TRANSPORT PROCESSES IN A MULTICOMPONENT ASSEMBLY II: FOKKER–PLANCK FORMALISM

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ABSTRACT

It has been pointed out by Landau, Spitzer et al and others that when the charged particles of multicomponent assembly interact according to comparatively long range Coulomb forces, small random deflections of the particles are very effective in the evolution of the distribution function. Such small random deviations are incorporated into the kinetic equation by using Fokker-Planck formalism. As has been done earlier in paper I (Devanathan, Uberoi and Bhatnagar, J. Indian Inst. Sci., 47, 106, 1965), the distribution functions are expanded in terms of the Generalized Hermite Polynomials and following Grad, consistent set of transport equations deduced. It is found that the Fokker-Planck term enhances the relaxation times of all the physical variables and thereby decreases the electrical conductivity. Apart from the anisotropy due to the magnetic field among the stresses, F-P terms introduce an additional bulk viscosity term. Further they introduce intense anisotropy among heat-flux tensors. Modified expressions for viscosity, heat conductivity and the electrical conductivity have been deduced.

1. INTRODUCTION

In a recent paper Devanathan, Uberoi and Bhatnagar[1], referred to as I in sequel, have studied the transport processes in a plasma based on the procedure developed by Grad[2]. The basic kinetic equations describing the assembly are the modified Maxwell-Boltzmann equations. The necessary modification is the replacing of the complicated collision integrals by simple, physically significant, tractable collision models. This was first developed by Bhatnagar, Gross and Krook[3] for a single component assembly and extended to multicomponent assemblies of charged particles by Bhatnagar[4], Bhatnagar and Devanathan[5]. These collision models have been extensively studied and found to take account of binary encounters completely and as effectively as the Maxwell-Boltzmann collision integral. However, these types of binary collisions are well defined only when the duration of collision between two particles is much smaller than the time of free flight of particles between successive encounters. But whenever the plasma is of sufficiently high density and is hot as in the case of stellar interiors or fusion reactors only weak multiple collisions take place apart from occasional strong binary collisions. Moreover, in plasmas, the charged particles interact according to Coulomb law which is comparatively long ranged. Hence, even at moderate temperatures many charged particles undergo simultaneous weak interactions. Thus, for plasmas, one must take account of these interactions, as in Brownian motion. The corresponding change in the distribution function has been independently discovered in various contexts, by many investigators, especially Fokker[6], Planck[7], Einstein[8], Smoulouchеskii[9], Chandrasekhar[10] and Landau[11]. An elegent expression for this was obtained from the Boltzmann equation itself by Allis[12] and Bhatnagar[13]. In principle, the collisions are separated into weak collisions and strong collisions whenever the impact parameter is greater than or less than a suitable critical value b_c. The effect of all simultanious weak collisions is to produce a random fluctuation $\Delta\vec{\xi}$ in the velocity $\vec{\xi}$ of any particle. Then, the Boltzmann equation is expanded in averages and correlations of the random variable $\Delta\vec{\xi}$. The critical impact parameter b_c for various types of encounters has been discussed in detail by Spitzer and Härm[14] and Cohen, Spitzer and McRoutly[15]. The resulting equation is the celebrated Fokker-Planck equation:

$$\frac{\partial f_\alpha}{\partial t} + \xi_{\alpha i}\frac{\partial f_\alpha}{\partial x_i} + \frac{F_{\alpha i}}{m_\alpha}\frac{\partial f_\alpha}{\partial \xi_{\alpha i}}$$

$$= \sum_{\beta=1}^{N}\left\{\frac{N_\beta}{\sigma_{\beta\alpha}}(N_\alpha \Phi_{\beta\alpha} - f_\alpha) - \frac{\partial}{\partial \xi_{\alpha i}}(f_\alpha < \Delta\,\xi_{\alpha i} >_\beta)\right.$$

$$\left. + \frac{1}{2}\cdot\frac{\partial^2}{\partial\xi_{\alpha i}\partial\xi_{\alpha j}}(f_\alpha < \Delta\,\xi_{\alpha i}\,\Delta\,\xi_{\alpha j} >_\beta) - \cdots\right\} \qquad [1.1]$$

Here we have used the notations of *I*. The symbol $< \, >_\beta$ stands for the average of the deflection in α-type particle due to all encounters with β-type particles. For the subsequent working we start with [1.1] as the basic kinetic equation.

Langevin[16] has pointed out that the averages $< \Delta\,\vec{\xi}_\alpha >_\beta$ and the correlations $< \Delta\,\vec{\xi}_\alpha\,\Delta\,\vec{\xi}_\alpha >_\beta$ should be evaluated in terms of the moments of distribution functions themselves and should not be taken as constants. Hence we have evaluated these exact expressions from the actual dynamics of collision following Jeans[17], after removing a small error in Jeans treatment. The details of it are given in Appendix I.

Following the procedure of Grad, in section 2 we establish the consistent set of transport equations and in section 3 we evaluate all the transport coefficients like diffusion coefficient, electrical conductivity, viscosity and heat conductivity coefficients, by interpreting the appropriate transport equations.

2. DERIVATION OF TRANSPORT EQUATIONS

In order to derive the transport equations governed by the kinetic equation, we introduce, following I, the non-dimensional velocity $\mathbf{v}_\alpha$ and the non-dimensional distribution function g_α given by

$$\mathbf{v}_\alpha = \left(\frac{m_\alpha}{KT_{\alpha\alpha}}\right)^{1/2} \vec{\xi}_\alpha \qquad [2.1]$$

and

$$g_\alpha = \frac{1}{N_\alpha}\left(\frac{KT_{\alpha\alpha}}{m_\alpha}\right)^{3/2} f_\alpha, \qquad [2.2]$$

where N_α is the number density and $T_{\alpha\alpha}$, the temperature of the α-th component and K is the Boltzmann constant. The non-dimensional distribution functions satisfy the kinetic equations,

$$\frac{\partial g_\alpha}{\partial t} + c_\alpha v_{\alpha i}\frac{\partial g_\alpha}{\partial x_i} + \frac{1}{c_\alpha m_\alpha}(F_{\alpha i} + e_\alpha E_i)$$

$$+ \frac{e_\alpha}{c\, m_\alpha} \epsilon_{ijk} v_{\alpha j} H_k \frac{\partial g_\alpha}{\partial v_{\alpha i}} + g_\alpha \left\{\left(\frac{\partial}{\partial t} + c_\alpha v_{\alpha i}\frac{\partial}{\partial x_i}\right)\left(\log \frac{N_\alpha}{c_\alpha^3}\right)\right\}$$

$$= \sum_\delta \left[\frac{N_\delta}{\sigma_{\delta\alpha}}(\Psi_{\delta\alpha} - g_\alpha) - \frac{1}{c_\alpha}\cdot\frac{\partial}{\partial v_{\alpha i}}(g_\alpha < \Delta\, \xi_{\alpha i} >_\delta)\right.$$

$$\left. + \frac{1}{2c_\alpha^2}\cdot\frac{\partial^2}{\partial v_{\alpha i}\partial v_{\alpha j}}(g_\alpha < \Delta\, \xi_{\alpha i}\, \Delta\, \xi_{\alpha j} >_\delta)\right] \qquad [2.3]$$

where

$$c_\alpha = \left(\frac{KT_{\alpha\alpha}}{m_\alpha}\right)^{1/2}, \qquad [2.4]$$

$$\Psi_{\delta\alpha} = \left(\frac{T_{\alpha\alpha}}{2\pi\, T_{\delta\alpha}}\right)^{3/2} \exp\left[-\frac{T_{\alpha\alpha}}{2\,T_{\delta\alpha}}(\mathbf{v}_\alpha - \mathbf{v}_{\delta\alpha})^2\right] \qquad [2.5]$$

In writiting the above equation, we have denoted by $\mathbf{v}_{\delta\alpha}$ and $T_{\delta\alpha}$ the non-dimensional cross velocities and temperatures denoting the average velocity and temperature of scattered α-type of particles due to their encounter with δ-type particles. Further, we have as usual, terminated the Fokker-Planck expressions at the second term, taking into account the effects of only diffusion and dispersion terms as for most of the investigations these two terms are sufficient. The detailed expressions for the averages $< \Delta\, \xi_{\alpha i} >_\delta$ and $< \Delta\, \xi_{\alpha i}\, \Delta\, \xi_{\alpha j} >_\delta$ are given in the Appendix I. Following *I*, we expand the non-dimensional distribution function g_α in the form

$$g_\alpha(\mathbf{v}_\alpha, \mathbf{r}, t) = \omega(\mathbf{v}_\alpha) \sum_{n=0}^{\infty} a_\alpha^{(n)}(\mathbf{r}, t)\, H^{(n)}(\mathbf{v}_\alpha), \qquad [2.6]$$

where $\omega(\mathbf{v}_\alpha)$ is the non-dimensional Maxwell's distribution function. By the orthogonality property of the generalized Hermite Polynomials $H^{(n)}(\mathbf{v}_\alpha)$, we have

$$a_\alpha^{(n)}(\mathbf{r}, t) = (1/X^{(n)}) \int H^{(n)}(\mathbf{v}_\alpha)\, g_\alpha\, d\mathbf{v}_\alpha, \qquad [2.7]$$

where
$$X^{(n)} = \int \omega(\mathbf{v}_\alpha)\,[H^{(n)}(\mathbf{v}_\alpha)]^2\, d\mathbf{v}_\alpha$$
$$= n_1\,!\; n_2\,!\; n_3\,!, \tag{2.8}$$

where n_1, n_2, n_3 are the number of suffixes of direction 1, 2 and 3 contained in the set (n), with $n_1 + n_2 + n_3 = n$. From the kinetic equations [2.3], the equations governing the coefficients $a_\alpha^{(n)}$ can be obtained by multiplying throughout by $H^{(n)}(\mathbf{v}_\alpha)$ and integrating. In Cartesian tensor product notation, we have

$$D^{(n)}\, a_\alpha^{(n)} = -\,\tau_\alpha\, X^{(n)}\, a_\alpha^{(n)} + \sum_\delta \left\{ \frac{N_\delta}{\sigma_{\delta\alpha}} \left(\frac{T_{\alpha\alpha}}{2\pi T_{\delta\alpha}} \right)^{3/2} A_{\delta\alpha}^{(n)} + \sum_l \sum_m a_\alpha^{(l)}\, a_\delta^{(m)}\, I_{\delta\alpha}^{\prime(l,\,m;\,n)} \right\} \tag{2.9}$$

where

$$\begin{aligned}
D^{(n)}\, a_\alpha^{(n)} &= X_\alpha^{(n)} \frac{\partial a_\alpha^{(n)}}{\partial t} \\
&+ \left[(n+3)\, X^{(n)}\, a_\alpha^{(n)} + 2\, X^{(n-2)}\, \delta^{(2)}\, a_\alpha^{(n-2)}\right] \frac{\partial}{\partial t}(\log c_\alpha) \\
&+ c_\alpha \Big\{ X^{(n+1)} \frac{\partial a_{\alpha i}^{(n+1)}}{\partial x_i} + X^{(n-1)}\, \delta_i^{(2)} \frac{\partial a_\alpha^{(n-1)}}{\partial x_i} \\
&+ (n+4)\left[X^{(n+1)}\, a_{\alpha i}^{(n+1)} + X^{(n-1)}\, \delta_i^{(2)}\, a_\alpha^{(n-1)}\right] \frac{\partial}{\partial x_i}(\log c_\alpha) \\
&+ 2X^{(n-3)}\, \delta_i^{(2)}\, \delta^{(2)}\, a_\alpha^{(n-3)} \frac{\partial}{\partial x_i}(\log c_\alpha) \Big\} \\
&- \frac{1}{c_\alpha m_\alpha} (F_{\alpha i} + e_\alpha E_i) X^{(n-1)}\, \delta_i^{(2)}\, a_\alpha^{(n-1)} \\
&- \frac{e_\alpha}{c m_\alpha}\, \epsilon_{ijk} H_k \left[X^{(n)}\, \delta_i^{(2)}\, a_{\alpha j}^{(n)} + X^{(n-2)}\, \delta_i^{(2)}\, \delta_j^{(2)}\, a_\alpha^{(n-2)}\right] \\
&+ \frac{\partial}{\partial t}\left(\log \frac{N_\alpha}{c_\alpha^3}\right) X^{(n)}\, a_\alpha^{(n)} \\
&+ c_\alpha \frac{\partial}{\partial x_i}\left(\log \frac{N_\alpha}{c_\alpha^3}\right) \left[X^{(n+1)}\, a_{\alpha i}^{(n+1)} + X^{(n-1)}\, \delta_i^{(2)}\, a_\alpha^{(n-1)}\right],
\end{aligned} \tag{2.10}$$

$$\tau_\alpha = \sum_\delta \frac{N_\delta}{\sigma_{\delta\alpha}}, \tag{2.11}$$

$$A_{\delta a}^{(n)} = \int H^{(n)} (\mathbf{w}_a + \mathbf{v}_{\delta a}) \exp\left[- \frac{T_{aa}}{2 T_{\delta a}} \mathbf{w}_a^2 \right] d\mathbf{w}_a , \qquad [2.12]$$

and

$$I_{\delta a}^{(l, m; n)} = \iiint H^{(n)} (\mathbf{v}_a) \, \omega (\mathbf{v}_a) \, \omega (\mathbf{v}_\delta) \times \left\{ \frac{\partial}{\partial v_{ai}} [H^{(l)} (\mathbf{v}_a) \, H^{(m)} (\mathbf{v}_\delta) \, (v'_{ai} - v_{ai})] \right.$$

$$- \frac{1}{2} \cdot \frac{\partial^2}{\partial v_{ai} \partial v_{aj}} [H^{(l)} (\mathbf{v}_a) \, H^{(m)} (\mathbf{v}_\delta) \, (v'_{ai} - v_{ai}) \, (v'_{aj} - v_{aj})]$$

$$\times g_{\delta a} \, b \, db \, d\epsilon \, d\mathbf{v}_a \, d\mathbf{v}_\delta , \qquad [2.13]$$

the primes denoting the non-dimensional velocities after collision.

We note the following points about the set of moment equations (2.9). The operator $D^{(n)}$ introduces coupling between the convective derivatives of $a^{(n+1)}$ with lower order moments. Further, the Fokker-Planck terms through $I^{(l, m; n)}$ preserve the non-linear collisional coupling in its complete generality. Hence, to obtain a complete set of transport equations, a suitable cut-off is necessary. Following *I*, and Grad[2], we terminate the expansion at $n = 3$. This adequately takes into account the effects of mean velocity, stresses and the heat flux. Again, on the same lines of Grad, we restrict the collisional terms to $l + m = 3$, thereby neglecting the intense interactions arising out of heat flux and stresses. It has been found justifiable to do so for most of the physical situations encountered, including shock phenomena[2], [*I*.] With the help of the above assumptions, we get a consistent system of equations. Since $a_a^{(n)}$ are nothing but linear combinations of the physical moments, converting back we obtain the following transport equations :

$$\frac{\partial N_a}{\partial t} + \frac{\partial}{\partial x_i} (N_a u_{aai}) = 0, \qquad [2.14]$$

$$\frac{1}{N_a} \cdot \frac{\partial}{\partial t} (N_a u_{aar}) + \frac{1}{N_a} \cdot \frac{\partial}{\partial x_i} (N_a P_{air}) - \frac{1}{m_a} (F_{ar} + e_a E_r) - \frac{e_a}{c m_a} \epsilon_{rjk} u_{aaj} H_k$$

$$= \sum_\delta \left\{ \frac{N_\delta}{\sigma_{\delta a}} \cdot \frac{A_{\delta a}}{m_a} (u_{\delta\delta r} - u_{aar}) + \frac{16}{15} \cdot \frac{K_{\delta a}}{m_a c_{\delta a}} [5 c_{\delta a}^2 (u_{\delta\delta r} - u_{aar}) \right.$$

$$\left. + (u_{\delta\delta r} P_{aii} - u_{aar} P_{\delta ii}) + 2 (u_{\delta\delta i} P_{air} - u_{aai} P_{\delta ir}) + 2 (S_{\delta r} - S_{ar})] \right\} \qquad [2.15]$$

$$\frac{1}{N_a} \cdot \frac{\partial}{\partial t} (N_a P_{arr}) + \frac{1}{N_a} \cdot \frac{\partial}{\partial x_i} (N_a S_{airr}) - \frac{2}{m_a} (F_{ar} + e_a E_r) u_{aar} - \frac{2 e_a}{c m_a} \epsilon_{rjk} P_{ajr} H_k$$

$$= \sum_\delta \left\{ - \frac{N_\delta}{\sigma_{\delta a}} \left(P_{arr} - u_{\delta ar}^2 - \frac{K T_{\delta a}}{m_a} \right) + \frac{64}{15} \cdot \frac{K_{\delta a}}{c_{\delta a}} [- 5 c_a^2 c_{\delta a}^2 \right.$$

$$+ \{c_a^2 + (5 c_\delta^2 + 7 c_a^2) \delta_{ir}\} u_{\alpha a l} u_{\delta \delta i} - c_a^2 (1 + 2\delta_{ir}) (P_{\delta i l} - c_\delta^2)$$
$$- \{c_a^2 + (10 c_\delta^2 + 12 c_a^2) \delta_{ir}\} (P_{a i i} - c_a^2)]$$
$$+ \frac{64}{15} L_{\delta a}\, c_{\delta a} [5 c_{\delta a}^2 + (1 + 12 \delta_{ir}) (P_{a i i} + P_{\delta i i} - c_{\delta a}^2)$$
$$- (1 + 12 \delta_{ir}) u_{\alpha a i} u_{\delta \delta i}]\Big\} \qquad [2.16]$$

$$\frac{1}{N_a} \cdot \frac{\partial}{\partial t} (N_a P_{ars}) + \frac{1}{N_a} \cdot \frac{\partial}{\partial x_i} (N_a S_{airs})$$
$$- \frac{1}{m_a} [(F_{ar} + e_a E_r) u_{a a s} + (F_{as} + e_a E_s) u_{a a r}] - \frac{e_a}{c m_a} (\epsilon_{rjk} P_{ajs} + \epsilon_{sjk} P_{ajr}) H_k$$
$$= \sum_\delta \Big\{ - \frac{N_\delta}{\sigma_{\delta a}} (P_{ars} - u_{\delta ar} u_{\delta as}) + \frac{32}{15} \cdot \frac{K_{\delta a}}{c_{\delta a}} [- 2c_a^2 P_{\delta rs} - 2 (5 c_\delta^2 + 6 c_a^2) P_{ars}$$
$$+ (5 c_a^2 + 7 c_\delta^2) (u_{a a r} u_{\delta \delta s} + u_{a a s} u_{\delta \delta r})]$$
$$+ \frac{64}{5} L_{\delta a}\, c_{\delta a} [2 (P_{ars} + P_{\delta rs}) - 2 (u_{a a r} u_{\delta \delta s} + u_{a a s} u_{\delta \delta r})]\Big\}, \qquad [2.17]$$

$$\frac{1}{N_a} \cdot \frac{\partial}{\partial t} (N_a S_{arrr}) + \frac{1}{N_a} \cdot \frac{\partial}{\partial x_i} (N_a Q_{airrr}) - \frac{3}{m_a} (F_{ar} + e_a E_r) P_{arr} - \frac{3 e_a}{c m_a} \epsilon_{rjk} S_{ajrr} H_k$$
$$= \sum_\delta \Big\{ - \frac{N_\delta}{\sigma_{\delta a}} \Big(S_{arrr} - u_{\delta ar}^3 - \frac{3 K T_{\delta a}}{m_a} u_{\delta ar} \Big) + A_{\delta arrr} \Big\} \qquad [2.18]$$

$$\frac{1}{N_a} \cdot \frac{\partial}{\partial t} (N_a S_{arrs}) + \frac{1}{N_a} \cdot \frac{\partial}{\partial x_i} (N_a Q_{airrs}) - \frac{1}{m_a} (F_{as} + e_a E_s) P_{arr} - \frac{2}{m_a} (F_{ar} + e_a E_r) P_{ars}$$
$$- \frac{e_a}{c m_a} (2 \epsilon_{rjk} S_{ajrs} + \epsilon_{sjk} S_{ajrr}) H_k$$
$$= \sum_\delta \Big\{ - \frac{N_\delta}{\sigma_{\delta a}} \Big(S_{arrs} - u_{\delta ar}^2 u_{\delta as} - \frac{K T_{\delta a}}{m_a} u_{\delta as}\Big) + B_{\delta arrs} \Big\}, \qquad [2.19]$$

$$\frac{1}{N_a} \cdot \frac{\partial}{\partial t} (N_a S_{arst}) + \frac{1}{N_a} \cdot \frac{\partial}{\partial x_i} (N_a Q_{airst}) - \frac{1}{m_a} (F_{ar} + e_a E_r) P_{ast} - \frac{1}{m_a} (F_{as} + e_a E_s) P_{atr}$$
$$- \frac{1}{m_a} (F_{at} + e_a E_t) P_{ars} - \frac{e_a}{c m_a} (\epsilon_{rjk} S_{ajst} + \epsilon_{sjk} S_{ajtr} + \epsilon_{tjk} S_{ajrs}) H_k$$
$$= \sum_\delta \Big\{ - \frac{N_\delta}{\sigma_{\delta a}} (S_{arst} - u_{\delta ar} u_{\delta ar} u_{\delta at}) + C_{\delta arst} \Big\}, \qquad [2.20]$$

where $$c_{\delta\alpha}^2 = c_\delta^2 + c_\alpha^2 .$$

The lengthy expressions $A_{\delta\alpha rrr}$, $B_{\delta\alpha rrs}$ and $C_{\delta\alpha rst}$ are glven in the Appendix II. The constants $K_{\delta\alpha}$ and $L_{\delta\alpha}$ arise from the diffusion and dispersion parts of the Fokker-Planck terms and are given by

$$K_{\delta\alpha} = \frac{m_\delta N_\delta (b_u^2 - b_l^2)}{m_\alpha + m_\delta} \sqrt{\left(\frac{\pi}{2}\right)}, \tag{2.21}$$

$$L_{\delta\alpha} = \frac{m_\delta^2 N_\delta (b_u^2 - b_l^2)}{(m_\alpha + m_\delta)^2} \sqrt{\left(\frac{\pi}{2}\right)}, \tag{2.22}$$

where b_u and b_l are the upper and lower cut-offs for the impact parameter during weak interactions between α and δ type particles. [refer *I*].

The following important points should be noted regarding the above transport equations: The momentum equations are not affeeted by the dispersion part of the Fokker-Planck terms, but are directly coupled to the stresses as well as the heat flux vectors. The stress equations are not coupled directly with heat-flux tensors. The heat-flux tensors are coupled directly with all the lower order moments. Consequently, even in the absence of magnetic field, we expect intense anisotropy in the medium.

3. Stationary Non-equilibrium Processes

In this section, we shall consider some simple stationary non-equilibrium processes to obtain explicit expressions for electrical conductivity, viscosity, heat conductivity and diffusitivity of the assembly, following the procedure of *I*.

In order to find the expression for electrical conductivity and diffusivity terms, we consider Lorentz problem. Expressing the higher order moments in terms of the equivalent lower order moments, the momentum equation reduces to

$$\frac{1}{N_\alpha m_\alpha} \cdot \frac{\partial}{\partial x_r} (N_\alpha K T_{\alpha\alpha}) - \frac{1}{m_\alpha} (F_{\alpha r} + e_\alpha E_r) - \frac{e_\alpha}{c m_\alpha} \epsilon_{rjk} u_{\alpha\alpha j} H_k$$

$$= \sum_\delta \left\{ \frac{N_\delta}{\sigma_{\delta\alpha}} \cdot \frac{A_{\delta\alpha}}{m_\alpha} (u_{\delta\delta r} - u_{\alpha\alpha r}) + \frac{32}{3} K_{\delta\alpha} c_{\delta\alpha} (u_{\delta\delta r} - u_{\alpha\alpha r}) \right\}, \tag{3.1}$$

The right hand side of the above equation can be rewritten as

$$\sum_\delta \left\{ \frac{N_\delta}{m_\alpha} \left(\frac{\mathcal{A}_{\delta\alpha}}{\sigma_{\delta\alpha}} + \frac{32}{3} \mathcal{K}_{\delta\alpha} c_{\delta\alpha} \right) (u_{\delta\delta r} - u_{\alpha\alpha r}) \right\}$$

$$\equiv \sum_\delta \frac{N_\delta}{m_\alpha} D_{\delta\alpha} (u_{\delta\delta r} - u_{\alpha\alpha r}), \tag{3.2}$$

where
$$D_{\delta\alpha} = \frac{\mathcal{A}_{\delta\alpha}}{\sigma_{\delta\alpha}} + \frac{32}{3}\mathcal{K}_{\delta\alpha}\, c_{\delta\alpha},$$

from which we conclude that the contribution of the Fokker-Planck term to the momentum equation for this simple case is to increase the number of collisions per unit volume per unit mass as expected. Consequently, for the Lorentz problem, the current density expressions remain unaltered and are given by [5.1] – [5.3] of *I* excepting that

$$\frac{\mathcal{A}_{\delta\alpha}}{\sigma_{\delta\alpha}} \text{ is replaced by } \frac{\mathcal{A}_{\delta\alpha}}{\sigma_{\delta\alpha}} + \frac{32}{3}\mathcal{K}_{\delta\alpha}\, c_{\delta\alpha},$$

where
$$\mathcal{K}_{\delta\alpha} = \frac{m_\alpha m_\delta}{m_\alpha + m_\delta}(b_u^2 - b_l^2)\sqrt{\left(\frac{\pi}{2}\right)} \qquad [3.3]$$

Hence, the electrical conductivity σ and the generalized diffusivities σ_α and σ_β are given by

$$\sigma = \frac{(e_\beta N_\beta - e_\alpha N_\alpha)^2}{4 N_\alpha N_\beta}\cdot\frac{D_2}{D_1}, \qquad [3.4]$$

$$\sigma_\alpha = \frac{(e_\beta N_\beta - e_\alpha N_\alpha)}{2 D_1} D_{\alpha\gamma}, \qquad [3.5]$$

$$\sigma_\beta = \frac{(e_\alpha N_\alpha - e_\beta N_\beta)}{2 D_1} D_{\beta\gamma}, \qquad [3.6]$$

where
$$D_1 = N_\alpha D_{\beta\alpha} D_{\alpha\gamma} + N_\beta D_{\gamma\beta} D_{\beta\alpha} + N_\gamma D_{\beta\gamma} D_{\gamma\alpha}, \qquad [3.7]$$

and
$$D_2 = N_\alpha D_{\alpha\gamma} + N_\beta D_{\beta\gamma}. \qquad [3.8]$$

From [3.4] we conclude that due to the diffusive effects of Fokker-Planck term, the electrical conductivity decreases, this is natural since the effect is to increase the number of collisions.

Concentrating on the gradient dependence of the stresses and rewriting the equations [2.16] and [2.17], we obtain,

$$(\tau_\alpha + 2\mathcal{M}_\alpha) P_{\alpha rr} + \mathcal{N}_\alpha P_{\alpha ll} - 2\omega_\alpha \epsilon_{rj3} P_{\alpha jr} = -\frac{1}{N_\alpha}\cdot\frac{\partial}{\partial x_l}(N_\alpha S_{\alpha lrr}), \qquad [3.9]$$

and
$$(\tau_\alpha + \mathcal{M}_\alpha) P_{\alpha rs} - \omega_\alpha(\epsilon_{rj3} P_{\alpha js} + \epsilon_{sj3} P_{\alpha jr}) = -\frac{1}{N_\alpha}\cdot\frac{\partial}{\partial x_l}(N_\alpha S_{\alpha lrs}) \qquad [3.10]$$

where
$$\omega_\alpha = \frac{e_\alpha H}{c m_\alpha}, \qquad [3.11]$$

$$\mathcal{M}_\alpha = \sum_\delta \left[\frac{64}{15} \cdot \frac{K_{\delta\alpha}}{c_{\delta\alpha}} (6 c_\alpha^2 + 5 c_\delta^2) - \frac{128}{5} L_{\delta\alpha} c_{\delta\alpha} \right] + \frac{32\sqrt{2}}{15} K_{\alpha\alpha} c_\alpha - \frac{128\sqrt{2}}{5} L_{\alpha\alpha} c_\alpha , \qquad [3.12]$$

and

$$\mathcal{N}_\alpha = \frac{64}{15} \sum_\delta \left[\frac{K_{\delta\alpha}}{c_{\delta\alpha}} c_\alpha^2 - L_{\delta\alpha} c_{\delta\alpha} \right] + \frac{32\sqrt{2}}{15} (K_{\alpha\alpha} - 2 L_{\alpha\alpha}) c_\alpha ; \qquad [3.13]$$

For simplicity of argument, we have chosen the z-axis along the direction of the magnetic field. As in *I*, we replace the heat-flux tensors by their equivalent moments and solving [3.9] and [3.10] we obtain

$$P_{\alpha 33} = - \mu_{\alpha 33}^{(0)} e_{\alpha 33} - \mu_{\alpha 33}^{(1)} \nabla T_{\alpha\alpha} - \mu_{\alpha 33}^{(2)} \nabla N_\alpha - \mu_{\alpha 33}^{(3)} (e_{\alpha 11} + e_{\alpha 22} + e_{\alpha 33}), \qquad [3.14]$$

where

$$\mu_{\alpha 33}^{(0)} = \frac{2 K T_{\alpha\alpha}}{m_\alpha} \cdot \frac{1}{\tau_\alpha + 2 \mathcal{M}_\alpha} , \qquad [3.15]$$

$$\mu_{\alpha 33}^{(1)} = \frac{K}{m_\alpha^{!} (\tau + 2 \mathcal{M}_\alpha)} (u_{\alpha\alpha 1}, u_{\alpha\alpha 2}, u_{\alpha\alpha 3}) - \frac{5 K \mathcal{N}_\alpha}{m_\alpha (\tau + 2 \mathcal{M}_\alpha)(\tau_\alpha + 2 \mathcal{M}_\alpha + 3 \mathcal{N}_\alpha)} (u_{\alpha\alpha 1}, u_{\alpha\alpha 2}, u_{\alpha\alpha 3}), \qquad [3.16]$$

$$\mu_{\alpha 33}^{(2)} = \frac{6 K T_{\alpha\alpha}}{N_\alpha m_\alpha (\tau_\alpha + 2 \mathcal{M}_\alpha)} (0, 0, u_{\alpha\alpha 3}) - \frac{K T_{\alpha\alpha} \mathcal{N}_\alpha}{N_\alpha m_\alpha (\tau_\alpha + 2 \mathcal{M}_\alpha)(\tau_\alpha + 2 \mathcal{M}_\alpha + 3 \mathcal{N}_\alpha)} (u_{\alpha\alpha 1}, u_{\alpha\alpha 2}, u_{\alpha\alpha 3}) \qquad [3.17]$$

and

$$\mu_{\alpha 33}^{(3)} = \mu_{\alpha 33}^{(0)} \frac{\mathcal{N}_\alpha}{\tau_\alpha + 2 \mathcal{M}_\alpha + 3 \mathcal{N}_\alpha} \qquad [3.18]$$

$$\begin{bmatrix} P_{\alpha 23} \\ P_{\alpha 31} \end{bmatrix} = - \mu_{\alpha 3}^{(0)} \begin{bmatrix} e_{\alpha 23} \\ e_{\alpha 31} \end{bmatrix} - \mu_{\alpha 3}^{(1)} \nabla T_{\alpha\alpha} - \mu_{\alpha 3}^{(2)} \nabla N_\alpha , \qquad [3.19]$$

where

$$\mu_{\alpha 3}^{(0)} = \frac{2 K T_{\alpha\alpha}}{m_\alpha \Delta_1} \begin{bmatrix} \tau_\alpha + \mathcal{M}_\alpha & - \omega_\alpha \\ \omega_\alpha & \tau_\alpha + \mathcal{M}_\alpha \end{bmatrix} \qquad [3.20]$$

$$\mu_{a3}^{(1)} = \frac{1}{2\,T_{aa}}\,\mu_{a3}^{(0)} \begin{bmatrix} 0 & u_{aa3} & u_{aa2} \\ u_{aa3} & 0 & u_{aa1} \end{bmatrix}, \qquad [3.21]$$

$$\mu_{a3}^{(2)} = \frac{T_{aa}}{N_a}\,\mu_{a33}^{(1)}, \qquad [3.22]$$

$$\Delta_1 = (\tau_a + \mathcal{M}_a)^2 + \omega_a^2, \qquad [3.23]$$

$$P_{a12} = -\,\mu_{a12}^{(0)} \begin{Bmatrix} e_{a11} \\ e_{a12} \\ e_{a22} \end{Bmatrix} - \mu_{a12}^{(1)}\,\nabla\,T_{aa} - \mu_{a12}^{(2)}\,\nabla N_a, \qquad [324]$$

where

$$\mu_{a12}^{(0)} = \frac{2K\,T_{aa}}{m_a\,\Delta_2}\left(-\,\omega_a,\ \tau_a + 2\,\mathcal{M}_a,\ \omega_a\right), \qquad [3.25]$$

$$\mu_{a12}^{(1)} = \frac{K}{m_a\,\Delta_2}\,(\tau_a + 2\,\mathcal{M}_a)\,u_{aa2} - 2\,\omega_a\,u_{aa1},\ (\tau_a + 2\,\mathcal{M}_a)\,u_{aa1} + 2\omega_a\,u_{aa2},\,0] \qquad [3.26]$$

$$\mu_{a12}^{(2)} = \frac{T_{aa}}{N_a}\,\mu_{a12}^{(1)}, \qquad [3.27]$$

$$\Delta_2 = (\tau_a + \mathcal{M}_a)^2 + 4\,\omega_a^2. \qquad [3.28]$$

$$P_{a11} = -\,\mu_{a11}^{(0)} \begin{Bmatrix} e_{a11} \\ e_{a12} \\ e_{a22} \end{Bmatrix} - \mu_{a11}^{(1)}\,\nabla\,T_{aa} - \mu_{a11}^{(2)}\,\nabla\,N_a$$
$$-\,\mu_{a11}^{(3)}\left(e_{a11} + e_{a22} + e_{a33}\right), \qquad [3\ 29]$$

where

$$\mu_{a11}^{(0)} = \frac{2\,K\,T_{aa}}{m_a\,(\tau_a + 2\mathcal{M}_a)\,\Delta_2}[(\tau_a + 2\mathcal{M}_a)(\tau_a + \mathcal{M}_a) + 2\omega_a^2,\,2(\tau_a + 2\,\mathcal{M}_a)\,\omega_a,\,2\omega_a^2] \qquad [3.30]$$

$$\mu_{a11}^{(1)} = \frac{K}{m_a\,(\tau_a + 2\,\mathcal{M}_a)\,\Delta_2}\,[\{(\tau_a + 2\,\mathcal{M}_a)\,(\tau_a + \mathcal{M}_a) + 2\,\omega_a^2\}\,(3\,u_{aa1},\,u_{aa2},\,0)$$
$$+\,2\,(\tau_a + 2\,\mathcal{M}_a)\,\omega_a\,(u_{aa2},\,u_{aa1},\,0)$$
$$+\,2\,\omega_a^2\,(u_{aa1},\,3\,u_{aa2},\,0) + \Delta_2\,(0,\,0,\,u_{aa3})]$$
$$-\,\frac{5\,K\mathcal{N}_a}{m_a\,(\tau_a + \mathcal{M}_a)\,(\tau_a + 2\,\mathcal{M}_a + 3\,\mathcal{N}_a)}\,(u_{aa1},\,u_{aa2},\,u_{aa3}), \qquad [3.31]$$

$$\mu^{(2)}_{a11} = \frac{2KT_{aa}}{m_a N_a(\tau_a + 2\mathcal{M}_a)\,\Delta_2}$$
$$\times [\{(\tau_a + \mathcal{M}_a)(\tau_a + 2\mathcal{M}_a) + \omega_a^2\} u_{aa1} + (\tau_a + 2\mathcal{M}_a)\,\omega_a u_{aa2},$$
$$(\tau_a + 2\mathcal{M}_a)\,\omega_a u_{aa1} + 2\,\omega_a^2 u_{aa2},\, 0]$$
$$- \frac{2KT_{aa}\mathcal{N}_a}{N_a m_a(\tau_a + 2\mathcal{M}_a)(\tau_a + 2\mathcal{M}_a + 3\mathcal{N}_a)}(u_{aa1},\, u_{aa2},\, u_{aa3}), \qquad [3.32]$$

and

$$\mu^{(3)}_{a11} = \frac{2KT_{aa}\mathcal{N}_a}{m_a(\tau_a + 2\mathcal{M}_a)(\tau_a + 2\mathcal{M}_a + 3\mathcal{N}_a)} \qquad [3.33]$$

with similar expressions for P_{a22}.

The following important points have to be noticed regarding the viscosity matrices: The Fokker-Planck terms introduce very high anisotropy even in the absence of magnetic field and the magnitudes of the viscosity coefficients in the principal directions are decreased. This is natural since the Fokker-Planck terms increase the collisions, as we have already pointed out. Further the diagonal terms of the stress tensor depend also on the dilatation term $(e_{a11} + e_{a22} + e_{a33})$ apart from other couplings. From this we conclude that one of the functions of Fokker-Planck terms is to increase the bulk viscosity of the plasma apart from introducing the anisotropy. Along the direction of the magnetic field, the main viscosity term is given by

$$\frac{2KT_{aa}}{m_a\tau_a}\left(1 - \frac{2\mathcal{M}_a}{\tau_a + 2\mathcal{M}_a}\right) \qquad [3.34]$$

Thus, the viscosity is decreased by a factor of

$$\left(1 - \frac{2\mathcal{M}_a}{\tau_a + 2\mathcal{M}_a}\right)$$

by the Fokker-Planck terms. As in *I*, the stresses in the plane containing the magnetic field are coupled by the magnetic field and the viscosity is given by

$$\frac{2KT_{aa}}{m_a\tau_a}\left(1 - \frac{\mathcal{M}_a}{\tau_a + \mathcal{M}_a}\right)\left(1 - \frac{\omega_a^2}{(\tau_a + \mathcal{M}_a)^2 + \omega_a^2}\right) \qquad [3.35]$$

The effect of the magnetic field and the Fokker-Planck terms are self-evident. The viscosity co-efficients corresponding to P_{a12} and P_{a11} and P_{a22} are given by

$$\frac{2KT_{aa}}{m_a\tau_a}\left[1 - \frac{\mathcal{M}_a^2 + 4\omega_a^2}{(\tau_a + \mathcal{M}_a)^2 + 4\omega_a^2}\right] \qquad [3.36]$$

and

$$\frac{2\,K\,T_{aa}\,[(\tau_a + 2\mathcal{M}_a)(\tau_a + \mathcal{M}_a) + 2\,\omega_a^2]}{m_a\,(\tau_a + 2\mathcal{M}_a)\,[(\tau_a + \mathcal{M}_a)^2 + 4\,\omega_a^2]}. \qquad [3.37)$$

The anisotropy is evident even in the absence of the magnetic field.

Proceeding in a similar fashion, for the heat flux vector, we obtain,

$$S_{a3} = -K_{a3}^{(0)}\,\nabla_3\,T_{aa} - K_{a3}^{(1)}\,\nabla_3\,N_a, \qquad [3.38]$$

$$\begin{bmatrix} S_{a1} \\ S_{a2} \end{bmatrix} = -K_a^{(0)}\,\nabla_1\,T_{aa} - K_a^{(1)}\,\nabla_1\,N_a, \qquad [3.39]$$

where

$$K_{a3}^{(0)} = \frac{5K^2\,T_{aa}}{m_a^2\,\tau_a}\left[1 - \frac{\tau_a(K_a + 2\,J_a) + J_a\,K_a}{(\tau_a + K_a)(\tau_a + K_a + J_a)}\right] \qquad [3.40]$$

$$K_{a3}^{(1)} = \frac{T_{aa}}{2\,N_a}\,K_{a3}^{(0)}, \qquad [3.41]$$

$$K_a^{(0)} = \frac{5\,K^2\,T_{aa}}{m_a^2[(\sigma_a + K_a + J_a)^2 + \omega_a^2]}\begin{bmatrix} \gamma_a + K_a + J_a & \omega_a \\ -\,\omega_a & \tau_a + K_a + J_a \end{bmatrix} \qquad [3.42]$$

$$K_a^{(1)} = \frac{T_{aa}}{2N_a}\,K_a^{(0)}, \qquad [3.43]$$

where

$$J_a = \sum_\delta\left[\frac{16\,K_{\delta a}}{35\,c_{\delta a}^3}\,(14\,c_\delta^2 + 11\,c_a^2)\,c_a^2 - \frac{32\,L_{\delta a}}{35\,c_{\delta a}}\,(7\,c_\delta^2 + 18\,c_a^2)\right]$$
$$+\left(\frac{12\sqrt{2}}{35}\,K_{aa} + \frac{176\sqrt{2}}{35}\,L_{aa}\right)c_a \qquad [3.44]$$

and

$$K_a = \sum_\delta\left[\frac{16\,K_{\delta a}}{105\,c_{\delta a}^3}\{299\,c_a^4 + 518\,c_a^2\,c_\delta^2 + 210\,c_\delta^4\}\right.$$
$$-\frac{32\,L_{\delta a}}{105\,c_{\delta a}}\{306\,c_a^2 + 259\,c_\delta^2\}$$
$$+\left(\frac{12\sqrt{2}}{35}\,K_{aa} + \frac{752\sqrt{2}}{105}\,L_{aa}\right)c_a. \qquad [3\,45]$$

As in the case of stresses, both the magnetic field and the Fokker-Planck terms introduce anisotropy, even though the intense coupling is mainly due to magnetic field. The coefficient of heat conductivity along the magnetic field is

$$\frac{5K^2 T_{\alpha\alpha}}{m_\alpha^2 \tau_\alpha}\left[1-\frac{\tau_\alpha(K_\alpha+2J_\alpha)+J_\alpha K_\alpha}{(\tau_\alpha+K_\alpha)(\tau_\alpha+K_\alpha+J_\alpha)}\right], \qquad [3.46]$$

while transverse to the magnetic field, it is

$$\frac{5K^2 T_{\alpha\alpha}}{m_\alpha^2 \tau_\alpha}\left[1-\frac{\tau_\alpha(J_\alpha+K_\alpha+J_\alpha)+\omega_\alpha^2}{(\tau_\alpha+K_\alpha+J_\alpha)^2+\omega_\alpha^2}\right], \qquad [3.47]$$

Two points are worthy of note : (i) The magnitudes of these coefficients are reduced mainly by the Fokker-Planck terms and the magnetic field. (ii) The ratio of the coefficients of heat conductivity and viscosity along the magnetic field is no longer a constant, as in the pure collisional case. This suggests that pure ideal gas approximation and adiabatic approximations may not be strictly valid for the high density, high temperature plasmas, or gases under long-range forces of interaction.

Appendix I

Following Jeans[17], we shall derive the change in the molecular velocity $\vec{\xi}_\alpha$ of the particle m_α due to its interaction with a particle m_δ due to its interaction with a particle m_δ moving with velocity $\vec{\xi}_\delta$.

Consider relative motion of m_δ with respect to m_α situated at O. Let the plane of interaction make an angle ϵ with the fixed plane XOG and let θ be the angle through which the relative velocity $g_{\delta\alpha}=\vec{\xi}_\delta-\vec{\xi}_\alpha$ turns after interaction. Let X, Y, Z, G, G' be the points where the coordinate axes and the relative velocities before and after collision intersect the unit sphere with centre at the origin. Denoting by prime the velocities after collision, from conservation of momentum, we have

$$m_\alpha\vec{\xi}_\alpha+m_\delta\vec{\xi}_\delta=m_\alpha\vec{\xi}'_\alpha+m_\delta\vec{\xi}'_\delta. \qquad [I.1]$$

Also, from the spherical triange XGG',

$$\cos XG' = \cos XG\cos GG' + \sin XG \sin GG' \cos \angle XGG',$$

or

$$\xi'_{\alpha x}-\xi'_{\delta x}=(\xi_{\delta x}-\xi_{\alpha x})\cos 2\theta+(g_{\delta\alpha y}^2+g_{\delta\alpha z}^2)^{1/2}\sin 2\theta\cos\epsilon. \qquad [I.2]$$

Eliminating $\xi'_{\delta x}$ between [I.1] and [I.2], we get

$$\Delta\, \xi_{\alpha x} = \xi'_{\alpha x} - \xi_{\alpha x}$$

$$= \frac{1}{m_\alpha + m_\delta}\left[m_\delta\,(1+\cos 2\,\theta)\,(\xi_{\delta x} - \xi_{\alpha x}) + m_\delta\,(g^2_{\delta\alpha y} + g^2_{\delta\alpha z})^{1/2}\sin 2\,\theta\cos\epsilon\right]. \qquad [I.3]$$

Similarly, from the spherical triangle YGG', we have

$$\xi'_{\alpha y} - \xi'_{\delta y} = (\xi_{\delta y} - \xi_{\alpha y})\cos 2\,\theta + (g^2_{\delta\alpha z} + g^2_{\delta\alpha x})^{1/2}\sin 2\,\theta$$

$$\times\,(\cos\omega_1\cos\epsilon + \sin\omega_1\sin\epsilon), \qquad [I.4]^*$$

where ω_1 is the angle between the planes XOG and YOG. From the spherical triangle XYG, we have

$$0 = (\xi_{\delta x} - \xi_{\alpha x})\,(\xi_{\delta y} - \xi_{\alpha y}) + [(g^2_{\delta\alpha} - g^2_{\delta\alpha x})\,(g^2_{\delta\alpha} - g^2_{\delta\alpha y})]^{1/2}\cos\omega_1. \qquad [I.5]$$

Eliminating $\xi'_{\delta y}$ and ω_1 from [I.1], [1.4] and [1.5] we get

$$\Delta\, \xi_{\alpha y} = \xi'_{\alpha y} - \xi_{\alpha y}$$

$$= \frac{m_\delta}{m_\alpha + m_\delta}\Bigg[(1+\cos 2\,\theta)\,(\xi_{\delta y} - \xi_{\alpha y})$$

$$+ \frac{\sin 2\,\theta}{(g^2_{\delta\alpha} - g^2_{\delta\alpha x})^{1/2}}\{-g_{\delta\alpha x}\,g_{\delta\alpha y}\cos\epsilon + g_{\delta\alpha}\,g_{\delta\alpha y}\sin\epsilon\}\Bigg], \qquad [1.6]$$

Similarly,

$$\Delta\, \xi_{\alpha z} = \xi'_{\alpha z} - \xi_{\alpha z}$$

$$= \frac{m_\delta}{m_\alpha + m_\delta}\Bigg[(1+\cos 2\,\theta)\,(\xi_{\delta z} - \xi_{\alpha z})$$

$$- \frac{\sin 2\,\theta}{(g^2_{\delta\alpha} - g^2_{\delta\alpha x})^{1/2}}\{g_{\delta\alpha x}\,g_{\delta\alpha z}\cos\epsilon + g_{\delta\alpha}\,g_{\delta\alpha y}\sin\epsilon\}\Bigg]. \qquad [1.7]$$

* Jeans has taken minus sign instead of $+ve$ sign in the last factor. This alters the expressions for $\Delta\xi_{\alpha y}$ and $\Delta\xi_{\alpha z}$.

APPENDIX II

$$A_{\delta\alpha rrr} = (16\,K_{\delta\alpha}/35 c_{\delta\alpha}^3)\,[\{-14\,(13\,c_\alpha^2 + 10\,c_\delta^2)\,c_\alpha^2\,u_{\alpha\alpha r} + 42\,c_\alpha^4\,u_{\delta\delta r}\}\,c_{\delta\alpha}^2$$

$$-(107\,c_\alpha^4 + 182\,c_\alpha^2\,c_\delta^2 + 70\,c_\delta^4)\,(S_{\alpha rrr} - 3\,c_\alpha^2\,u_{\alpha\alpha r}) - 5\,c_\alpha^4\,(S_{\delta rrr} - 3\,c_\delta^2\,u_{\delta\delta r})$$

$$+(139\,c_\alpha^4 + 224\,c_\alpha^2\,c_\delta^2 + 70\,c_\delta^4)\,(P_{\alpha rr} - c_\alpha^2)\,u_{\delta\delta r}$$

$$-(27\,c_\alpha^4 + 42\,c_\alpha^2\,c_\delta^2)\,(P_{\delta rr} - c_\delta^2)\,u_{\alpha\alpha r} + \sum_i\{-3\,c_\alpha^4\,(S_{\delta iir} - c_\delta^2\,u_{\delta\delta r})$$

$$+(11\,c_\alpha^4 + 14\,c_\alpha^2\,c_\delta^2)\,(S_{\alpha iir} - c_\alpha^2\,u_{\alpha\alpha r}) - 3\,c_\alpha^4\,(P_{\alpha ii} - c_\alpha^2)\,u_{\delta\delta r}$$

$$+(11\,c_\alpha^4 + 14\,c_\alpha^2\,c_\delta^2)\,(P_{\delta ii} - c_\delta^2)\,u_{\alpha\alpha r} + (22\,c_\alpha^4 + 28\,c_\alpha^2\,c_\delta^2)\,P_{\alpha ir}\,u_{\delta\delta i}$$

$$+6\,c_\alpha^4\,P_{\delta ir}\,u_{\alpha\alpha i}\}] + (32\,L_{\delta\alpha}/35\,c_{\delta\alpha})[14\,c_{\delta\alpha}^2\,\{(5\,c_\delta^2 + 18\,c_\alpha^2)\,u_{\alpha\alpha r} - 13\,c_\alpha^2\,u_{\delta\delta r}\}$$

$$-23\,c_\alpha^2\,(S_{\delta rrr} - 3\,c_\delta^2\,u_{\delta\delta r}) + (91\,c_\delta^2 + 114\,c_\alpha^2)\,(S_{\alpha rrr} - 3\,c_\alpha^2\,u_{\alpha\alpha r})$$

$$+(91\,c_\delta^2 + 160\,c_\alpha^2)\,(P_{\delta rr} - c_\delta^2)\,u_{\alpha\alpha r} - (182\,c_\delta^2 + 251\,c_\alpha^2)\,(P_{\alpha rr} - c_\alpha^2)\,u_{\delta\delta r}$$

$$+\sum_i\{(7\,c_\delta^2 + 18\,c_\alpha^2)\,(S_{\alpha iir} - c_\alpha^2\,u_{\alpha\alpha r}) - 11\,c_\alpha^2\,(S_{\delta iir} - c_\delta^2\,u_{\delta\delta r})$$

$$+(7\,c_\delta^2 + 18\,c_\alpha^2)\,(P_{\delta ii} - c_\delta^2)\,u_{\alpha\alpha r} - 11\,c_\alpha^2\,(P_{\alpha ii} - c_\alpha^2)\,u_{\delta\delta r}$$

$$+22\,c_\alpha^2\,P_{\delta ir}\,u_{\alpha\alpha i} - (14\,c_\delta^2 + 36\,c_\alpha^2)\,P_{\alpha ir}\,u_{\delta\delta i}\}]$$

$$B_{\delta\alpha rrs} = (16\,K_{\delta\alpha}/105\,c_{\delta\alpha}^3)[14\,c_\alpha^2\,c_{\delta\alpha}^2\,\{3\,c_\alpha^2\,(u_{\delta\delta s} - u_{\alpha\alpha s}) - 10\,c_{\delta\alpha}^2\,u_{\alpha\alpha s}\}$$

$$-3\,c_\alpha^4\,(S_{\delta sss} - S_{\alpha sss}) - 14\,c_\alpha^2\,c_{\delta\alpha}^2\,S_{\alpha sss} + 9\,c_\alpha^4\,c_\delta^2\,(u_{\alpha\alpha s} + u_{\delta\delta s})$$

$$+33\,c_{\delta\alpha}^2\,c_\alpha^4\,u_{\alpha\alpha s} + (19\,c_\alpha^4 + 28\,c_\alpha^2\,c_\delta^2)\,(P_{\alpha ss} - c_\alpha^2)\,u_{\delta\delta s}$$

$$-(5\,c_\alpha^4 + 11\,c_\alpha^2\,c_\delta^2)\,(P_{\delta ss} - c_\delta^2)\,u_{\alpha\alpha s} + \sum_i\{3\,c_\alpha^4\,(S_{\delta iis} - c_\delta^2\,u_{\delta\delta s})$$

$$+(11\,c_\alpha^4 + 14\,c_\alpha^2\,c_\delta^2)\,(S_{\alpha iis} - c_\alpha^2\,u_{\alpha\alpha s}) + 6\,\delta_{ir}\,c_\alpha^4\,(S_{\delta iis} - c_\delta^2\,u_{\delta\delta s})$$

$$+(288\,c_\alpha^4 + 504\,c_\alpha^2\,c_\delta^2 + 210\,c_\delta^4)\,\delta_{ir}\,(S_{\alpha iis} - c_\alpha^2\,u_{\alpha\alpha s})$$

$$-3\,c_\alpha^4(P_{\alpha ii} - c_\alpha^2)\,u_{\delta\delta s} - (11\,c_\alpha^4 + 14\,c_\alpha^2\,c_\delta^2)\,(P_{\delta iis} - c_\delta^2)\,u_{\alpha\alpha s}$$

$$+(70\,c_\delta^4 + 196\,c_\alpha^2\,c_\delta^2 + 120\,c_\alpha^4)\,\delta_{ir}\,(P_{\alpha ii} - c_\alpha^2)\,u_{\delta\delta s}$$

$$- (22\, c_\alpha^4 + 28\, c_\alpha^2 c_\delta^2)\, \delta_{ir} (P_{\delta ii} - c_\delta^2)\, u_{\alpha\alpha s} + (22\, c_\alpha^4 + 28\, c_\alpha^2 c_\delta^2)\, P_{\alpha is} u_{\delta\delta i}$$

$$+ 6\, c_\alpha^4 P_{\delta is} u_{\alpha\alpha i} + (240\, c_\alpha^4 + 392\, c_\alpha^2 c_\delta^2 + 140\, c_\delta^4)\, \delta_{ir} P_{\alpha is} u_{\delta\delta i}$$

$$- (44\, c_\alpha^4 + 56\, c_\alpha^2 c_\delta^2)\, \delta_{ir} P_{\delta is} u_{\alpha\alpha i}\}]$$

$$+ (32\, L_{\delta\alpha}/105\, c_{\delta\alpha}) [14\, c_{\delta\alpha}^2 \{(18\, c_\alpha^2 + 5\, c_\delta^2)\, u_{\alpha\alpha s} - 13\, c_\alpha^2 u_{\delta\delta s}\}$$

$$+ (7\, c_\delta^2 + 18\, c_\alpha^2)(S_{\alpha sss} - 3\, c_\alpha^2 u_{\alpha\alpha s}) - 11\, c_\alpha^2 (S_{\delta sss} - 3\, c_\delta^2 u_{\delta\delta s})$$

$$+ (7\, c_\delta^2 + 40\, c_\alpha^2)(P_{\delta ss} - c_\delta^2)\, u_{\alpha\alpha s} - (14\, c_\delta^2 + 47\, c_\alpha^2)(P_{\alpha ss} - c_\alpha^2)\, u_{\delta\delta s}$$

$$+ (7\, c_\delta^2 + 18\, c_\alpha^2)(P_{\delta ss} - c_\delta^2)\, u_{\alpha\alpha s} - 11\, c_\alpha^2 (P_{\alpha ss} - c_\alpha^2)\, u_{\delta\delta s}$$

$$+ \sum_i \{(7\, c_\delta^2 + 18\, c_\alpha^2)(S_{\alpha iis} - c_\alpha^2 u_{\alpha\alpha s}) - 11\, c_\alpha^2 (S_{\delta iis} - c_\delta^2 u_{\delta\delta s})$$

$$+ (252\, c_\delta^2 + 228\, c_\alpha^2)\, \delta_{ir} (S_{\alpha iis} - c_\alpha^2 u_{\alpha\alpha s}) - 36\, c_\alpha^2 \delta_{ir} (S_{\delta iis} - c_\delta^2 u_{\delta\delta s})$$

$$+ 22\, c_\alpha^2 P_{\delta is} u_{\alpha\alpha i} - (14\, c_\delta^2 + 36\, c_\alpha^2)\, P_{\alpha is} u_{\delta\delta i}$$

$$+ (84\, c_\delta^2 + 120\, c_\alpha^2)\, \delta_{ir} (P_{\delta ss} - c_\delta^2)\, u_{\alpha\alpha s}$$

$$- (168\, c_\delta^2 + 204\, c_\alpha^2)\, \delta_{ir} (P_{\alpha ss} - c_\alpha^2)\, u_{\delta\delta s} + (240\, c_\alpha^2 + 168\, c_\delta^2)\, \delta_{ir} P_{\delta is} u_{\alpha\alpha i}$$

$$- (336\, c_\delta^2 + 408\, c_\alpha^2)\, \delta_{ir} P_{\alpha is} u_{\delta\delta i}\}]$$

$$C_{\delta arst} = - (32\, K_{\delta\alpha}/105\, c_{\delta\alpha}^3)[- 3\, c_\alpha^4 S_{\delta rst} + (105\, c_\delta^4 + 252\, c_\alpha^2 c_\delta^2 + 144\, c_\alpha^4)\, S_{\alpha rst}$$

$$+ (11\, c_\alpha^4 + 14\, c_\alpha^2 c_\delta^2)(P_{\delta 23} u_{\alpha\alpha 1} + P_{\delta 31} u_{\alpha\alpha 2} + P_{\delta 12} u_{\alpha\alpha 3})$$

$$- (60\, c_\alpha^4 + 98\, c_\alpha^2 c_\delta^2 + 35\, c_\delta^4)(P_{\alpha 23} u_{\delta\delta 1} + P_{\alpha 31} u_{\delta\delta 2} + P_{\alpha 12} u_{\delta\delta 3})]$$

$$+ (64\, L_{\delta\alpha}/35\, c_{\delta\alpha}) [(42\, c_\delta^2 + 48\, c_\alpha^2)\, S_{\alpha rst} - 6\, c_\alpha^2 S_{\delta rst}$$

$$+ (14\, c_\delta^2 + 20\, c_\alpha^2)(P_{\delta 23} u_{\alpha\alpha 1} + P_{\delta 31} u_{\alpha\alpha 2} + P_{\delta 12} u_{\alpha\alpha 3})$$

$$- (28\, c_\delta^2 + 34\, c_\alpha^2)(P_{\alpha 23} u_{\delta\delta 1} + P_{\alpha 31} u_{\delta\delta 2} + P_{\alpha 12} u_{\delta\delta 3})].$$

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