CONVERGENCE OF TYPE DISTRIBUTION IN A GENERAL GROWTH MODEL

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ABSTRACT

Athreya and Kaplan proved the convergence of the age distribution in a supercritical one-dimensional Bellman-Harris process. In this paper the techniques of that paper are applied to a general growth model introduced by Jagers. The results are also specialized to age-dependent birth and death processes.

Key words : Age-dependent branching process, type-distribution, Jagers' model, birth and death process convergence.

1. INTRODUCTION

In a recent paper [1] Athreya and Kaplan established the convergence of the age distribution in a supercritical one dimensional Bellman-Harris process under fairly general conditions. The techniques of that paper are amenable to a great degree of generalization and in this paper, we apply them to prove a corresponding result about a general growth model introduced by Jagers [5]. We also specialize our results to the age-dependent birth and death process.

2. THE MODEL

The distinguishing feature in Jagers' model is that the offspring production does not have to wait till the death of the parent as in Bellman-Harris processes. Jagers postulates that to every individual x entering the processes there is an associated pair of objects (λ_x, μ_x) where λ_x is a non-negative random variable denoting the life time of the individual and μ_x a point process on $[0, \infty)$ such that $\mu_x[\lambda_x, \infty) = 0$ with probability one. It is assumed that $\mu_x (0, \lambda_x) < \infty$ w.p.1. It is not assumed that λ_x and μ_x are independent. Finally, it is assumed that the pairs (λ_x, μ_x) as x varies over all the individuals are mutally independent. The rigorous construction of such a process along the lines of Harris' family histories treatment [4] is done by Jagers in [5] and more recently in his book [6].

3. Preliminaries

We shall follow the notation in [5].

Let Z_t denote the total number of particles in the system and $Y_t = (\theta_1, \theta_2, \ldots \theta_{Z_t}$ denote the 'type' chart at time t where θ_i is an element of the type space Θ and consists of the present age and history of the associated particle production μ upto the present. Thus Θ

$$= \{(a, \mu(v) : v \leqslant a), \ a \geqslant 0,$$

$\mu(v) \geqslant 0$ is a nondecreasing integer valued right continuous function$\}$.

A moment's reflection shows that the stochastic process $\{Y_t : t \geqslant 0\}$ is a Markov process with stationary transition probabilities. The state space consists of

$E = \overset{\infty}{\underset{n=0}{U}} \Theta^n / \sim$ where Θ^n is the n-fold cartesian product of Θ and $\sim$ denotes the equivalence relation on Θ^n equating two vectors if they have the same components.

Let $\mathscr{B}$ be the σ-algebra on Θ generated by cylinder sets of the form

$\{\theta : \theta = \{a, \mu(v) : v \leqslant a\};$

$a \leqslant r, \ \mu(A_1) = r_1, \ \ldots . \mu(A_k) = r_k\}$

where r is a +ve number

r_i's are+ve integers

and A_i's are intervals in [0, a].

Define now the type distribution at time t by

$$A(t, B, \omega) = \frac{1}{Z_t} \sum_{i=1}^{Z} \chi_B(\theta_i) \qquad (*)$$

where $B \epsilon \mathscr{B}$ and $(\theta_1, \theta_2, \ldots \theta_{Z_t})$ is the type chart at time t. Clearly, $A(t, \cdot, \omega)$ is a random probability measure.

We now study the convergence of this measure as $t \to \infty$ assuming the following self explanatory conditions : For each individual x.

(A·1) $P\{\mu_x(\lambda_x) = 0\} = 0$

(A·2) $P\{\lambda_x > 0\} = 1$

(A·3) $E\{ u_x (\lambda_x)\} < \infty$

(A·4) $P\{\mu_x \lambda_x, \infty) = 0\} = 1.$

Here $(\lambda_x, \mu_x(v) : v \geqslant 0)$ denote the life time variable and the particle production process associated with individual x. Recall the assumption of Jagers that the distributions of $(\lambda_x, \mu_x(\cdot))$ over all the individuals entering the population are i.i.d.

Our main result is that $A(t, \ . \ , \omega)$ converges to a deterministic probability measure as $t \to \infty$ provided some mild regularity conditions are satisfied. Let $Z(\theta, s, \mathrm{B})$ denote the random variable = number of particles in the set B at time s in a colony starting from one particle of type θ at time 0.

Let $M(\theta, s, B) = E\{Z(\theta, s, B)\}$.

The following integral equation for $M(\theta, s, B)$ is now immediate from the definition of a Jagers process.

$$M(\theta, s, B) = q(\theta, s) + \int_0^s M(O, s-u, B)\, E\big(d\mu(u+a) \mid (a, \mu(v): v \leqslant a)\big) \tag{1}$$

where $\theta = (a, \mu(v): v \leqslant a)$ and

$q(\theta, s) = P\{$a type θ particle lives beyond time s and its type at time s lies in $B\}$

By specializing (1) to $\theta = 0$, we get

$$M(0, S, B) = q(0, s) + \int_0^s M(0, s-u, B)\ dE\mu(u). \tag{2}$$

We have used the convention that when $\theta = 0$, the conditional measure given θ is the same as the unconditional measure for a brand new particle. Now define the *Malthusian parameter* α by the relation

$$\int_0^\infty e^{-\alpha u}\, dE\mu(u) = 1.$$

(Recall the assumptions A.1 — A.4 here. Note that α must statisfy $0 < \alpha < \infty$).

Multiplying both sides of (2) by $e^{-\alpha s}$ and using standard renewal theory, (*see* Chapter 4, [2]), we get

$$\lim_{s \to \infty} e^{-\alpha s}\ M(0, s, B) \equiv \tilde{A}(B)$$

$$= \frac{\int_0^\infty q(0, u)\, e^{-\alpha u} du}{\int_0^\infty u e^{-\alpha u}\; dE\mu(u)} \tag{3}$$

Multiplying both sides of (1) by $e^{-\alpha s}$, we get

$$|e^{-\alpha s} \quad M(\theta, s, B) - \tilde{A}(B)\, V(\theta)\,|$$

$$\leqslant e^{-\alpha s} + \int_0^{s-k} |\, M(0, s, -u, B)\, e^{-\alpha(s-u)} - \tilde{A}(B) \;|\; e^{-\alpha u}\, dE$$

$$\left(\mu(u + a|\,(\theta)\right) \;+ \text{const.} \int_{s-k}^{\infty} e^{-\alpha u} \quad dE(\mu(u + a \mid \theta)) \tag{4}$$

where

$$V(\theta) = \int_0^\infty e^{-\alpha u}\; dE\,\mu(u + a \mid \theta)$$

$$\theta \;= (a, \mu(v) : v \leqslant a)$$

Now let us make the following assumptions in addition to (A.1) - (A.4).

(A.5) $\sup_\theta \left(\int_0^\infty e^{-\alpha v} \quad dE(\mu(a + v) \mid \theta) \right) < \infty$

(A.6) $\sup_\theta \quad \int_s^\infty e^{-\alpha u} \quad dE(\mu(a + v) \mid \theta) \to 0$

as $s \longrightarrow \infty$

(A.7) For each $0 < s < \infty$

$\sup_\theta \; E_\theta \{ \mu(s + a) - \mu(a) \colon \; \mu(s + a) - \mu(a) \geqslant k \} \downarrow 0$ as $k \uparrow \infty$

where a and θ are related as before.

F.om (4), ·hoosing first k and then s, we can read off the following :

Lemma 1: Under assumptions A.1–A.6

$$\sup_{\theta, B} \;|\; e^{-\alpha s} M(\theta, s, B) - \mathrm{V}(\theta)\, \tilde{A}(B) \;| \;\to 0$$

as $s \to \infty$.

Fix $0 < s < \infty$ and $B \in \mathscr{B}$. Consider

$$X_t \equiv \frac{1}{Z_t} \sum_{i=1}^{Z_t} \{ Z(\theta_i, s, B) - M(\theta_i, s, B) \} \tag{5}$$

We now prove the following basic,

Lemma 2 : For fixed $0 < s < \infty$ and $B \in \mathscr{B}$, $X_t \xrightarrow{p} 0$ where X_t is defined by (5), provided (A.1)–(A.7) hold.

Proof : Since the random variables $Z(\theta, s, B)$ are non-negative and $\sup_{\theta, B} M(\theta, s, B) < \infty$, we can employ moment generating functions.

To prove the lemma, it sufficies to show that for each $\gamma \geqslant 0$

$E\{\bar{e}^{\gamma x_t} | F_t\} \rightarrow 1$ in probability as $t \rightarrow \infty$ where F is the σ-algebra generated by family histories upto time t.

In view of the assumptions A.1–A.7

$$P\{Z(\theta, s, B) \geqslant K\} \leqslant P\{1 + \sum_{j=1}^{N(\theta)} Z_j(s) \geqslant K\} \tag{6}$$

where $N(\theta)$, $Z_j(s)$, j = 1, 2,.... are independent with Z_j's having the same distribution as $Z(0, s, \Theta)$ and $N(\theta)$ having the distribution $\mu(s + a) - \mu(a)$ Conditioned on θ.

This makes the family $\{Z(\theta, s, B),\ \theta \in \Theta,\ B \in \mathscr{B}\}$ uniformly integrable.

Let $F(\gamma, \theta, s, B) = E(e^{-\gamma Z(\theta, s, B)})$.

By uniform integrability

$$\sup_{\theta, B} \left| \frac{1 - F(\gamma, \theta, s, B)}{\gamma} - M(\theta, s, B) \right| \rightarrow 0$$

as $\gamma \downarrow 0$

and $\sup_{\theta, B} M(\theta, s, B) < \infty$.

Using the fact $\log(1 - h) = -h + o(h)$

as $h \downarrow 0$, we conclude that

$$E\{e^{-\gamma x_t} | F_t\}$$

$$= \exp\left\{\sum_{i=1}^{Z_t} \frac{\gamma}{Z_t} M(\theta_i s, B) + \log F\left(\frac{\gamma}{Z_t}, \theta_i, s, B\right)\right\}$$

$\rightarrow 1$ w.p.1 since $Z_t \rightarrow \infty$ as $t \rightarrow \infty$ w.p.1. q.e.d.

Let us now introduce one further assumption :

(A.8) $\inf_{\theta} V(\theta) > 0$.

4. *The Main result* :

We are ready to state our main result.

Theorem 1 : Assume A.1–A.8 hold. Then for any initial distribution with finite number of particles, for each

$$B \in \mathscr{B},\ A(t, B\ \omega) \to A(B) = \frac{\tilde{A}(B)}{\tilde{A}(\Theta)}$$

where $A(t, B, \omega)$ and $\bar{A}(B)$ are respectively as defined by (*) and (3) in §3.

Proof : It sufficies to consider the case of starting with one particle. of type θ_0 Then by the additivity of the process

$$Z(\theta_0, t+s, B) = \sum_{i=1}^{Z_t} Z(\theta_i, s, B)$$

Where $\{\theta_i,\ i = 1, 2, \cdots Z_t\}$ is the type chart at time t.

$$A(t+s, B, \omega) = \frac{Z(\theta_0, t+s, B)}{Z(\theta_0, t+s, \Theta)}$$

$$\frac{1}{Z_t} \times \text{The Numerator} = \Big\{\frac{1}{Z_t}\sum_{i=1}^{Z_t}[Z(\theta_i, s, B) - M(\theta_i, s, B)]e^{-\alpha s}$$

$$+ \frac{1}{Z_t}\sum_{i=1}^{Z_t}[M(\theta_i, s, B)e^{-\alpha s} - V(\theta_i)\tilde{A}(B)]$$

$$+ \frac{1}{Z_t}\sum_{i=1}^{Z_t} V(\theta_i)\tilde{A}(B)\Big\}$$

$$\frac{1}{Z_t} \times \text{The Denominator} = \Big\{\frac{1}{Z_t}\sum_{i=1}^{Z_t}[Z(\theta_i, s, \Theta) - M(\theta_i, s, \Theta)]e^{-\alpha s}$$

$$+ \frac{1}{Z_t}\sum_{i=1}^{Z_t} M(\theta_i\ s\ \Theta)\, e^{-\alpha s}$$

$$- V(\theta_i)\tilde{A}\ \Theta) + \frac{1}{Z_t}\sum_{i=1}^{Z_t} V(\theta_i)\tilde{A}(\Theta).\Big\}$$

Given an $\epsilon > 0$ *there exists by lemma 1 an* $s_0 > 0$ *such that*

$$\sup_{B,\{\theta_i\},\ t} \frac{1}{Z_t}\sum_{i=1}^{Z_t}[M(\theta_i\ s, B)e^{-\alpha s_0} - V(\theta_i)\tilde{A}(B)] < \epsilon$$

Now let $t \to \infty$ and appeal to lemma 2, to make the first term of the numerator and denominator to go to zero in probability. Finally use (A.8) to see that

$$\inf_t \frac{1}{Z_t} \sum_{i=1}^{Z_t} V(\theta_i)$$

is bounded below and hence may be removed as a common factor. q.e.d.

5. AGE-DEPENDENT BIRTH AND DEATH PROCESS

In age-dependent Birth and Death Processes introduced first by Kendall [7]), each object produces a random number of off-springs, born at various times throughout its life ; the process is specified by the birth rate $\lambda(.)$ and the death rate $\mu(.)$; the conditional probability that a particle alive and of age x at time t gives birth to an offspring in the time interval $[t, t+dt]$ is $\lambda(x)dt$ and the probability of the particle dying in the same interval is $\mu(x)\,dt$ and that these probabilities are independent of the past once the age is known.

In this section, we obtain conditions on the birth rate $\lambda(.)$ and the death rate $\mu(.)$ which ensure that Theorem 1 of section § 4 holds for age-dependent Birth and Death processes. We assume that λ and μ are continuous functions on $[0, \infty)$, that λ and μ are non-negative and $\int_0^\infty \mu(y)\,dy = \infty$. We use notations similar to those in the previous sections : In this case, the type space is $[o, \infty)$, $Z(x, y, s, \omega)$ denotes the number of particles of age $\leqslant y$ at time s, initiated by a single particle of age x at time o, in the "family ω": $M(x, y, s)$ is the expected value of $Z(x, y, s, \omega)$. We write $Z(x, s, \omega)$ and $M(x, s)$ for $Z(x, \infty, s, \omega)$ and $M(x, \infty, s)$ respectively. We assume in what follows that the probability of extinction is zero. We define

$$A(x, t, \omega) = \frac{Z_0(x, t, \omega)}{Z_0(t, \omega)}$$

$$A(x) = \frac{\int_x^\infty e^{-as} f(o, s)\,ds}{\int_0^\infty e^{-as} f(o, s)\,ds}$$

$f(x, s) = P$ (initial particle lives upto time s/ its age at time zero $= x$)

$$V(x) = \int_0^\infty e^{-au}\, \lambda(x+u) \exp\left[-\int_x^{x+u} \mu(y)\,dy\right] du$$

where, the Malthusian parameter α is defined by the relation

$$\int_0^\infty e^{-\alpha u} \lambda(u) \exp\left[-\int_0^u \mu(y)\, dy\right] du = 1$$

The expected number $C_x(t)$ of first generation offspring produced in time by a particle of age x at time o is given by

$$C_x(t) = \int_0^t \exp\left[-\int_x^{x+u} \mu(y)\, dy\right] \lambda(x+u)\, du.$$

It is clear that 'α' is determined by $\int_0^\infty e^{-\alpha u}\, dC_0(u) = 1$

The integral equation for $M(x, y, s)$ is :

$$M(x, y, s) = f(x, s)\, J(x+s-y) + \int_0^s M(o, y, s-u)\, dC_x(u)$$

Where $J(u) = \begin{cases} 1 \text{ if } u \leqslant 0 \\ 0 \text{ if } u > 0 \end{cases}$

Theorem 2 :

Assume that λ is bounded,

$$\underline{\lim}_{x \to \infty} \lambda(x) > 0 \text{ and } \overline{\lim}_{x \to \infty} \int_x^{x+u} \mu(y)\, dy < \infty$$

$$\text{Then, } A(t, x, \omega) \xrightarrow{p} A(x).$$

Proof.—From the integral equation satisfied by $M(x, y, s)$, it is clear by renewal theory [2] that $\lim_{s \to \infty} e^{-\alpha s} M(o, y, s) = \bar{A}(y)$,

where

$$\bar{A}(y) = \frac{\int_0^y e^{-\alpha s} f(o, s)\, ds}{\int_0^\infty t e^{-\alpha t}\, d\, C_0(t)}$$

Also,

$$| e^{-\alpha s} M(x, y, s) - \bar{A}(y) V(x) | \leqslant e^{-\alpha s} + \int_0^{s-k} | M(o, y, s-u)$$

$$e^{-\alpha(s-u)} - \bar{A}(y) |\, e^{-\alpha u}\, d\, C_x(u) + \text{Constant} \int_{s-k}^\infty e^{-\alpha u}\, d\; C_x(u).$$

Now, $\left| \int_0^{s-k} e^{-\alpha u} \, d\, C_x(u) \right|$

$$= \left| \int_0^{s-k} e^{-\alpha u} \lambda(x+u) \exp\left(-\int_x^{x+u} \mu(y)\, dy\right) du \right|$$

$$\leqslant \frac{M}{\alpha}\left(1 - e^{-\alpha(s-k)}\right)$$

$$\leqslant \frac{M}{\alpha}$$

if $|\lambda(t)| \leqslant M \; \forall \; t \in [o \; \infty)$.

Hence, $\sup_x \left[\int_0^{s-k} | M(o, y, s-u) \, e^{-\alpha(s-u)} - A(y) | \, e^{-\alpha y} \, d\, C_x(y) \right]$

$\rightarrow 0$ as $(s-k) \rightarrow \infty$.

Also, $\left| \int_{s-k}^{\infty} e^{-\alpha u} \, dC_x(u) \right.$

$$= \left| \int_{s-k}^{\infty} e^{-\alpha u} \lambda(x+u) \exp\left(-\int_x^{x+u} \mu(y)\, dy\right) du \right|$$

$\leqslant \frac{M}{\alpha} e^{-\alpha(s-k)}$, so that

$$\sup_x \left| \int_{s-k}^{\infty} e^{-\alpha u} \, d\, C_x(u) \right| \rightarrow 0 \text{ as } (s-k) \rightarrow \infty.$$

It follows, therefore, that

$$\sup_x \left| e^{-\alpha s} M(x, y, s) - \bar{A}(y) \, V(x) \right| \rightarrow 0 \text{ as } s \rightarrow \infty.$$

From the integral equation :

$$M(x, s) = f(x, s) + \int_0^s M(o, s-u) \, d\, C_x(u)$$

satisfied by $M(x, s) \equiv M(x, \infty, s)$, we see that

$$\sup_x \left| e^{-\alpha s} M(x, s) - \bar{A}(\infty) \, V(x) \right| \rightarrow 0 \text{ as } s \rightarrow \infty.$$

By the argument in Lemma, 2, § 3, it follows that

$$X_t \equiv \frac{1}{Z(t)} \sum_{i=1}^{Z(t)} \{ Z_{xi}(s, y) - M_{xi}(s, y) \} \xrightarrow{p} 0 \text{ for}$$

fixed s in $(o \; \infty)$ and y in $(o \; \infty)$.

Now, $V(x) = \int_0^{\infty} e^{-\alpha u} \; \lambda(x+u) \exp\left[-\int_x^{x+u} \mu(y)\, dy\right] du,$

Since λ is continuous, so is V, by bounded convergence theorem. Since $\underline{\lim}_{x \to \infty} \lambda(x) > 0$, λ does not have compact support : hence $V(x) > 0$ for each fixed x. By continuity, $\inf_{x \in [0\ k]} V(x) > 0$ for each $k < \infty$.

By Fatou's lemma,

$$\underline{\lim}_{x \to \infty} V(x) \geqslant \int_0^{\infty} e^{-\alpha u} \underline{\lim}_{x \to \infty} \lambda(x+u) \exp\left(-\overline{\lim}_{x \to \infty} \int_x^{x+u} \mu(y)\, dy\right) du > 0$$

since $\underline{\lim}_{x \to \infty} \lambda(x) > 0$ and $\lim_{x \to \infty} \int_x^{x+u} \mu(y)\, dy < \infty$ for some u.

Thus, $\inf_{0 \leqslant x < \infty} V(x) > 0$. It follows that $\frac{1}{z(x)} \sum_{i=1}^{Z(t)} V(x_i)$ is bounded below in probability.

The theorem now follows.

Theorem 3 : If $\lim_{x \to \infty} \lambda(x) = 0$, then $A(t, x, \omega) \xrightarrow{p} A(x)$

provided, $E(\wedge(y) \log |\wedge(y)|) < \infty$, where $\wedge(x) = \int_0^x e^{-\alpha y} \lambda(y)\, dy$ and y is a random variable having distribution F, where

$$F(t) = \begin{cases} 1 - \exp.\ \left(-\int_0^t \mu(y)\, dy\right) & \text{if } t > 0 \\ 0 & \text{if } t \leqslant 0. \end{cases}$$

Proof.—We have,

$$\frac{Z(t+s)}{Z(t)} e^{-\alpha s} = \frac{1}{Z(t)} \sum_{j=1}^{Z(t)} Z_j(s)\, e^{-\alpha s},$$ where $Z_j(s)$ = number of particles at time $(s+t)$ originated from the j^{th} particle at time t.

Then

$$\frac{Z(t+s)}{Z(t)} e^{-\alpha s} = \frac{1}{Z(t)} \sum_{a_j \leqslant k} Z_j(s)\, e^{-\alpha s} + \frac{1}{Z(t)} \sum_{a_j > k} Z_j(s)\, e^{-\alpha s},$$

where, as usual,

$\{x_j\}_{j=1}^{Z(t)}$ is the 'age-chart' at time t.

Now, $E(Z_j(s)\, e^{-\alpha s} = E(M(x_j, s)\, e^{-\alpha s}$.

Since $\sup_x \; | M(x, s) e^{-\alpha s} - \tilde{A}(\infty) V(x) | \to 0$ as $s \to \infty$ and since

$$V(x) = \int_0^\infty \exp(-\alpha u) \lambda(x+u) \exp\left(-\int_x^{x+u} \mu(y)\, dy\right) du$$

$\to 0$ as $x \to \infty$, the last term :

$\frac{1}{Z(t)} \sum_{x_j > k} Z_j(s)$ can be made arbitrarily small (in probability by choosing s large and k large.

Also, $\frac{1}{Z(k, t)} \sum_{x_j \leqslant k} Z_j(s) e^{-\alpha s}$

is bounded above in probability since $M(x, s) \; e^{-\alpha s}$ is bounded for x and in (o, ∞).

We now show that $\frac{Z(t+s) e^{-\alpha s}}{Z(t)}$ is bounded below in probability. For $0 < \eta < 1$, we have $P \frac{Z(t+s) e^{-\alpha s}}{Z(t)} < (\eta)$

$$\leqslant P\,(Z(t+s) e^{-\alpha(s+t)} \geqslant \delta_1, Z(t) e^{-\alpha t} \leqslant \delta_2, \; \delta_1 < \eta \, \delta_2)$$
$$+ P(Z(t+s) e^{-\alpha(s+t)} < \delta_1)$$
$$+ P(Z(t) e^{-\alpha t} > \delta_2)$$
$$= P(Z(t+s) e^{-\alpha(s+t)} < \delta_1)$$
$$+ P(Z(t) e^{-\alpha t} > \delta_2)$$

if δ_1, δ_2, are positive and $\delta_1 \geqslant \delta_2$, Now $P(Z(t+s) e^{-\alpha(s+t)} < (\delta_1)$

$\to 0$ as $t \to \infty$ and $\delta_1 \to 0$, since $\frac{Z(t)}{e^{\alpha t}}$ converges in distribution to a positive random variable W with a continuous distribution function. (*See* Doney [3], Theorem 7·7)—

Also since $M(o, t) e^{-\alpha t}$ is bounded above, it is lear that

$$P(Z(t) e^{-\alpha t} > \delta_2) \to 0 \text{ as } t \to \infty \text{ and } \delta_2 \to \infty.$$

Thus,

$$P\left(\frac{Z(t+s) e^{-\alpha s}}{Z(t)} \geqslant \eta\right) \to 1 \text{ as } \eta \to 0 \text{ and } t \to \infty,$$

uniformly in s.

From the equation

$$\frac{Z(t+s)\,e^{-\alpha s}}{Z(t)} = \frac{Z(k,\,t)}{Z(t)}\,\frac{1}{Z(k,\,t)}\sum_{x_j \leqslant k} Z_j(s)$$

$$+\frac{1}{Z(t)}\sum_{x_j > k} Z_j(s),$$ it now follows that

$$P\Big(\frac{Z(k,\,t)}{Z(t)} \geqslant \eta)\Big) \to 1$$ for sufficiently small η.

as $k \to \infty$ and $t \to \infty$. It now follows, as in Athreya, Kaplan ([1], Corollary 2, 4) that

$$\frac{1}{Z(t)}\sum_{j=1}^{Z(t)} V(x_j)$$ is bounded away from 0 in probability as $t \to \infty$.

Since $\sup_x \; V(x) < \infty$ and

$$\sup_x \int_T^\infty e^{-\alpha u}\,\lambda(x+u)\,\exp\,(-\int_x^{x+u}\mu(y)\,dy)\,du \to 0$$

as $T \to \infty$, it follows exactly as in the previous cases that

$$\sup_y \;|(M_y(x,s)\,e^{-\alpha s} - V(y)\,\tilde{A}(x)\,| \to 0 \text{ as } s \to \infty$$

$$\sup_y \;|\,M_y(\infty,\,s)\,e^{-\alpha s} - \tilde{A}(\infty)\,V(y)\,| \to 0 \text{ as } s \to \infty$$

and

$$\frac{1}{Z(t)}\sum_{i=1}^{Z(t)}\{Z_{xi}(x,\,s) - M_{xi}(x,\,s)\} \xrightarrow{p} 0 \text{ as } t \to \infty.$$

This completes the proof of theorem 3.

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