NOTE ON THE DEFLECTION OF A HEATED ELLIPTIC PLATE

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ABSTRACT

Deflection of an elliptic plate with clamped edges under stationary temperature distribution and subjected to a uniform load has been obtained in terms of Mathieu functions of the first kind of order zero.

Keywords: Elliptic plate, Mathieu functions.

INTRODUCTION

The problems of determination of thermal stresses and deflections in plates have got wide practical applications in air-craft and machine structures. Forray and Newmann [1] have obtained the thermal stresses in circular plate with different edge conditions for a particular temperature distribution. Forray and Zaid [2] have introduced stress functions and obtained the stresses. Quasi-static thermal deflection in a solid circular plate in the axisymmetric case has been investigated by Sarkar, S. K. [3].

In this paper the deflection of a heated elliptic plate with clamped edges under stationary temperature distribution and subjected to uniform load has been obtained in terms of Mathieu functions of the first kind of order zero and with usual limiting process the corresponding result for a circular plate has been deduced.

NOTATIONS

ω = Deflection in the normal direction of the plate;

ν = Poisson's ratio;

α_t = co-efficient of thermal expansion;

$D = \dfrac{Eh^3}{12(1-\nu^2)}$ = flexual rigidity;

E = Young's modulus;

$2d$ = Interfocal distance of the elliptic plate;

p = Intensity of load per unit area;

$A_0^{(0)}, A_2^{(0)}$ = Fourier co-efficients in the expansion of $Ce\,(\eta, -q)$

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{2}{d\,(\cos h2\xi - \cos 2\eta)}\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right)$$

= two-dimensional Laplacian operator in ellipitic co-ordinates,

(ξ, η) = elliptic co-ordinates;

C_{2m}, C_0, C_1, C_2, K = constants.

Governing equation.—An elliptic plate of thicikness h is taken with the centre of the plate in the middle surface as origin and z-axis downwards.

The equilibrium equation for the deflection of a heated plate is given by

$$\nabla^4 \omega + (1 + \nu)\,\alpha_t\,\nabla^2 T = \frac{p}{D} \tag{1}$$

Analysis.—If there is no source of heat inside the plate the temperature distribution given by

$$T'(x, y, z) = T_0(x, y) + zT(x\ y), \tag{2}$$

satisfies the following differential equations (Nowacki [5] p. 439)

$$\nabla^2 T_0 - \epsilon T_0 = -\frac{\epsilon_0}{2}(\theta_1 + \theta_2) \tag{3}$$

$$\nabla^2 T - \frac{12\,(1 + \epsilon)}{h^2}\,T = -\frac{12\epsilon}{h^3}(\theta_1 - \theta_2) \tag{4}$$

in which θ_1 and θ_2 denote temperatures at the lower and upper media of the plate respectively.

If $\theta_1 - \theta_2$ = constant, the general solution of equation (4) in elliptic co-ordinates defined by $x + iy = d\cosh(\xi + i\eta)$ is given by

$$T(\xi, \eta) = \sum_{m=0}^{\infty} C_{2m}\,Ce_{2m}(\xi, -q)\,ce_{2m}(\eta, -q) - \frac{\lambda_1}{\lambda_2} \tag{5}$$

in which the summation of terms is the complimentary function and λ_1/λ_2 is the particular integral, $Ce_{2m}(\xi, -q)$ and $Ce_{2m}(\eta, -q)$ being the modified Mathieu function and ordinary Mathieu function of the first kind of order $2m$, and

$$q = \frac{\lambda^2 d^2}{4} \tag{6}$$

$$\lambda^2 = \frac{12(1+\epsilon)}{h^2} \tag{7}$$

$$\lambda_1 = \frac{12\epsilon}{h^3}(\theta_1 - \theta_2). \tag{8}$$

While solving a problem of bending of a plate with elliptic hole, instead of taking Mathieu functions of all orders, taking a single Mathieu function of the second order, Naghdi [6] has shown that the results are satisfactory for larger elliptic hole. In this present problem also, similar approximation is made by taking Mathieu functions of order zero. Therefore (5) reduces to

$$T(\xi, \eta) = C_0\, Ce_0(\xi, -q)\, ce_0(\eta, -q) - \frac{\lambda_1}{\lambda_2} \tag{9}$$

If $T = K$ on $\xi = \xi_0$ then (9) results into

$$K + \frac{\lambda_1}{\lambda_2} = C_0\, Ce_0(\xi_0, -q)\, ce_0(\eta, -q) \tag{10}$$

Multiplying the equation (10) by $Ce_0(\eta, -q)$ and integrating with respect to η from 0 to 2π and using orthogonality relation and normalisation (Maclachlan [4] one gets

$$C_0 = 2A_0^{(0)}\left(K + \frac{\lambda_1}{\lambda_2}\right)\Big/ Ce_0(\xi_0, -q). \tag{11}$$

Using $T(\xi, \eta)$ given by (9), equation (1) can be written as

$$\nabla^4 \omega = \nabla^2 \left\{\frac{p}{4D}(x^2 + y^2) - (1+\nu)a_t\, C_0\, Ce_0(\xi, -q)\, C_0(\eta, -q) + (1+\nu)\, a_t \frac{\lambda_1}{\lambda^2}\right\} \tag{12}$$

The complementary of equation (12) is given by

$$\omega = C_1 + C_2(\cosh 2\xi + \cos 2\eta). \tag{13}$$

Now the particular integral of

$$\nabla^2 \omega = \frac{p}{4D}(x^2 + y^2) - (1 + \nu)\,\alpha_t\, C_0 Ce_0(\xi, -q) Ce_0(\eta, -q) + (1 + \nu)\alpha_t \frac{\lambda_1}{\lambda^2} \tag{14}$$

is necessarily the particular solution of (12). Clearly the particular integral is given by

$$\omega = \frac{pd^4}{48D}(\cosh^4 \xi \cos^4 \eta + \sinh^4 \xi \sin^4 \eta) + \frac{1}{4}(1 + \nu)\alpha_t \times \frac{\lambda_1 d^2}{\lambda^2}(\cosh^2 \xi \cos^2 \eta + \sinh^2 \xi \sin^2 \eta) - \frac{(1 + \nu)\alpha_t}{\lambda^2} C_0\, Ce_0(\xi, -q)\, Ce_0(\eta, -q). \tag{15}$$

Therefore the general solution of equation (12) is given by

$$\omega = C_1 + C_2(\cosh 2\xi + \cos 2\eta) + \frac{pd^4}{48D}\cosh^4 \xi \cos^4 \eta + \sinh^4 \xi \sin^4 \eta) + \frac{1}{4}(1 + \nu)\alpha_t \frac{\lambda_1 d^2}{\lambda^2}(\cosh^2 \xi \cos^2 \eta + \sinh^2 \xi \sin^2 \eta) - \frac{(1 + \nu)\,\alpha_t\, C_0}{\lambda^2} Ce_0(\xi, -q)\, Ce_0(\eta, -q). \tag{16}$$

If the outer boundary of the plate $\xi = \xi_0$ be clamped, then

$$\omega = 0 = \frac{\delta\omega}{\delta\xi} \quad \text{when} \quad \xi = \xi_0. \tag{17}$$

Multiplying the two equations obtained by introducing the above boundary conditions into (16) by $Ce_0\ (\eta, -q)$ and integrating with respect to η from 0 to 2π and using orthogonality relation and normalisation, one gets

$$C_1 = \left(\cosh 2\xi + \frac{1}{2}\frac{A_2^{(0)}}{A_0^{(0)}}\right)\left\{\frac{pd^4}{128D}(\cosh^2 \xi_0 + \sinh^2 \xi_0) + \frac{1}{8}(1 + \nu)\,\alpha_t \frac{\lambda_1 d^2}{\lambda^2} - \frac{1}{4}(1 + \nu)\,\alpha_t \frac{C_0\, Ce'_0(\xi_0, -q)}{\lambda^2 A_0^{(0)} \sinh 2\xi_0}\right\}$$

$$- \frac{pd^4}{128D}(\sinh^4 \xi_0 + \cosh^4 \xi_0)$$

$$- \frac{1}{8}(1 + \nu)\alpha_t \frac{\lambda_1 d^2}{\lambda^2}(\sinh^2 \xi_0 + \cosh^2 \xi_0)$$

$$+ \frac{(1 + \nu)\alpha_t\, C_0\, Ce_0(\xi_0, -q)}{2\lambda^2 A_0^{(0)}} \tag{18}$$

$$C_2 = -\frac{pd^4}{128D}(\cosh^2 \xi_0 + \sinh^2 \xi_0) - \frac{1}{8}(1 + \nu)\,\alpha_t \frac{\lambda_1}{\lambda^2} d^2$$

$$+ \frac{(1 + \nu)}{4\lambda^2} \cdot \alpha_t \cdot \frac{C_0\, Ce'_0(\xi_0, -q)}{A_0^{(0)} \sinh 2\xi_0}. \tag{19}$$

Thus the deflection $\omega(\xi, \eta)$ is completely determined.

Limiting case.—In the limiting case when $d \to 0$, $\xi \to \infty$, the ellipse degenerates to a circle or radius a (say). In that case

$$Ce_0(\eta, 0) \to 2^{-1/2} = A_0^{(0)}, \quad A_2^{(0)} \to 0,$$
$$Ce_0(\xi, -q) \to P_0' I_0(\lambda r), \quad Ce_0(\xi_0, -q) \to P_0' I_0(\lambda a)$$
$$Ce_0'(\xi_0, -q) \to P_0' \lambda a\, I_0'(\lambda a),$$
$$\frac{d^2(\cosh 2\xi + \cos 2\eta)}{2} \to r^2, \quad d^2 \sinh 2\xi_0 \to 2a^2, \ \sinh \sim \cosh$$

where

$$P_0' = Ce_0(0, -q)\, Ce_0(\pi/2, -q)/A_0^{(0)}$$

Therefore equation (16) reduces to

$$w(r) = \frac{pa^2}{32D}(a^2 - r^2) + \frac{p}{64D}(r^4 - a^4)$$

$$+ \frac{(1 + \nu)\,\alpha_t\left((k + \frac{\lambda_1}{\lambda^2}\right)}{\lambda^2 I_0(\lambda a)}\left\{\left(\frac{r^2 - a^2}{2a}\right)\lambda I_0'(\lambda a) + I_0(\lambda a) - I_0(\lambda r)\right\} \tag{20}$$

which gives the corresponding thermal deflection for a clamped circular plate.

Numerical calculation.—Using the data $\xi = 1$, $\eta = \pi/2$, $\xi_0 = 3$, $d = 1$, $h = 1$, $p = 10$, $E = 2 \times 10^{12}$, $\alpha_t = 1 \cdot 2 \times 10^{-5}$, $\nu = 0 \cdot 3$, $\epsilon = 0 \cdot 05$, one sets from equation (16)

$$\theta_1 - \theta_2 = 200 \text{ (in absolute degrees)}$$

$$K = 600 \text{ (in absolute degrees)}$$

$$\frac{10\omega}{d} = 0 \cdot 018 \text{ (approx.)}.$$

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