ELECTROMAGNETIC BOUNDARY VALUE PROBLEM OF THE DIELECTRIC-COATED CONDUCTING SPHERE EXCITED BY DELTA-FUNCTION ELECTRIC AND MAGNETIC SOURCES

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ABSTRACT

The electromagnetic boundary-value problem of the dielectric-coated conducting sphere excited by delta-function electric and magnetic sources applied normally across an arbitrary plane, has been solved. The possibility of symmetric as well as unsymmetric TE, TM and hybrid modes have been investigated.

Key words : Electromagnetic scattering, Dielectric coated conducting sphere, Boundary value problem.

1. INTRODUCTION

Scattering of electromagnetic waves from conducting, dielectric and dielectric-coated conducting spheres have been studied by several authors [1-7]. The problem of forced oscillations of a conducting sphere which is excited in an infinite number of symmetric *TM* modes by a delta-function electric source field applied normally across the equatorial plane, has been considered by Stratton and Chu [8] in 1941. In an earlier paper [9] the present author has studied the electromagnetic boundary-value problem of the dielectric sphere excited by delta-function electric and magnetic sources applied normally across an arbitrary plane, and discussed the possibility of exciting *TM*, *TE* and hybrid modes. The present author with others has also studied the problem of radiation from a dielectric-coated metal spherical antenna excited in the unsymmetric hybrid mode [10].

In this paper, the electromagnetic boundary-value problem of a dielectric-coated conducting sphere excited by delta function electric and magnetic sources applied normally across an arbitrary plane has been solved. The possibility of exciting both symmetric and unsymmetric *TM*, *TE* and hybrid modes have been investigated.

2. STATEMENT OF THE PROBLEM

The geometry of the structure is given in Fig. 1. Spherical co-ordinates r, θ, ϕ are used. A perfectly conducting sphere of radius 'a' and constants ϵ_0, μ_0, $\sigma = \infty$, coated by a dielectric of constants ϵ_1, μ_1, σ_1 and of thickness '$b - a$', is embedded in another dielectric medium of constants ϵ_0, μ_0, σ_0. The dielectric-coated conducting sphere is excited by delta-function electric and magnetic field sources in a direction normal to the plane $z = z_1 = b \cos \theta_1$.

The object of this paper is to solve the electromagnetic boundary-value problem and to discuss the possibility of the existence of hybrid, *TM* and *TE* modes.

3. HYBRID MODES

Let the excitation of the dielectric-coated conducting sphere be a combination of an electric field $E' e^{-j\omega t}$ and a magnetic field $H' e^{-j\omega t}$ applied uniformly over the plane $z = z_1 = b \cos \theta_1$, and in a direction normal to this plane.

Let

$$E' = E_0 \cos m\phi \tag{1}$$

$$H' = H_0 \cos m\phi \tag{2}$$

Both E' and H' have components $E_{r'}$, $E_{\theta'}$ and $H_{r'}$, $H_{\theta'}$ in the r and θ directions respectively. These components of the applied electric and magnetic fields can be expanded in series of spherical harmonies as follows:

$$E_r'(r, \theta, \phi) = \frac{-n(n+1)}{k_1} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} D_{mn}(r) \cos(m\phi) P_n^m(\cos\theta) e^{-j\omega t} \tag{3}$$

$$\sin\theta\, E_\theta'(r, \theta, \phi) = -\frac{1}{k_1} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_{mn}(r) \cos(m\phi) P_n^m(\cos\theta) e^{-j\omega t} \tag{4}$$

$$H_r'(r, \theta, \phi) = \frac{-n(n+1)}{j\omega\mu_1} \sum_{m=0}^{\infty} D'_{mn}(r) \sin(m\phi) P_n^m(\cos\theta) e^{-j\omega t} \tag{5}$$

$$= \frac{\mu_0}{\mu_1\mu_0} N'_{m,\,m+1} h_{m+1}^{(1)}(k_0 b) \tag{58}$$

$$\frac{1}{\mu_1}\{L'_{m,\,m+1} j_{m+1}(k_1 a) + M'_{m,\,m+1} y_{m+1}(k_1 a)\} = 0. \tag{59}$$

Equations (48)–(59) are twelve equations in the twelve unknown amplitude coefficients L_{mm}, M_{mm}, N_{mm}, L'_{mm}, M'_{mm}, N'_{mm}, $L_{m,m+1}$, $N_{m,m+1}$, $M_{m,m+1}$, $L'_{m,m+1}$, $M'_{m,m+1}$, $N'_{m,m+1}$. Hence these unknown eoefficients can be solved for.

Putting $n = m+1, m+2, m+3, \cdots$, etc., in equations (39)–(47), the other higher order coefficients $L_{m,m+2}$, $M_{m,m+2}$, $N_{m,m+2}$, $L'_{m,m+2}$, $M'_{m,m+2}$, $N'_{m,m+2}, \cdots$,etc., can be solved for. Since equations (39)–(47) contain the amplitude Coefficients L_{mn}, M_{mn}, N_{mn}, L'_{mn}, M'_{mn}, N'_{mn} $L_{m,n-1}$, $M_{m,n-1}$, $N_{m,n-1}$, $L'_{m,n-1}$, $M'_{m,n-1}$, $N'_{m,n-1}$, $L_{m,n+1}$, $M_{m,n+1}$, $N_{m,n+1}$, $L'_{m,n+1}$, $M'_{m,n+1}$, $N'_{m,n+1}$, it is not possible to separate out only the coefficients L_{mn}, M_{mn}, N_{mn}, L'_{mn}, M'_{mn}, and N'_{mn} for the same value of n.

Hence the boundary conditions are satisfied not for a single value of n but for $n-1$, n, and $n+1$ combined together. This shows that for a hybrid mode, for any particular value of m, the electric and magnetic field components consist of an infinite number of terms for values of n varying from m to ∞.

Since for each value of m, the infinite number of amplitude coefficients L_{mn}, L'_{mn}, M_{mn}, M'_{mn}, N_{mn}, N'_{mn}, can be solved for $n = m$ to ∞, it can be concluded that for each value of m, there exists a corresponding hybrid mode.

4. Unsymmetric *TM* and *TE* Modes

For an unsymmetric *TM* mode ($m \neq 0$), let the excitation be $E'e^{-jwt}$ applied uniformly over the plane $z = z_1 = a\cos\theta_1$, and in a direction normal to this plane.

Let

$$E' = E_0 \cos(m\phi) \tag{60}$$

E' has components E_r' and E_θ' in the r and θ directions. These components E_r' and E_θ' can be expanded in series of spherical harmonies as given by equations (3), (4), (7), (9), (13) and (15).

The field components inside the dielectric coating are:

$$E_r^i = -\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{n(n+1)}{k_1 r}\cos(m\phi)\,P_n^m(\cos\theta)\,\{L_{mn}j_n(k_1 r) + M_{mn}y_n(k_1 r)\}\,e^{-j\omega t} + E_r' \tag{61}$$

$$E_\theta^i = -\sum_{n=0}^{\infty}{}'\sum_{m=0}^{\infty}\frac{d}{dt}\{P_n^m(\cos\theta)\}\cos(m\phi)\,\frac{1}{k_1 r}\{L_{mn}[k_1 r\,j_n(k_1 r)]' + M_{mn}[k_1 r y_n(k_1 r)]'\}\,e^{-j\omega t} + E_\theta' \tag{62}$$

$$E_\phi^i = \frac{1}{\sin\theta}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}P_n^m(\cos\theta)\,m\sin(m\phi)\,\frac{1}{k_1 r}\{L_{mn}[k_1 r\,j_n(k_1 r)]' + M_{mn}[k_1 r y_n(k_1 r)]'\}\,e^{-j\omega t} \tag{63}$$

$$H_r^i = 0 \tag{64}$$

$$H_\theta^i = \frac{k_1}{j\omega\mu_1}\frac{1}{\sin\theta}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}P_n^m(\cos\theta)\,m\sin(m\phi)\,\{L_{mn}j_n(k_1 r) + M_{mn}y_n(k_1 r)\}\,e^{-j\omega t} \tag{65}$$

$$H_\phi^i = \frac{k_1}{j\omega\mu_1}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{d}{d\theta}\{P_n^m(\cos\theta)\}\cos(m\phi)\,\{L_{mn}j_n(k_1 r) + M_{mn}y_n(k_1 r)\}\,e^{-j\omega t} \tag{66}$$

The field components outside the dielectric coating are

$$E_r^e = -\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}n(n+1)\,N_{mn}\cos(m\phi)\,P_n^m(\cos\theta) \times \frac{h_n^{(1)}(k_0 r)}{k_0 r}\,e^{-j\omega t} \tag{67}$$

$$E_\theta^e = -\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{d}{d\theta}\{P_n^m(\cos\theta)\}\,N_{mn}\cos(m\phi)\,\frac{1}{k_0 r} \times [k_0 r h_n^{(1)}(k_0 r)]'\,e^{-j\omega t} \tag{68}$$

$$E_\phi^e = \frac{1}{\sin\theta}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}N_{mn}P_n^m(\cos\theta)\,m\sin(m\phi)\,\frac{1}{k_0 r} \times [k_0 r h_r^{(1)}(k_0 r)]'\,e^{-j\omega t} \tag{69}$$

$$H_r^e = 0 \tag{70}$$

$$H_\theta^e = \frac{k_0}{j\omega\mu_0 \sin\theta} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} m\, N_{mn} P_n^m(\cos\theta) \sin(m\phi)\, h_n^{(1)}(k_0 r)\, e^{-j\omega t} \tag{71}$$

$$H_\phi^e = \frac{k_0}{j\omega\mu_0} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} N_{mn} \frac{d}{d\theta} \{P_n^m(\cos\theta)\} \cos(m\phi)\, h_n^{(1)}(k_0 r)\, e^{-j\omega t}. \tag{72}$$

Applying the boundary conditions that

(i) $E_\theta^i = E_\theta^e,\ \ E_\phi^i = E_\phi^e,\ \ H_\theta^i = H_\theta^e,\ \ H_\phi^i = H_\phi^e$ at $r = b$

and

(ii) $E_\theta^i = 0,\ \ E_\phi^i = 0$ at $r = a$

we obtain

$$\frac{1}{k_1 b} \{L_{mn} [k_1 b j_n(k_1 b)]' + M_{mn} [k_1 b y_n(k_1 b)]'\} + \frac{C_{mn}(b)}{k_1}$$

$$= \frac{N_{mn}}{k_0 b} [k_0 b h_n^{(1)}(k_0 b)]' \tag{73}$$

$$\frac{1}{k_1} \{L_{mn} [k_1 b j_n(k_1 b)]' + M_{mn} [k_1 b y_n(k_1 b)]'\}$$

$$= \frac{N_{mn}}{k_0} [k_0 b h_n^{(1)}(k_0 b)]' \tag{74}$$

$$\frac{k_1}{\mu_1} \{L_{mn} j_n(k_1 b) + M_{mn} y_n(k_1 b)\} = \frac{k_0}{\mu_0} N_{mn} h_n^{(1)}(k_0 b) \tag{75}$$

$$\frac{k_1}{\mu_1} \{L_{mn} j_n(k_1 b) + M_{mn} y_n(k_1 b)\} = \frac{k_0}{\mu_0} N_{mn} h_n^{(1)}(k_0 b) \tag{76}$$

$$\frac{1}{k_1 a} \{L_{mn} [k_1 a j_n(k_1 a)]' + M_{mn} [k_1 a y_n(k_1 a)]'\} + \frac{C_{mn}(a)}{k_1} = 0 \tag{77}$$

$$\frac{1}{k_1} \{L_{mn} [k_1 a j_n(k_1 a)]' + M_{mn} [k_1 a y_n(k_1 a)]'\} = 0. \tag{78}$$

Equations (75) and (76) are the same and equations (77) and (78) are the same. Equations (73), (74), (75) and (77) are four equations in the three unknowns L_{mn}, M_{mn}, and N_{mn}. Hence there is no unique solution for L_{mn}, M_{mn} and N_{mn}.

This shows that forced unsymmetric *TM* modes are not possible on the dielectric coated metal sphere.

It can be shown in a similar manner that forced unsymmetric TE modes are also not possible on the dielectric coated metal sphere.

5. Symmetric TM and TE Modes

For a symmetric TM mode ($m = 0$), let the excitation be $E' e^{-j\omega t}$ applied uniformly over the plane $z = z_1 = a \cos \theta_1$, and in a direction normal to this plane. Let $E' = E_0$, and E' have components.

$$E_r' = E_0 \cos \theta_1 = E_{r_0} \quad \text{and} \quad E_\theta' = E_{\theta_0} = -E_0 \sin \theta,$$

in the r and θ directions. These components E_r' and E_θ' can be expanded in series of spherical harmonics as given below:

$$E_{\theta_0} = -\frac{1}{k_1} \sum_{n=0}^{\infty} C_{on}(r) P_n^1(\cos \theta) e^{-j\omega t} \tag{77}$$

$$E_{r_0} = -\frac{1}{k_1} \sum_{n=0}^{\infty} n(n+1) D_{on}(r) P_n(\cos \theta) e^{-j\omega t} \tag{78}$$

where

$$C_{on}(r) = \frac{-k_1(2n+1)}{2\pi} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} E_{\theta_0} P_n^1(\cos \theta) \sin \theta \, d\theta d\phi \tag{79}$$

$$D_{on}(r) = \frac{-k_1(2n+1)}{n(n+1) 2\pi} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} E_{r_0} P_n(\cos \theta) \sin \theta \, d\theta d\phi. \tag{80}$$

If E_0 is a δ-function source given by equations (11) and (12), then

$$C_{on}(r) = C_{on}(a) = C_{on}(b)$$
$$= \frac{-V}{b} \frac{k_1(2n+1)}{2n(n+1)} \sin \theta_1 P_n^1(\cos \theta_1) \tag{81}$$

$$D_{on}(r) = D_{on}(a) = D_{on}(b)$$
$$= \frac{V}{b} \cos \theta_1 P_n(\cos \theta_1) \frac{k_1(2n+1)}{2n(n+1)}. \tag{82}$$

The field components inside the sphere are

$$E_r^i = -\sum_{n=0}^{\infty} \frac{n(n+1)}{k_1 r} P_n(\cos \theta) \{L_{on} j_n(k_1 r)$$
$$+ M_{on} y_n(k_1 r)\} e^{-j\omega t} + E_{r_0} \tag{83}$$

$$E_\theta{}^i = -\sum_{n=0}^{\infty} \frac{d}{d\theta}\{P_n(\cos\theta)\}\frac{1}{k_1 r}\{L_{on}[k_1 r j_n(k_1 r)]'$$
$$+ M_{on}[k_1 r y_n(k_1 r)]'\} e^{-j\omega t}$$
$$= -\sum_{n=0}^{\infty} P_n{}^1(\cos\theta)\{L_{on}[k_1 r j_n(k_1 r)]' + M_{on}[k_1 r y_n(k_1 r)]'\} e^{-j\omega t} \tag{84}$$

$$H_\phi{}^i = \frac{k_1}{j\omega\mu_1}\sum_{n=0}^{\infty} P_n{}^1(\cos\theta)\{L_{on} j_n(k_1 r) + M_{on} y_n(k_1 r)\} e^{-j\omega t}. \tag{85}$$

The field components outside the sphere are

$$E_r{}^e = -\sum_{n=0}^{\infty} n(n+1) N_{on} P_n(\cos\theta)\frac{h_n{}^{(1)}(k_0 r)}{k_0 r} e^{-j\omega t} \tag{86}$$

$$E_\theta{}^e = -\sum_{n=0}^{\infty} P_n{}^1(\cos\theta)\frac{1}{k_0 r} N_{on}[k_0 r h_n{}^{(1)}(k_0 r)]' e^{-j\omega t} \tag{87}$$

$$H_\phi{}^e = \frac{k_0}{j\omega\mu_0}\sum_{n=0}^{\infty} P_n{}^1(\cos\theta) N_{on} h^{(1)}(k_0 r) e^{-j\omega t}. \tag{88}$$

Applying the boundary conditions that

(i) $E_\theta{}^e = E_\theta{}^i$ and $H_\phi{}^e = H_\phi{}^i$ at $r = b$

and

(ii) $E_\theta{}^i = 0$ at $r = a$,

we obtain

$$\frac{1}{k_1 b}\{L_{on}[k_1 b j_n(k_1 b)]' + M_{on}[k_1 b y_n(k_1 b)]'\} + \frac{C_{on}(b)}{k_1}$$
$$= \frac{N_{on}}{k_0 b}[k_0 b h_n{}^{(1)}(k_0 b)]' \tag{89}$$

$$\frac{k_1}{\mu_1}\{L_{on} j_n(k_1 b) + M_{on} y_n(k_1 b)\} = \frac{k_0}{\mu_0} N_{on} h_n{}^{(1)}(k_0 b) \tag{90}$$

and

$$\frac{1}{k_1 a}\{L_{on}[k_1 a j_n(k_1 a)]' + M_{on}[k_1 a y_n(k_1 a)]'\} + \frac{C_{on}(a)}{k_1} = 0. \tag{91}$$

Using equation (81) in equations (89), (90) and (91), the amplitude coefficients L_{on}, M_{on} and N_{on} can be uniquely determined for each value of n.

The field is thus uniquely determined both inside and outside the dielectric coated conducting sphere for each value of n. This shows that M_{0n} modes exist for $n = 0, 1, 2,$ for this structure.

If

$$Z_n = \begin{vmatrix} \frac{1}{k_1 b}[k_1 b j_n(k_1 b)]', & \frac{1}{k_1 b}(k_1 b y_n(k_1 b)]', & -\frac{1}{k_0 b}[k_0 b h_n^{(1)}(k_0 b)]' \\ \frac{k_1}{\mu_1} j_n(k_1 b), & \frac{k_1}{\mu_1} y_n(k_1 b), & \frac{-k_0}{\mu_0} h_n^{(1)}(k_0 b) \\ \frac{1}{k_1 a}[k_1 a j_n(k_1 a)]', & \frac{1}{k_1 a}[k_1 a y_n(k_1 a)]', & 0 \end{vmatrix}$$

then when $Z_n = 0$, free oscillations of the dielectric coated conducting sphere results. Hence the roots of the equation $Z_n = 0$ determine the characteristic or resonant frequencies of the natural modes of oscillations.

It can similarly be shown that symmetric TE_{0n} modes also exist for $n = 0, 1, 2, \ldots$ for the dielectric coated conducting sphere.

6. CONCLUSIONS

The following conclusions can be drawn from the above investigations on the dielectric-coated conducting sphere:

(i) It is not possible to excite unsymmetric *TM* and *TE* modes on the dielectric coated conducting sphere.

(ii) It is possible to excite symmetric *TM* and *TE* modes, as well as symmetric and unsymmetric hybrid modes on the structure.

Numerical calculations and experimental verification of the results obtained will be reported in subsequent papers.

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