\omega_0'' W_1'' + \omega_1'' W_0''] - \frac{2}{\lambda} [\omega_0'' W_1' + \omega_1'' W_0'] + 4 [\omega_0 \omega_1''' + \omega_1 \omega_0''']$$

$$- [W_0 \omega_1^{iv} + W_1 \omega_0^{iv}] + \frac{4}{\lambda} [\omega_0 \omega_1'' + \omega_1 \omega_0''] - \frac{4}{\lambda^2} [\omega_0 \omega_1' + \omega_1 \omega_0']$$

$$- \omega_0' \left(W_1''' + \frac{1}{\lambda} W'' - \frac{1}{\lambda^2} W' \right) + \frac{1}{\lambda^2} [W_0 \omega_1'' + W_1 \omega_0'']$$

$$- \omega_1' \left(W_0''' + \frac{1}{\lambda} W_0'' - \frac{1}{\lambda^2} W_0' \right) - \frac{1}{\lambda} [W_0 \omega_1''' + W_1 \omega_0'''] \Big\} = 0 \qquad [3.6]$$

and so on.

The conditions to be satisfied by w_n and W_n are

$$\omega_n [1] = \omega_n' [0] = 0 \text{ for } n = 0, 1, 2, \cdots \qquad]3.7]$$

$$W_0 [1] = 1 , \qquad W_n [1] = 0 \text{ for } n \geqslant 1 . \qquad [3.8]$$

We can easily check that the various order solutions are given by

$$\omega^{(0)} [\lambda] = -4 [\lambda^2 - 1] \qquad [3.9]$$

$$W^{(0)} [\lambda] = -\lambda [\lambda^2 - 2] ; \qquad [3.10]$$

$$\omega^{(1)} [\lambda] = -4 [\lambda^2 - 1] + R \left(\frac{2}{9} - \lambda^2 + \lambda^4 - \frac{2}{9} \lambda^6 \right) \qquad [3.11]$$

$$W^{(1)} [\lambda] = -\lambda [\lambda^2 - 2] + R \left(\frac{1}{9} \lambda - \frac{1}{4} \lambda^3 + \frac{1}{6} \lambda^5 - \frac{1}{36} \lambda^7 \right) ; \qquad [3.12]$$

$$\omega^{(2)} [\lambda] = -4 [\lambda^2 - 1] + R \left(\frac{2}{9} - \lambda^2 + \lambda^4 - \frac{2}{9} \lambda^6 \right)$$

$$+ R^2 \left[\frac{83}{1350} - \frac{38}{135} \lambda^2 + \frac{11}{36} \lambda^4 - \frac{1}{9} \lambda^6 + \frac{1}{36} \lambda^8 - \frac{1}{450} \lambda^{10} \right.$$

$$\left. + \frac{8\beta}{a^2} \left(\frac{1}{5} - \frac{83}{90} \lambda^2 + \lambda^4 - \frac{1}{3} \lambda^6 + \frac{1}{18} \lambda^8 \right) \right] \qquad [3.13]$$

$$W^{(2)} [\lambda] = -\lambda [\lambda^2 - 2] + R \left(\frac{1}{9} \lambda - \frac{1}{4} \lambda^3 + \frac{1}{6} \lambda^5 - \frac{1}{36} \lambda^7 \right)$$

$$+ R^2 \left[\frac{83}{2700} \lambda - \frac{19}{270} \lambda^3 + \frac{11}{216} \lambda^5 - \frac{1}{72} \lambda^7 + \frac{1}{360} \lambda^9 \right.$$

$$\left. - \frac{1}{5400} \lambda^{11} + \frac{8\beta}{a^2} \left(\frac{1}{10} \lambda - \frac{83}{360} \lambda^3 + \frac{1}{6} \lambda^5 - \frac{1}{24} \lambda^7 + \frac{1}{180} \lambda^9 \right) \right] \qquad [3.14]$$

and so on.

Using the non-dimensional parameters

$$N_{Re} = [a\,\bar{u}_0/\alpha] = \text{inlet flow Reynolds number} \qquad [3.15]$$

and

$$K = \beta/a^2 \qquad [3.16]$$

the expressions for the velocity components in the non-dimensional form are given by

$$\frac{u[x,\lambda]}{\bar{u}_0} = \left(\frac{1}{2} - \frac{x}{a}\cdot\frac{R}{N_{Re}}\right)\Bigg[4\,[1-\lambda^2] + R\left(\frac{2}{9} - \lambda^2 + \lambda^4 - \frac{2}{9}\lambda^6\right)$$

$$+ R^2\left\{\frac{83}{1350} - \frac{38}{135}\lambda^2 + \frac{11}{36}\lambda^4 - \frac{1}{9}\lambda^6 + \frac{1}{36}\lambda^8 - \frac{1}{450}\lambda^{10}\right.$$

$$\left.\left. + 8K\left(\frac{1}{5} - \frac{83}{90}\lambda^2 + \lambda^4 - \frac{1}{3}\lambda^6 + \frac{1}{18}\lambda^8\right)\right\}\right] \qquad [3.17]$$

$$\frac{v[\lambda]}{v_\omega} = \lambda\,[2-\lambda^2] + R\left(\frac{1}{9}\lambda - \frac{1}{4}\lambda^3 + \frac{1}{6}\lambda^5 - \frac{1}{36}\lambda^7\right)$$

$$+ R^2\left\{\frac{83}{2700}\lambda - \frac{19}{270}\lambda^3 + \frac{11}{216}\lambda^5 - \frac{1}{72}\lambda^7 + \frac{1}{360}\lambda^9 - \frac{1}{5400}\lambda^{11}\right.$$

$$\left. + 8K\left(\frac{1}{10}\lambda - \frac{83}{360}\lambda^3 + \frac{1}{6}\lambda^5 - \frac{1}{24}\lambda^7 + \frac{1}{180}\lambda^9\right)\right\} \qquad [3.18]$$

From [2.21], [2.22], [3.13] and [3.14] we get the pressure distribution :

$$\int_0^x \frac{\partial p[x,\lambda]}{\partial x}\,dx = p\,[x,\lambda] - p\,[0,\lambda] \qquad [3.19]$$

$$\int_0^\lambda \frac{\partial p[x,\lambda]}{\partial \lambda}\,d\lambda = p\,[x,\lambda] - p\,[x,0] \qquad [3.20]$$

From [3.19] and [3.20] it follows that

$$\int_0^x \frac{\partial p}{\partial x}\,dx + \left(\int_0^\lambda \frac{\partial p}{\partial \lambda}\,d\lambda\right)_{\lambda=0} = p\,[x,\lambda] - p\,[0,0] \qquad [3.21]$$

where $p\,[0,0]$ is the pressure at the entrance of the pipe on the axis.

It is interesting to calculate the pressure drop in the axial direction. From [3.21] we have

$$\frac{p[0,\lambda]-p[x,\lambda]}{\frac{1}{2}\rho\bar{u}_0^2}=\frac{x}{a N_{Re}}\left(1-\frac{x}{a}\cdot\frac{R}{N_{Re}}\right)\Bigg[16-R\{12+192\,K[1-2\lambda^2]\}$$

$$-R^2\left\{\frac{88}{135}+32\,K\left(\frac{41}{45}-6\lambda^2+10\lambda^4-\frac{28}{9}\lambda^6\right)\right\}\Bigg] \quad [3.22]$$

This completes the solution for the laminar flow of visco-elastic fluids in a porous pipe. The results hold for $|R|\leqslant 1$. Positive values of R [or v_w] represent suction, while its negative values represent injection at the wall. We may also deduce the solutions for the Newtonian flluid as particular cases from the above solutions by taking $K=0$.

Discussion of the Results

4. No doubt our expansion method is strictly valid when $|R|<1$, but in our numerical work we find that even when $|R|=1$, the various order corrections decrease rapidly. Hence we have taken $R=1$ in the following discussion.

(i) *Velocity field*: It can be seen from [3.18] that the radial component of velocity depends only on the radial distance and is independent of the axial distance.

In the absence of cross flow, *i.e.*, $R=0$, the axial velocity is given by

$$\frac{u[x,\lambda]}{\bar{u}_0}=\frac{u[\lambda]}{\bar{u}_0}=2[1-\lambda^2] \quad [4.1]$$

which describes the well-known flow in a solid wall pipe. The profile is parabolic with its maximum occuring at the centre of the pipe. This maximum value is given by

$$\left[\frac{u[x,o]}{\bar{u}_0}\right]_{R=0}=2 \quad [4.2]$$

For small suction flows $[0<R\leqslant 1]$, we find from [3.17] that the centre line value of the volocity decreases as the distance along the axis increases and that the presence of visco-elasticity causes the axial velocity to decrease much more rapidly than in the case of Newtonian fluids. For injection at the pipe wall $[-1\leqslant R<0]$, the axial velocity increases as the distance along the axis increases.

If we compare the axial velocity profiles with the parabolic profile of the Poiseuille flow for a solid wall pipe [Fig. I], we find that for small suction

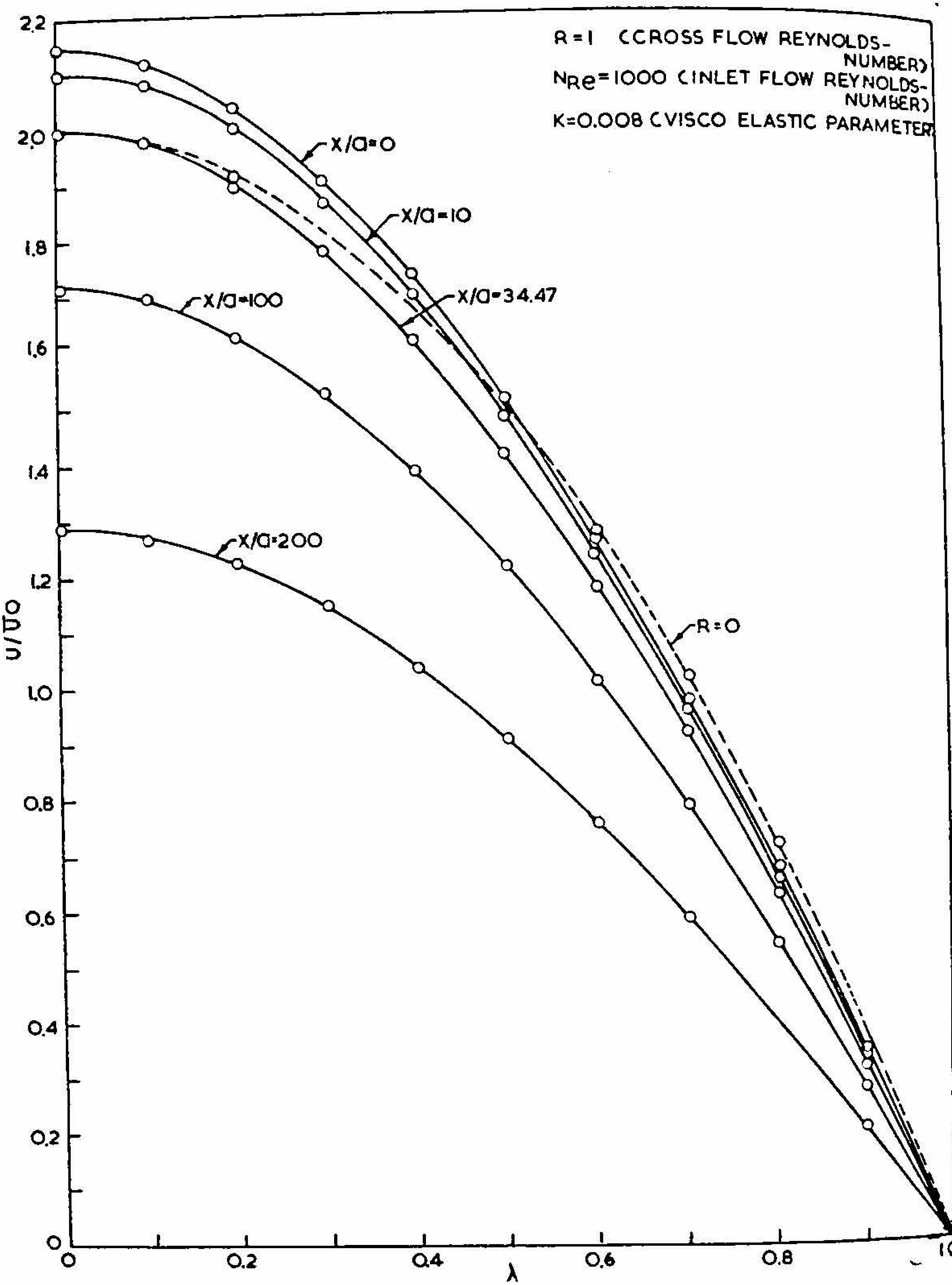

FIG. I

Axial velocity profile for suction ($R=1$) at the entrance of the pipe and far away from the entrance at the pipe wall, the profiles bulge out at the centre of the pipe more than the Poiseuille parabola upto a certain distance along the axis after which they start flattening and become rod-like till all the fluid is sucked out. We may call this point as a separation point, since a transition occurs in the behaviour of the velocity profile at this point, although the centre line value of the velocity at this point is same as that for the Poiseuille flow.

We may calculate the distance from the inlet of the pipe to the separation point by solving for $[x/a]$ from [3.17] by putting $\lambda = 0$ on both sides of the equation and then making use of [4.2]. For example, in the case of $R = 1$, $N_{Re} = 1000$ and $K = .008$, we find that x/a is equal to 34.47. Thus upto this distance the velocity profiles bulge more than the parabolic profile, while beyond this distance the profiles flatten. For injection flows also, a similar phenomenon takes place.

For example in the case of $R = -1$, $N_{Re} = 1000$ and $K = .008$, the separation takes place at $x/a = 19.24$ Thus for $x/a < 19.24$, the velocity profiles remain more flattened than the parabolic profile, while beyond this distance, the centre line value of the velocity is more than the parabolic profile value.

Further, we notice from [3.17], that since $[x/a\,N_{Re}]$ is small, as we increase the suction in the range $0 < R \leqslant 1$, the centre-line value of the velocity increases and the slope of the velocity profiles decreases at the wall while in the case of small injection' the centre-line value of the velocity decreases, thereby increasing the slope of the profile at the pipe wall. This may also be noticed from Figs. I and II. In this respect the visco-elastic fluids are similar to the Newtonian fluids [Berman[4]].

(ii) *Pressure field*: Fig. III gives graphically the pressure drop along the axis as well as along the pipe walls for small suction and injection flows of visco-elastic fluids for an arbitrary choice of $N_{Re} = 1000$ and $K = .008$. The graph also illustrates the x-dependence of pressure in a solid wall pipe as well as the x-dependence of pressure in a pipe for small injection and suction for Newtonian fluids. We notice that in the case of suction, the pressure drop along the axis is less, while the pressure along the pipe walls is more in the case of visco-elastic fluids, than for Newtonian fluids having the same entrance Reynolds number N_{Re}. In the case of injection, the reverse is true. Thus for suction, the effect of visco-elasticity is to diminish the pressure drop along the axis thereby decreasing the velocity here, while for injection, it is to increase the axial drop of pressure thereby increasing the velocity at the axis.

In case of suction, the effect of visco-elasticity is to increase the pressure drop at the wall thereby increasing the velocity at the pipe walls. The reverse is true for injection.

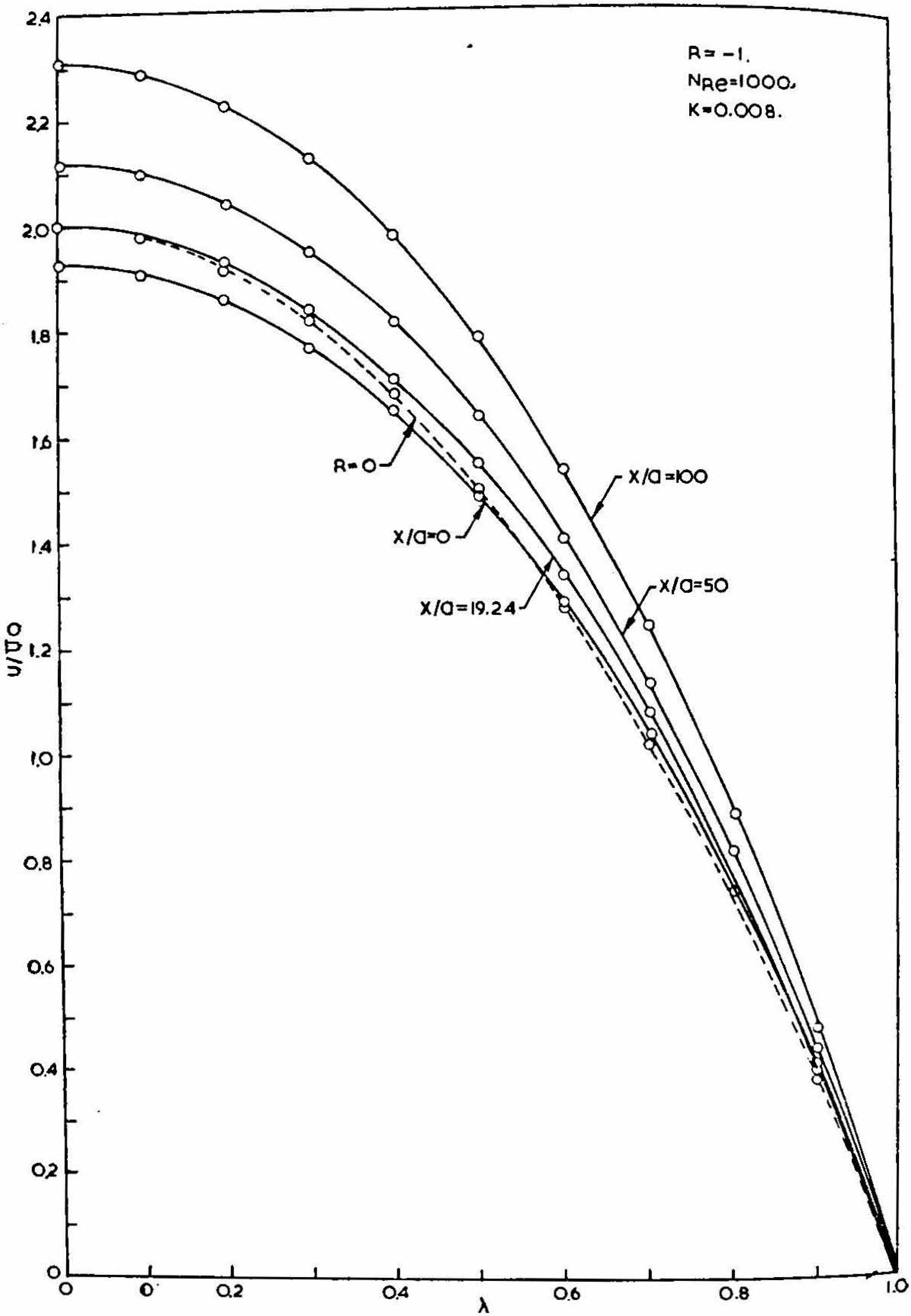

FIG. II
Axial velocity profile for injection (R=1) at the entrance of the pipe and far away from it

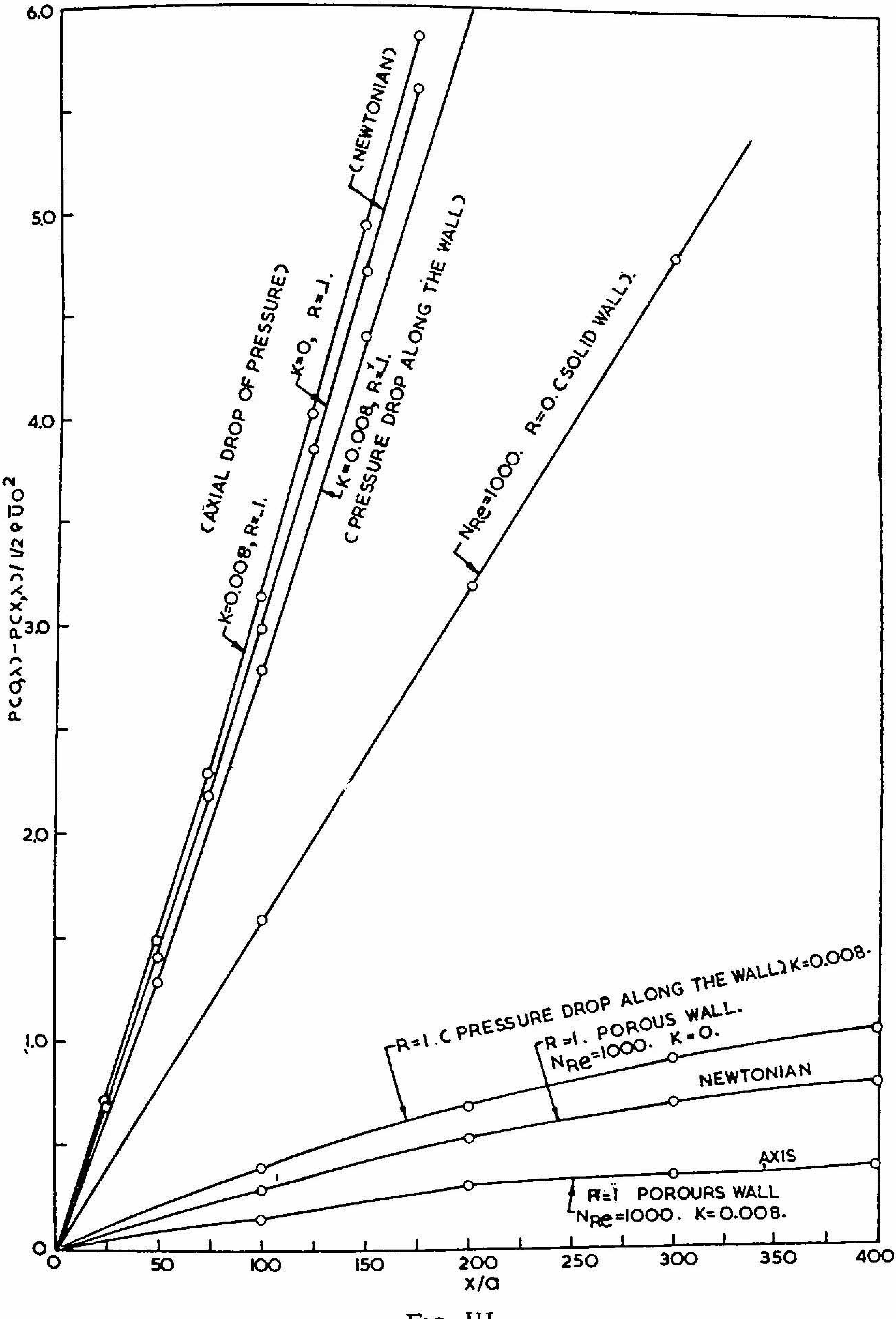

FIG. III
Pressure drop in the axial direction and along the wall in pipes

FLOW VISUALISATION

5. To visualize the flow pattern, we have drawn in Fig. IV the stream-lines in a meridian plane starting from the inlet with the help of the stream function in the non-dimensional form :

$$\Psi[x,\lambda]=\frac{\psi[x,\lambda]}{a^2\bar{u}_0}=\left(\frac{1}{2}-\frac{x}{a}\cdot\frac{R}{N_{Re}}\right)\lambda^2\left[[2-\lambda^2]+R\left(\frac{1}{9}-\frac{1}{4}\lambda^2\right.\right.$$
$$\left.+\frac{1}{6}\lambda^4-\frac{1}{36}\lambda^4\right)+R^2\left\{\frac{83}{2700}-\frac{19}{270}\lambda^2+\frac{11}{216}\lambda^4-\frac{1}{72}\lambda^6\right.$$
$$\left.\left.+\frac{1}{360}\lambda^8-\frac{1}{5400}\lambda^{10}+8K\left(\frac{1}{10}-\frac{83}{360}\lambda^2+\frac{1}{6}\lambda^4-\frac{1}{26}\lambda^6+\frac{1}{180}\lambda^8\right)\right\}\right] \quad [5.1]$$

when $R=1$, $N_{Re}=1000$ and $K=.008$

STRESSES IN THE FLOW FIELD

6. The stress components are given by

$$T_{x\theta}=T_{r\theta}=0,\quad \text{and } T_{xx}=-p \quad [6.1]$$

$$T_{\theta\theta}=-p\,\rho\alpha\left(\frac{2v}{a\lambda}\right)+\rho\beta\left(v^2+v\lambda\frac{\partial v}{\partial\lambda}\right) \quad [6.2]$$

$$T_{rr}=-p\frac{2\alpha\rho}{a}\cdot\frac{\partial v}{\partial\lambda}+2\rho\beta\left[\frac{v}{a^2}\cdot\frac{\partial^2 v}{\partial\lambda^2}+\frac{2}{a^2}\left(\frac{\partial v}{\partial\lambda}\right)^2+\frac{1}{a^2}\left(\frac{\partial u}{\partial\lambda}\right)^2\right] \quad [6.3]$$

$$T_{rx}=\frac{\alpha\rho}{a}\cdot\frac{\partial u}{\partial\lambda}+\frac{\rho\beta}{a^2}\left[v\frac{\partial^2 u}{\partial\lambda^2}+\frac{\partial u}{\partial\lambda}\cdot\frac{\partial v}{\partial\lambda}\right] \quad [6.4]$$

The shearing stress on the wall rendered dimensionless with the help of $\bar{u}_0\,\alpha\,\rho/a$ is given by

$$\frac{T_{rx}]_{\lambda=1}}{\bar{u}_0\,\alpha\,\rho/a}=\left(\frac{1}{2}-\frac{x}{a}\cdot\frac{R}{N_{Re}}\right)\left[-8+\frac{2}{3}R+R^2\left(\frac{26}{135}+\frac{24}{5}K\right)\right.$$
$$\left.+KR^2\left\{\frac{8}{3}+R\left(\frac{14}{15}+\frac{64}{3}K\right)\right\}\right] \quad [6.5]$$

In Fig. V, the shearing stress on the wall is plotted against x/a for an arbitrary choice of $K=0$, 0.3 and $N_{Re}=1000$ for suction and injection flows. We notice that for a solid wall pipe, the shearing stress along the wall is constant, its modulus being equal to 4. The effect of visco-elasticity, both for suction and injection, is to decrease the shearing stress at the wall. But in both the cases, the shearing stress increases, as the axial distance along the wall increases, as seen from the graph.

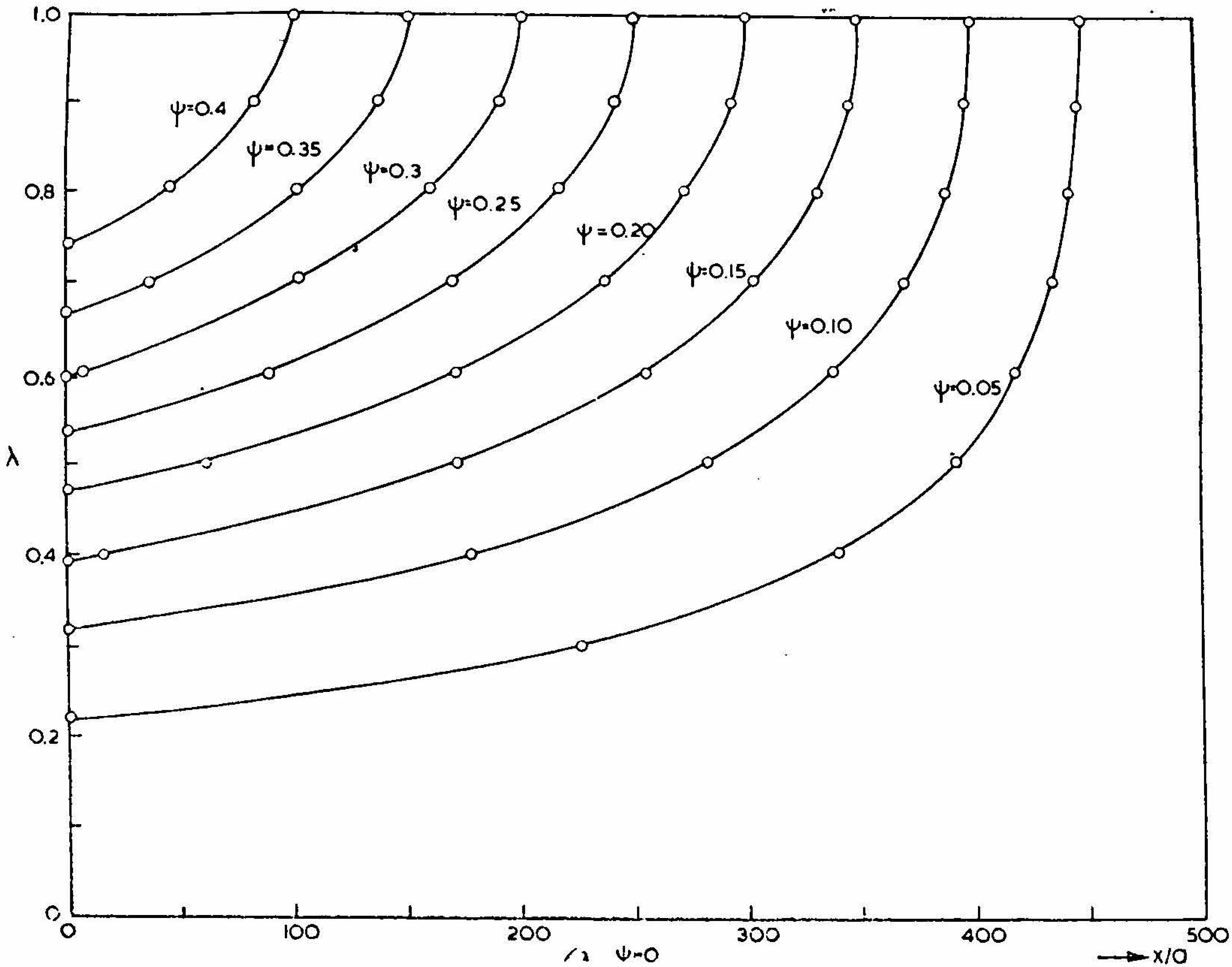

FIG. IV
Non Newtonian flow lines in the Meridian Plane for a pipe with porous walls $R=1$ and N_{Re} 1000

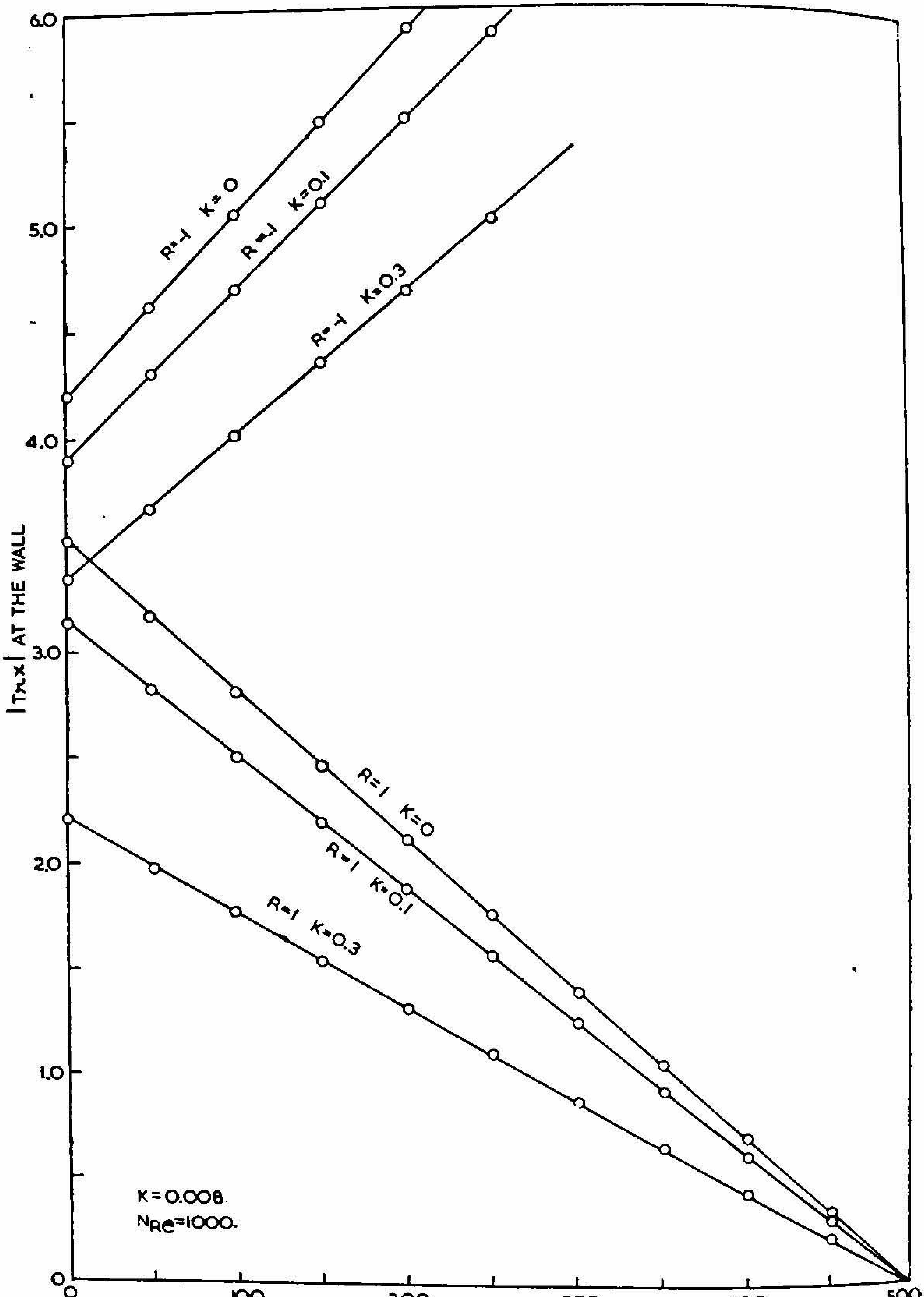

FIG. V

Variation of Shearing stress at the wall

The skin friction on the wall of a cylinder of length l and radius a is given by

$$D = \frac{1}{2}\, l \left[a - l\,\frac{R}{N_{Re}}\right] \left[-8 + \frac{2}{3} R + R^2\left(\frac{26}{135} + \frac{24}{5} K\right) + K R^2 \left\{\frac{8}{3} + R\left(\frac{14}{15} + \frac{64}{3} K\right)\right\}\right] \qquad [6.6]$$

PART B

FLOW IN AN ANNULUS

7. We shall now study the steady laminar flow of visco-elastic fluid in the region bounded by two porous walls of two concentric tubes of radii a and b $[< a]$. We assume that the rate of fluid injection at one wall is equal to the rate of fluid suction at the other wall. This condition is satisfied

if
$$b\, v_b = a\, v_a \qquad [7.1]$$

where v_b and v_a are the radial velocities at the walls of the smaller and larger tubes respectively, and then the axial velocity does not depend on x.

If we choose a cylindrical system of coordinates with x-axis along the common axis of of the cylinders, the equations of continuity and momentum are same as [2.2[, [2.3] and [2.4], where now $\lambda = r/a$. The range for the dimensionless parameter λ is given by

$$\sigma = [b/a] \leqslant \lambda \leqslant 1 \qquad [7.2]$$

From [2.4] and [7.1] we have

$$v\,[\lambda] = \frac{v_a}{\lambda} = \sigma\,\frac{v_b}{\lambda} \qquad [7.3]$$

The equations [2.2] and [2.3] reduce to

$$\frac{a^2}{\phi_1}\cdot\frac{\partial p}{\partial x} = \frac{\beta R}{a^2 \lambda}\, u''' + \left(1 - \frac{\beta}{a^2}\cdot\frac{R}{\lambda^2}\right) u'' - \frac{u'}{\lambda}\left[\,[R-1] - \frac{\beta}{a^2}\cdot\frac{R}{\lambda^2}\right] \qquad [7.4]$$

and
$$\frac{\partial p}{\partial \lambda} = \frac{\rho\, v_a^2}{\lambda^3} + \phi_2\left[\frac{-24\, v_a^2}{a^2 \lambda^5} + \frac{2}{a^2 \lambda}\,[u'^2 + 2\,\lambda\, u'\, u'']\right] \qquad [7.5]$$

From [7.5] it is clear that $[\partial^2 p/\partial x\, \partial\lambda] = 0$ [7.6]

Thus if we put $[a^2/\phi_1]\,[\partial p/\partial x] = c$ [7.7] the equation [7.4] reduces to

$$S u''' + \frac{d}{d\lambda}[u' \lambda] - S \frac{d}{d\lambda}\left(\frac{u'}{\lambda}\right) - u' R = c\lambda \qquad [7.8]$$

where $KR = S.$

The boundary conditions on u are

$$u[\lambda] = 0 \text{ at } \lambda = \sigma \text{ and } \lambda = 1. \qquad [7.9]$$

Taking S as the perturbation parameter and regarding it as small, we assume the following series expansion for the axial component of the velocity:

$$u = \sum_{m=0}^{\infty} S^m u_m \qquad [7.10]$$

where $u_m[1] = u_m[\sigma] = 0$ for all m.

From [7.8] and [7.10] we have

$$\frac{d}{d\lambda}[\lambda u_0'] - R u_0' = c\lambda \qquad [7.12]$$

$$\frac{d}{d\lambda}[\lambda u_m'] - R u_m' = \frac{d}{d\lambda}\left(\frac{u_{m-1}'}{\lambda}\right) - u_{m-1}''', \quad \text{for } m \geqslant 1 \qquad [7.13]$$

Integrating the equations [7.12] and [7.13] and using the boundary conditions [7.11], we have the following expressions for the various order approximations to u:

$$u_0 = \frac{c}{2[2-R]}\left[-[1-\lambda^2] + \frac{1-\sigma^2}{1-\sigma^R}[1-\lambda^R]\right] \qquad [7.14]$$

$$u_1 = -\frac{CR}{4}\left(\frac{1-\sigma^2}{1-\sigma^R}\right)\left[[1-\lambda^{R-2}] - \left(\frac{1-\sigma^{R-2}}{1-\sigma^R}\right)[1-\lambda^R]\right] \qquad [7.15]$$

and

$$u_2 = -\frac{CR[R-2]}{16} \cdot \frac{1-\sigma^2}{[1-\sigma^R]^2}\left[\lambda^R\left\{[R-4][1-\sigma^{R-4}] - \frac{2R[1-\sigma^{R-2}]^2}{1-\sigma^R}\right\}\right.$$
$$+ 2R[1-\sigma^{R-2}]\lambda^{R-2} - [R-4][1-\sigma^R]\lambda^{R-4}$$
$$\left. + \sigma^{R-4}[1-\sigma^2]\left\{[R-4][1+\sigma^2] - 2R\left(\frac{1-\sigma^{R-2}}{1-\sigma^R}\right)\sigma^2\right\}\right] \qquad [7.16]$$

In general, we have u_m in the form

$$u_m = \lambda^R \int \frac{1}{\lambda^R}\left\{\frac{u'_{m-1}}{\lambda^2} - \frac{u''_{m-1}}{\lambda}\right\} d\lambda - \frac{A_m}{R} + B_m \lambda^R \qquad [7.17]$$

where A_m and B_m can be determined with the help of the boundary conditions [7.11]. Retaining powers of S upto the second only we have,

$$\frac{u}{c} = \frac{1}{2[2-R]}\Bigg[-[1-\lambda^2] + \frac{[1-\sigma^2]}{[1-\sigma^R]}[1-\lambda^R]\Bigg]$$

$$-\frac{KR^2}{4}\left(\frac{1-\sigma^2}{1-\sigma^R}\right)\left[[1-\lambda^{R-2}] - \frac{1-\sigma^{R-2}}{1-\sigma^R}[1-\lambda^R]\right]$$

$$-\frac{K^2R^3[R-2]}{16}\cdot\frac{1-\sigma^2}{[1-\sigma^R]^2}\Bigg[\lambda^R\Bigg\{[R-4][1-\sigma^{R-4}] - \frac{2R[1-\sigma^{R-2}]^2}{1-\sigma^R}\Bigg]$$

$$+2R[1-\sigma^{R-2}]\lambda^{R-2} - [R-4][1-\sigma^R]\lambda^{R-4}$$

$$+\sigma^{R-4}[1-\sigma^2]\left\{[R-4][1+\sigma^2] - 2R\frac{1-\sigma^{R-2}}{1-\sigma^R}\sigma^2\right\}\Bigg] \qquad [7.18]$$

The average axial velocity in the annulus is given by

$$\bar{u} = \frac{2}{\sigma^2 - 1}\int_1^\sigma \lambda u \, d\lambda \qquad [7.19]$$

Using [7.18] in [7.19] we have,

$$\frac{\bar{u}}{c} = -\frac{1}{2[4-R^2]}\left[\frac{[2+R]+\sigma^2[2-R]}{2} - R\frac{1-\sigma^2}{1-\sigma^R}\right]$$

$$-\frac{KR}{4[2+R][1-\sigma^R]^2}\{R^2[1-\sigma^2]^2\sigma^{R-2} - 4[1-\sigma^R]^2\}$$

$$-\frac{K^2R^3[R-2]}{8[1-\sigma^R]^2}\Bigg[\frac{R}{R-2}[1-\sigma^R][1-\sigma^{R-2}] - \frac{1}{2+R}\cdot\frac{1-\sigma^{R-2}}{[1-\sigma^R]}\{R\sigma^{R-4}[1-\sigma^2]^2$$

$$+R[1-\sigma^{R-2}]^2 + 4[1-\sigma^R][1-\sigma^{R-4}]\} + \frac{1}{2}\cdot\frac{\sigma^{R-4}[1-\sigma^2]^2}{1-\sigma^R}\times$$

$$\times\{R[1-\sigma^2][1+\sigma^R] - 4[1+\sigma^2][1-\sigma^R]\}\Bigg]. \qquad [7.20]$$

From [7.18] and [7.20] we can easily get the axial velocity component in terms of the average velocity.

The velocity field is now fully defined by [7.3] for the radial component and [7.18] for the axial component.

The pressure field can be obtained after the integration of [7.5] and [7.7]. The radial pressure variation is given by

$$\frac{p[x,1]-p[x,\lambda]}{[\rho v_a^2/2]}$$
$$=\left(\frac{1}{\lambda^2}-1\right)-12K\left(\frac{1}{\lambda^4}-1\right)+\frac{4Kc^2}{v_a^2}\left[\frac{3}{2[R-2]^2}[1-\lambda^2]\right.$$
$$+\frac{2R-1}{2R-2}\cdot\frac{R^2}{4}\left(\frac{\sigma^2-1}{\sigma^R-1}\right)^2\left\{\frac{1}{[R-2]^2}+\frac{3K^2R^4}{4}\left(\frac{\sigma^{R-2}-1}{\sigma^R-1}\right)^2-\frac{KR^2}{R-2}\cdot\frac{\sigma^{R-2}-1}{\sigma^R-1}\right.$$
$$\left.-\frac{K^2R^3[R-4]}{4}\cdot\frac{\sigma^{R-4}-1}{\sigma^R-1}\right\}[1-\lambda^{2R-2}]-\frac{R+1}{R-2}\cdot\frac{\sigma^2-1}{\sigma^R-1}\left\{\frac{1}{R-2}-\frac{KR^2}{2}\cdot\frac{\sigma^{R-2}-1}{\sigma^R-1}\right.$$
$$\left.+\frac{K^2R^4[R-2]}{4}\cdot\left(\frac{\sigma^{R-2}-1}{\sigma^R-1}\right)^2-\frac{K^2R^3[R-2][R-4]}{8}\cdot\frac{\sigma^{R-4}-1}{\sigma^R-1}\right\}[1-\lambda^R]$$
$$-\frac{KR^2}{2}\cdot\frac{R-1}{R-2}\cdot\frac{\sigma^2-1}{\sigma^R-1}\left\{1-\frac{KR^2}{2}[R-2]\frac{\sigma^{R-2}-1}{\sigma^R-1}\right\}$$
$$-\frac{K^2R^3}{8}[R-3][R-4]\frac{\sigma^2-1}{\sigma^R-1}[1-\lambda^{R-4}]$$
$$+\frac{K^2R^3[2R-3]}{8}\cdot\left(\frac{\sigma^2-1}{\sigma^R-1}\right)^2\left\{\frac{1}{R-2}-KR^2\frac{\sigma^{R-2}-1}{\sigma^R-1}\right\}[1-\lambda^{2R-4}]$$
$$\left.+\frac{K^2R^4[R-2]^2}{16}\cdot\frac{2R-5}{2R-6}\left(\frac{\sigma^2-1}{\sigma^R-1}\right)^2[1-\lambda^{2R-6}]\right] \qquad [7.21]$$

Integrating [7.7] with respect to x we get the axial pressure variation in the form

$$\left\{\frac{p[0,\lambda]-p[x,\lambda]}{[\rho\bar{u}^2/2]}\right\}\left(\frac{\bar{u}a}{8\alpha}\right)\left(\frac{a}{x}\right)$$
$$=\left[\frac{2}{4-R^2}\left\{\frac{[2+R]+\sigma^2[2-R]}{2}-\frac{R[1-\sigma^2]}{1-\sigma^R}\right\}\right.$$
$$+\frac{KR}{[2+R][1-\sigma^R]^2}\{R^2[1-\sigma^2]^2\sigma^{R-2}-4[1-\sigma^R]^2\}$$
$$+\frac{K^2R^3[R-2]}{2[1-\sigma^R]^2}\left\{\frac{R}{R-2}[1-\sigma^R][1-\sigma^{R-2}]\right.$$

$$-\frac{1}{2+R}\cdot\frac{1-\sigma^{R+2}}{1-\sigma^{R}}\{R[1-\sigma^{R-2}]^2+R[1-\sigma^2]^2\sigma^{R-4}+4[1-\sigma^R][1-\sigma^{R-4}]\}$$

$$+\frac{1}{2}\sigma^{R-4}\frac{[1-\sigma^2]^2}{1-\sigma^R}\{R[1+\sigma^R][1-\sigma^2]-4[1+\sigma^2][1-\sigma^R]\}\Big]^{-1} \qquad [7.22]$$

$\equiv g[\sigma, R]$, [say]

Positive values of R [or v_a and v_b] represent suction at the outer wall and injection at the inner wall, while the negative values of R represent sunction at the inner wall and injection at the outer wall of the annulus. It may be noted that Berman's solution [1958] for the Newtonian fluid can be obtained as a particular case from our solutions by putting $K = o$.

Discussion of the Results

8. The radial component of velocity given by [7.3] varies inversely proportional to the radial distance.

It is interesting to study the behaviour of the axial component of velocity given by [7.18]. In the absence of cross flow, *i.e.*, $R = 0$, the axial velocity becomes

$$\frac{u[\lambda]}{\bar{u}} = 2\,\frac{[1-\lambda^2]\,ln\,\sigma-[1-\sigma^2]\,ln\,\lambda}{[1+\sigma^2]\,ln\,\sigma+[1-\sigma^2]} \qquad [8.1]$$

as can be seen by taking the limit of [7.18] and [7.20] as R tends to zero. We note that [8.1] does not involve any term containing viscoelasticity. Thus it is same as in the case of Newtonian fluids. Hence for a solid wall annulus, $u[\lambda]/\bar{u}$ is same whether we consider the visco-elastic fluid or the Newtonian fluid. This profile is not symmetrical and the maximum velocity occurs closer to the inner wall of the annulus. When $R = 1$, in the case of non-Newtonian fluids, [injection at the inner wall and suction at the outer wall] the u-profile is non-symmetrical regardless of the radius ratio σ of the annulus as may be easily seen from Fig. VI. This is in contrast to the symmetrical parabolic profile for $R = 1$ in the case of Newtonian fluids.

In Fig. VI we have plotted the axial velocity profiles for various cross flows against a normalised radial distance parameter $\xi = [\lambda-\sigma]/[1-\sigma]$ for a particular value 0.2 of σ and $K = 0005$.

For a rapid convergence of the series solution we restrict S approximately to 0.1. Then if we take $K = .0005$ we can give values to R upto 200 or so.

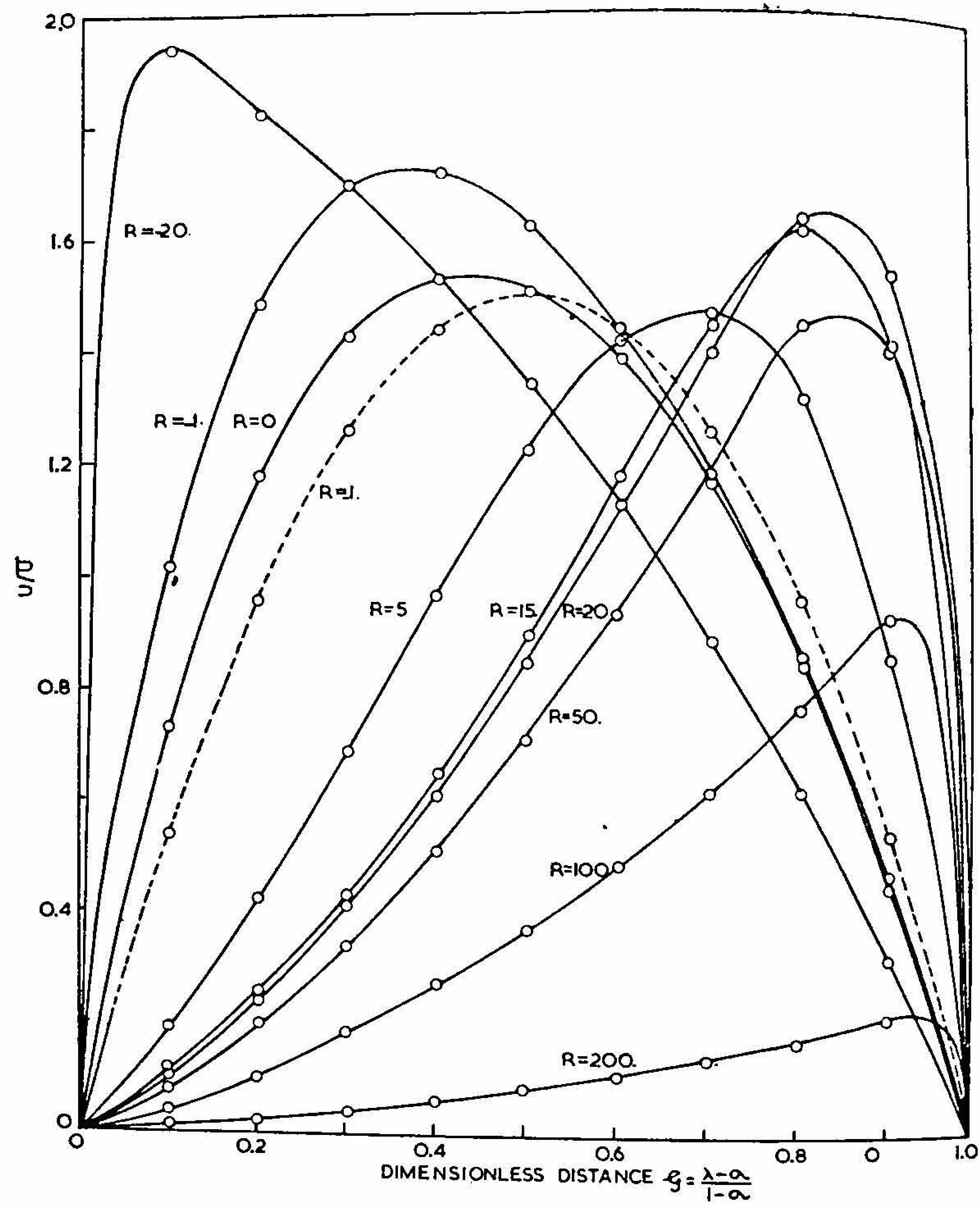
R=-20.
R=-1.
R=0
R=1.
R=5
R=15
R=20
R=50.
R=100
R=200.
u/ū
DIMENSIONLESS DISTANCE $\xi = \frac{\lambda - \alpha}{1-\alpha}$
2.0
1.6
1.2
0.8
0.4
0
0.2
0.4
0.6
0.8
1.0

FIG. VI

Retaining first two terms in [7.18], the points of inflexion of the axial velocity profile are given by

$$\frac{1-\sigma^R}{1-\sigma^2} - \left\{\frac{1}{2} + \frac{KR^2}{4}\cdot\frac{1-\sigma^{R-2}}{1-\sigma^R}[2-R]\right\} R\,[R-1]\,\lambda^{R-2}$$

$$-\frac{KR^2}{4}[R-2]^2\,[R-3]\,\lambda^{R-4} = 0 \qquad [8.2]$$

For $\sigma = .2$, from the numerical evaluation of the roots of the above equation, it is found that there exists a critical value $Rc = 1.7$ above which a point of inflexion in the u-profile occurs in the required range $.2 \leqslant \lambda \leqslant 1$. For negative R, inflexion points do not appear in the u-profile. When the fluid is injected at the outer wall and removed at the inner wall [that is for R negative] the velocity profiles become more asymmetrical as the cross flow velocity increases. For higher cross flow velocities, the maximum occurs nearer to the inner wall.

In the case of fluid injection at the inner wall and removal at the outer wall, the velocity profiles become more asymmetrical as the cross flow velocities increase. From Fig. VI it is quite clear that for a particular case when $\sigma = .2$ and $K = .0005$ the point where the maximum velocity occurs moves closer and closer to the outer wall of the aunulus, as the cross flow velocity increases. Again, we observe that the velocity profiles for R positive lie to the right of the profile corresponding to $R = 0$ and the profiles for R negative lie to the left of it, no matter how large R may be.

(ii) *Pressure field*: The radial pressure variation given by [7.21] depends upon the direction of the cross flow since the values of this variation will be different for R positive and R negative. In this respect, the visco-elastic fluids are different from the Newtonian fluids.

Fig. VII gives the ratio of the pressure drop along the axis with cross flow to that for a solid wall annulus as a function of the cross flow Reynolds number R, for constant average axial velecity. The curves are drawn for $\sigma = 0.1$, 0.2, 0.3, for the visco-elastic fluid $K = .005$ as well for the Newtonian fluid $[K = 0]$ for comparison. It is clear that the axial pressure drop when cross flow is present, is greater than that for solid wall annulus. This difference becomes more significant as σ decreases. This is also the case with the Newtonian fiuids as shown in the graph. We also notice that the effect of visco-elasticity is to increase the pressure drop.

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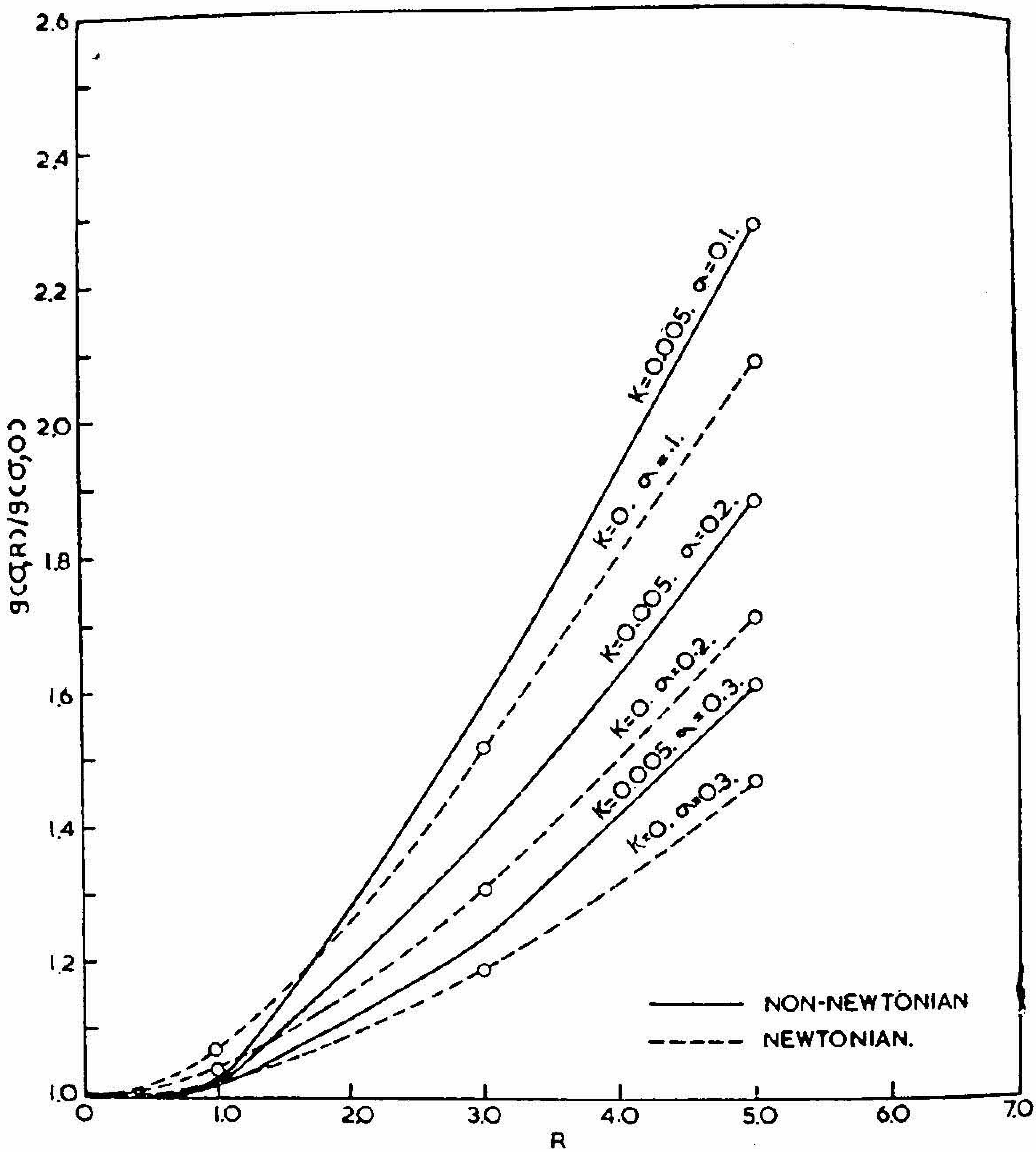

FIG. VII

Ratio of axail pressure function with cross flow to the axial pressure function in a solid wall annulus

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