THERMALLY INDUCED VIBRATIONS OF A RIGHT-ANGLED ISOTROPIC ISOSCELES TRIANGULAR PLATE ON ELASTIC FOUNDATION

B. M. KARMAKAR

(Department of Physics, Government Engineering College, Jalpaiguri, West Bengal, India)

Received on August 5, 1975

ABSTRACT

The solution to the thermal shock-induced vibration, of a right-angled isotropic isosceles triangular plate on elastic foundation, is obtained in a closed form. Nodal lines are also located. Incidentally this confirms the truth of the assumed boundary conditions.

In this paper vibrations of an isotropic right-angled isosceles triangular plate, due to a thermal shock, have been investigated. The solution presented is a rigorous one, since it is not based on assumptions of the type underlying strength of materials analyses.

The plate is considered free of external tractions. The problem is solved in terms of a double trigonometric series. The complete solution is derived from the sum of two deflections—quasi-static and dynamic. The dynamic solution is obtained by the method of Laplace transform.

The results obtained are exhibited in graphs which are found to be qualitatively similar to those of standard works.

Location of nodal lines confirm the validity of the assumed boundary conditions.

Keywords: Boundary value problems; Closed form solution; Foundation; Isotropic; Induced; Nodes; Plates; Shock; Thermal; Trigonometric series; Traingular; Tones; Vibration.

INTRODUCTION

Thermally induced vibrations are of interest in aircraft and machine designs, in chemical and nuclear engineering and even in astronautical engineering.

The general theory of the transverse vibrations of a circular plate was obtained by Kirchoff [1] who gave a full numerical discussion of the results. The problem has also been discussed very extensively by Lord Rayleigh [2]. In the problems of non-stationary quasi-static stresses in plates, the temperature field varies very slowly with time. But in case of sudden heating or for temperature fields varying harmonically with time, the elastic plates undergo some vibrations. Nowacki [3] and Boley and Weiner [4] independently investigated vibrations of rectangular isotropic plates due to thermal shocks. Both set of workers have exihibited results graphically which are qualitatively similar in nature.

In this paper the author has investigated thermally induced vibration of an isoscales right-angled triangular plate placed on elastic foundation. Problems on thermal vibrations on elastic foundations of any plate-shape are not found in earlier works. So the present author has chosen the problem of a triangular plate on an elastic foundation. A traingular plate is more irregular in shape than a rectangular or a circular plate. The form of the deflection w, as also a closed form solution are new features in the present authors' work.

ANALYSIS

Let ABC be the plate (Fig. 1–Inset) bounded by the space defined by

$$x = a, \qquad y = 0 \qquad \text{and} \qquad x = y. \tag{1}$$

The z-axis is through the thickness of the plate, and perpendicular to both the x and y axes. The face $z = +h/2$ is exposed to a step heat input Q, constant along the plate surface; the other face $z = -h/2$ together with all edges are insulated.

For plates of uniform thickness, we start with the following heat conduction equation [3], governing the deflection w, *viz.*,

$$\nabla^4 w(x, y;\, t) + (1 + \nu)\,\alpha_t \nabla^2 \tau(x, y;\, t) + cw(x, y;\, t) + \frac{\rho h}{N}\ddot{w} = 0 \tag{2}$$

where

ν = Poisson's ratio;

α_t = temperature co-efficient of the plate material;

τ = absolute temperature of the plate;

$c = \frac{K_1}{N}$, K_1 = foundation modulus;

$N = \dfrac{Eh^3}{12\,(1-\nu^2)}$ N = flexural rigidity;

E = Young's modulus of the plate material;

ρ = density of the plate material;

h = thickness of the plate;

w = deflection of the plate;

∇^2 = Laplacian operator.

The boundary conditions for solving equation (2) do not involve derivative with respect to time, and they simply play the role of parameters only.

The temperature distribution is assumed to be of the following form, *viz.*,

$$\tau(x, y;\, t) = \sum_{m=2,4,\ldots}^{\infty} \; \sum_{n=1,3,\ldots}^{\infty} \tau_{mn}(t) \cdot G \tag{3}$$

where

$$G = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a} - \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{a}. \tag{4}$$

The co-efficients $\tau_{mn}(t)$ are given by

$$\tau_{mn}(t) = \frac{16m}{\pi^2 n\,(n^2 - m^2)}\,\tau \tag{5}$$

$\tau(x, y;\, t)$ clearly satisfies the following boundary conditions:

$$\tau(x, y;\, t) = 0 \quad \text{at} \quad x = a, \quad y = 0 \quad \text{and} \quad x = y. \tag{6}$$

The deflection is found as the sum of two deflections, *viz.*,

$$w = w_s + w_d \tag{7}$$

where

w_s = quasi-static deflection

w_d = dynamic deflection.

First we proceed to find the quasi-static deflection where we disregard the inertia effect and assume that the temperature varies very slowly with time; and re-write equation (2) modified as

$$\nabla^4 w_s + (1+\nu)\,\alpha_t\,\nabla^2\,\tau(x, y;\ t) + c w_s = 0. \tag{8}$$

Let us assume the solution to the static deflection in the form

$$w_s = \sum_{m=2,4,\dots}^{\infty}\ \sum_{n=1,3,\dots}^{\infty} A_{mn}\cdot G \tag{9}$$

This represents the amplitude of a stationary wave. w_s evidently satisfies the simply supported boundary conditions along the edges of the plate. The co-efficients A_{mn} are given by

$$A_{mn} = C^2_{mn}\cdot\tau(t) \tag{10}$$

where

$$C^2_{mn} = \frac{16\,ma^2(1+\nu)\,\alpha_t\,(n^2+m^2)}{n(n^2-m^2)\,[\pi^4(n^2+m^2)^2+a^4 c]} \tag{11}$$

so that the quasi-static deflection becomes

$$w_s = \sum_{m=2,4,\dots}^{\infty}\ \sum_{n=1,3,\dots}^{\infty} C^2_{mn}\cdot\tau(t)\cdot G \tag{12}$$

$\tau(t)$ is given by [3]

$$\tau(t) = \frac{Q}{2\lambda}\left[1-\frac{96}{\pi^4}\sum_{j=1,3,\dots}^{\infty}\frac{1}{j^4}\,e^{-j^2\beta t}\right] \tag{13}$$

where $\lambda =$ co-efficient of heat conduction; $\beta = \dfrac{k\pi^2}{h^2}$.

A_{mn} assumes a constant magnitude in a very short time, since it is approximately invariant with time.

The solution which takes inertia into account, for the case of sudden heating, will be referred to as a "dynamic" solution. The equation now to be solved is

$$\nabla^4 w_d + c w_d + \frac{\rho h}{N}\,C^2_{mn}\,\ddot{\tau}(t) + \frac{\rho h}{N}\,\ddot{w}_d = 0 \tag{14}$$

where we assume the solution to the deflection in the form

$$w_d = \sum_{m=2,4,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} B_{mn}(t) \cdot G. \tag{15}$$

The solution of equation (14) is

$$\ddot{B}_{mn}(t) + \omega_{mn}^2 B_{mn}(t) = -C_{mn}^2 \ddot{\tau}(t) \tag{16}$$

where

$$\omega_{mn} = \frac{1}{a^2}\sqrt{\frac{N}{\rho h} \sum_{m=2,4,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} [\pi^4 (n^2+m^2)^2 + a^4 c]} \tag{17}$$

$$= B_1^2 \left[\left\{\pi^4 \sum_{m=2,4,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} (n^2+m^2)^2 + a^4 c\right\}\right]^{1/2} \frac{k}{h^2} \tag{18}$$

where

ω_{mn} = angular frequency of vibration of the plate;

$\phi = \frac{kt}{h^2}$, a dimensionless parameter;

k = thermal diffusivity;

$B_1 = \frac{h}{a\sqrt{k}} \cdot \left(\frac{N}{\rho h}\right)^{1/4}$

To find $B_{mn}(t)$, we apply Laplace transform method [3] and obtain

$$B_{mn}(t) = C_{mn}^2 \cdot \frac{Q}{2\lambda}\left[\frac{12\beta}{\pi^2 \omega_{mn}} \sin \omega_{mn} t \right.$$

$$+ \frac{96\beta^2}{\pi^4} \sum_{j=1,3,\ldots}^{\infty} \frac{1}{j^4 \beta^4 + \omega_{mn}^2}$$

$$\left. \times \left(\cos \omega_{mn} t + \frac{\omega_{mn}}{j^2 \beta} \sin \omega_{mn} t - e^{-j^2 \beta t}\right)\right] \tag{19}$$

For maximum value of $B_{mn}(t)$, we shall have (neglecting higher powers of t),

$$t = \frac{\dfrac{96\,\beta\omega_{mn}^2}{\pi^4 (j^6 \beta^4 + j^2 \omega_{mn}^2)} - \dfrac{12\beta}{\pi^2} + \dfrac{96\, j^2 \beta^3}{\pi^4 (j^4 \beta^4 + \omega_{mn}^2)}}{\dfrac{96\,\beta^2 \omega_{mn}^2}{\pi^4 (j^4 \beta^4 + \omega_{mn}^2)} + \dfrac{96\, j^4 \beta^4}{\pi^4 (j^4 \beta^4 + \omega_{mn}^2)}} \tag{20}$$

$= t_1$ (say),

Thus

$$[B_{mn}(t)]_{\max}$$

$$= C_{mn}^2 \cdot \frac{Q}{2\lambda} \cdot \frac{3(1-\beta^2)}{2}\left[\frac{12\beta t_1}{\pi^2} + \frac{96\beta^2}{\pi^4} \sum_{j=1,3,\ldots}^{\infty} \left\{\frac{1-j^2\beta t_1}{j^4\beta^4+\omega_{mn}^2}\right.\right.$$

$$- \frac{96\beta^2}{\pi^4} \sum_{j=1,3,\ldots}^{\infty} \frac{1}{j^4\beta^4+\omega_{mn}^2} - \frac{96\beta\omega_{mn}^2 t_1}{\pi^4}$$

$$\left.\left. \times \sum_{j=1,3,\ldots}^{\infty} \frac{1}{j^6\beta^4+j^2\omega_{mn}^2}\right\}\right] \tag{21}$$

As

$$\omega_{mn} \to 0, \qquad r_1 = \frac{\pi^2}{8}\left(\frac{1}{\beta} - \beta\right) \tag{22}$$

and then

$$[B_{mn}(t)]_{\max} = \frac{6\beta^2 - 3\beta^4 + 3}{2\beta^2} \tag{23}$$

For this value of t_1 given by equation (22), we have,

$$[\tau]_{\max} = \frac{3(1-\beta^2)}{2} \tag{24}$$

Further, $\beta \to \infty$ as $\omega_{mn} \to 0$, so that

$$\underset{Lt\ \omega_{mn}\to 0}{\left[\frac{w_{d\max}}{w_{s\max}}\right]} = \underset{Lt\ \beta\to\infty}{\left[\frac{w_{d\max}}{w_{s\max}}\right]} = 1. \tag{25}$$

and hence [4]

$$\frac{w_{s\ \max} + w_{d\ \max}}{w_{s\ \max}} = 2. \tag{26}$$

We now proceed to find the nodal lines [2] and assume the general solution in the form

$$w = w_s + w_d$$

$$= \sum_{m=2,4,\ldots}^{\infty} \sum_{n=2,3,\ldots}^{\infty} G \cdot D_{mn} \tag{27}$$

where

$$D_{mn} = A_{mn} + B_{mn} \tag{28}$$

The lowest tone is found by putting $m = 2$ and $n = 1$ in equation (27) so that

$$D_{21} + (-D_{21}) = 0 \tag{29}$$

and we obtain the edges of the triangle for the nodal lines as follows. With the condition given by equation (29),

$$w \infty (2) \cdot \sin \frac{\pi x}{a} \sin \frac{\pi y}{a} \left(\cos \frac{\pi x}{a} - \cos \frac{\pi y}{a} \right) \tag{30}$$

which vanishes giving the locations of the nodal lines at the edges,

$$x = a, \qquad y = 0 \quad \text{and} \quad x = y. \tag{31}$$

For the lowest tone the number of interior nodal lines parallel to the x-axis is nil, and that parallel to the y-axis is one.

Discussion

Dynamic systems are often subjected to the abrupt application of excitation. The term "shock" generally denotes a rapid application of excitation, having a short duration, to a system.

Our analysis rests on the following assumptions [4]: the temperature can be determined independently of the deflection of the plate, that the deflections are small, and that the material behaves elastically at all times. The first of these assumptions requires the omission of mechanical coupling terms in the heat conduction equation. The second implies that the displacements are sufficiently small as also the displacement gradients, so that their products may be neglected. The third assumption imples that neither the temperature changes nor the stresses are too large.

Equations (17) and (18) reveal that the frequency of the vibrating plate is higher when the plate is placed on an elastic foundation than when not.

The ratio of the greatest deflection at the centroid of the plate to the greatest quasi-static deflection for the fundmental mode is plotted in Fig. 1 for various values of the parameter B_1. The second abscissa scale (Labelled h) gives the plate-thickness alongside each value of B_1. Such a diagram may be used to estimate the error committed by adopting the static solution alone [4].

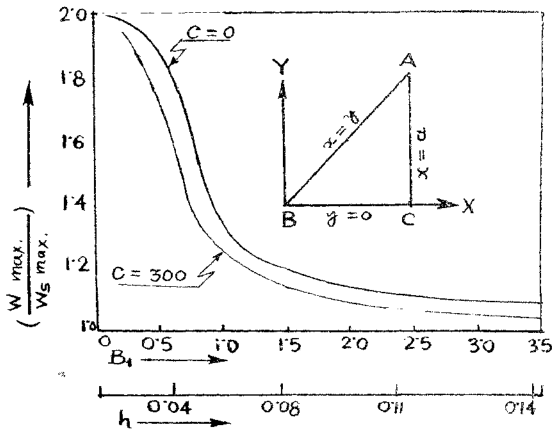

FIG. 1. Ratio of maximum dynamic to maximum quasi-static deflection (Inset:–Geometry of the aluminium plate).

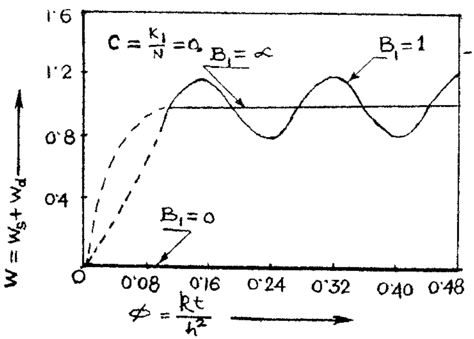

FIG. 2. Deflection history of the heated plate (The curve for $B_1 = \infty$ not to scale).

Non-dimensional plot of deflection against time for $B_1 = 1$, shown in Fig. 2, indicates that the dynamic solution oscillates about the quasi-static one. The importance of the inertia effects increases as B_1 becomes smaller. In effect, they prevent deflection for $B_1 = 0$. On the other hand, as $B_1 — \infty$, the inertia forces disappear and the qusi-static solution alone remains [4].

The nodal-lines investigation establishes the validity of the assumed boundary conditions. Similar inference may also be drawn from the investigations of the subsequent overtones.

ACKNOWLEDGEMENT

The author wishes to thank Dr. B. Banerjee, Department of Mathematics, for his guidance in the preparation of this paper.

REFERENCES

[1] Kirchoff .. *J. F. Maths (Crelle)*, Bd. 40, 1850.

[2] Rayleigh .. *Theory of Sound*, Vol. I, MacMillan & Co., Ltd , London, 1926.

[3] Nowacki, W. .. *Thermoelasticity*, Int. Series of Mono. in Aero and Astro., PWN. Polish Science Puhlishers, Worszaw., 1962.

[4] Boley, A. Bruno, and Weier, H. Jerome *Theory of Thermal Stresses.* Wiley, London, 1960.