MODIFIED LIAPUNOV-RAZUMIKHIN STABILITY CONDITION FOR EXTENDED RANGE OF APPLICABILITY

J. S. ANSARI

(*School of Automation, Indian Institute of Science, Bangalore* 12, *India*)

Received on June 25, 1975 and in revised form on October 6, 1975

ABSTRACT

Razumakhin-Liapunov theorem for functional differential equations is modified to increase its range of applicability. In the form presented here, it is more readily applicable to integral equations.

1. INTRODUCTION

Let

$$\dot{y}(t) = f(t, y_t) \tag{1}$$

represent a vector functional differential equation of the retarded type, as defined in [1]. y_t in the argument of f indicates the dependence of f on $y(s)$, $t - T(t) \leqslant s \leqslant t$, where $T(t) \geq 0$ and $t - T(t)$ is nondecreasing for $t \geqslant 0$. Interval $[t - T(t), t)$ will be represented by $I(t)$. Let $\| y \|$ represent Euclidean norm in R^n. Let $y = 0$ be a solution of equation (1). If $V(t, y)$ is a Liapunov function, Razumikin-Liapunov theorem requires that for *all* $y(t)$ within a region, $\| y \| < H_0$,

$$\dot{V}(t, y) \leqslant - \omega < 0, \text{ for all } t \geq 0, \text{ if} \tag{2}$$

$$V(s, y(s)) < V(t, y, (t)) + \delta, \; s \in I(t) \; ; \; \delta > 0.$$

It is shown here that it is enough if condition (2) is satisfied only for $H_2(t) \leqslant \| y \| \leqslant H_1(t)$, where $H_1(t) \to 0$. For example consider,

$$\dot{y}(t) = - y(t) + \tfrac{1}{2} e^{-t} [y(t - T) + y^{1 \cdot 3}(t - T)] \tag{4}$$

One normally uses a Liapunov function, $V(y)$, such that condition (1) of the theorem stated below is satisfied. For such a $V(y)$, with y scalar, there is a $\delta > 0$ such that

$$\frac{dV(y)}{dy^2} > 0, \text{ for } y^2 < \delta \tag{4}$$

It is seen from equation (3) that if

$$y^2(s) = y^2(t),\ s \in I(t) \tag{5}$$

we have

$$\frac{dy^2}{dt} > 0,\ \text{for}\ y^2 < (2e^t - 1)^{-3}.$$

It follows from relations (4) and (5) that, if

$$V(y(s)) = V(y(t)),\ s \in I(t) \tag{6}$$

we have

$$\dot{V}(y(t)) > 0,\ \text{for}\ y^2 < (2e^t - 1)^{-3} < \delta.$$

Hence, Razumikhin-Liapunov theorem cannot establish asymptotic stability with any V. However, the theorem stated below establishes attractivity Also, relaxation of condition (2) renders the theorem more readily applicable to integral equation. Although, Razumikhin-Liapunov theorem has been applied to integral equations [2[1]], the proposed theorem is expected to lead to weaker conditions on the structure of the system to ensure asymptotic stability.

2. THEOREM

2. *Theorem.*—Let $u(s)$ and $v(s)$ be continuous and non-decreasing for $0 \leqslant s \leqslant H_0$, with $0 < u(s) \leqslant v(s)$ for $s > 0$ and $u(0) = v(0) = 0$. Suppose, there exist scalar continuous functions $V(t, y)$ and $H(t, t_0)$, $t \geq t_0 \geq 0$, such that,

(i) $u(\| y \|) \leqslant V(t, y) \leqslant v(\| y \|)$

for $0 \leqslant \| y \| \leqslant H_0$, $t > -T(0)$.

(ii) (*a*) $V(t, y)$ is differentiable with respect to t and all elements, y_i, of the vector y ; and $\delta V/yt_i$, $\delta V/\delta y_i$ are continuous for $t > t_0$ and $\| y \| < H_0$.

(*b*) $\dot{H}(t, t_0) \equiv \dfrac{\delta H(t, t_0)}{\delta t}$ is continuous for $t > t_0$.

(iii) For any continuous vector $p(t)$ of order n, with continuous $\dot{p}(t)$

$$\frac{\delta V(t, p)}{\delta t} + \frac{\delta V(t, p)}{\delta p} \cdot f(t, pt) < \dot{H}(t, t_0)$$

if $p(t)$ is such that for all $t_0 > 0$
$V(t, p) = H(t, t_0)$; $V(s, p(s)) < H(s, t_0)$, $s \epsilon I(t)$,
$t > t_0$

(iv) $H(t, t_0) \leqslant H_0$, for $t \geq t_0 - T(t_0) \geqslant - T(0)$

(v) $\lim_{t \to \infty} H(t, t_0) = 0$, for all $t_0 \geqslant 0$

Then $V(t, y) \to 0$ bounded by $H(t, t)$ as $t \to \infty$, provided in the initial period

(vi) $0 \leqslant V(s, y(s)) < H(t_0, t_0) \leqslant H_0$, for $s \epsilon I(t_0)$, $s = t_0$

Lemma.—Let $f(t)$ and $g(t)$ be two differentiable, continuous functions with continuous derivatives. If

$$f(t) > g(t), \text{ for } t_0 \leqslant t < t_1 \tag{L1}$$

and

$$f(t_1) = g(t_1) \tag{12}$$

then

$$\frac{df(t_1)}{dt} \leqslant \frac{dg(t_1)}{dt} \tag{L3}$$

Proof.—Suppose, there exist on $\epsilon > 0$, such that

$$\frac{df(t_1)}{dt} - \epsilon = \frac{dg(t_1)}{dt} \tag{L4}$$

For a small increment $\delta > 0$

$$f(t_1 - \delta) = f(t_1) - \delta \dot{f}(t_1) + \delta \epsilon_1(\delta) \tag{L5}$$

and

$$g(t_1 - \delta) = g(t_1) - \delta \dot{g}(t_1) + \delta \epsilon_1(\delta) \tag{L6}$$

where ϵ_1 and ϵ_4 tend to zero with δ [3]. From relations (L4), (L5) and (L6).

$$g(t_1 - \delta) - f(t_1 - \delta) = \delta(\epsilon + \epsilon_2(\delta) - \epsilon_1(\delta)) \tag{L7}$$

For any given ϵ, we can choose a sufficiently small δ such that $\epsilon + \epsilon_2 - \epsilon_1 > 0$. Hence, there exist a $\delta > 0$ such that

$$g(t_1 - \delta) - f(t_1 - \delta) > 0 \tag{L8}$$

Since inequalities (L1) and (L8) are contradictory, assumption (L4) is no true. Hence, inequality (L3) must be true.

Proof of the theorem.—From condition (vi)

$$V(t_0, y(t_0)) < H(t_0, t_0) \tag{P1}$$

Suppose, there exist a t_1, such that t_1 is the smallest $t > t_0$ such that

$$V(t_1, y(t_1)) = H(t_1, t_0) \tag{P2}$$

where $y(t)$ satisfies equation (1). Then by definition of t_1

$$V[t, y(t)] < H(t, t_0), \text{ for } t_0 \leq t < t_1 \tag{P3}$$

It follows from the Lemma that

$$\dot{V}[t_1, y(t_1)] \geq \dot{H}(t_1, t_0) \tag{P4}$$

However, from relations (P2), (P4) and conditions (iii) and (vi), it follows that for $t = t_1$

$$\dot{V}[t, y(t)] < \dot{H}(t, t_0) \tag{P5}$$

Since, inequalities (P4) and (P5) are contradictory, assumption (P2) is not true. Hence, if conditions (i)–(vi) are satisfied $V[t, y(t)] < H(t, t_0)$ for all $t > t_0$. Since, $H(t, t_0)$ tends to zero, $V(t, y)$ also tends to zero.

3. EXAMPLES

Example 1.—Consider equation (3). Choose H as follows ;

$$H(t, t_2, a) = \begin{cases} 1, & \text{for } -T \leq t \leq t_4 \\ \text{Exp } [-2a(t - t_2)], & \text{for } t > t_2 \end{cases} \tag{7}$$

where $0 \leq a \leq 1$, $t_2 > 0$. Let

$$V(y) = y^2 \tag{8}$$

It is seen that if

$$(1 - a) \geq e_{2}^{-t_2} \tag{9}$$

all conditions of the above theorem are met. Hence,

$$V(t) < H(t,\ t_4,\ a), \text{ for } t > 0,\ 0 \leq a \leq 1 - e^{-t_4} \tag{10}$$

Define $H_m(t)$ for $t \geq 0$ as

$$H_m(t) = \min_{0 \leq a \leq 1} [H(t,\ \log\ (1-a)^{-1},\ a)] \tag{11}$$

Then $V(t)$ tends to zero bounded by $H_m(t)$.

Example 2.—Consider the stability of the solution $y = 0$ of the equation

$$\dot{y}(t) = -g(y(t)) + \int_0^t y(t-t')f(t')\,dt' \tag{12}$$

where $g(0) = 0$. Choose

$$H(t) = A^2 e^{-2bt},\ b > 0,\ A^2 > y^2(0) \tag{13}$$

$$V(t) = y^2(t) \tag{14}$$

For $V(t) = H(t)$ and $V(s) < H(s)$, $0 \leqslant s < t$, we have

$$\dot{H} - \dot{V} > -2A^2e^{-2bt} \mp 2Ae^{-bt}[-g(\pm Ae^{-bt})] + \int_0^t \pm Ae^{-b(t-t')}\,|f(t')|\,dt' \tag{15}$$

Let us define x as

$$x = Ae^{-bt} \tag{16}$$

From inequality (15) it is seen that condition (iii) requires that for asymptotic stability of equation (12)

$$\frac{g(\pm x)}{\pm x} > b + \int_{t=0}^{1/b \log A/x} e^{bt}\,|f(t')|\,dt',\ |x| < A \tag{17}$$

For example, if

$$f(t) = Be^{-ct}\ \cos\ \omega t,\ a > 0 \tag{18}$$

condition (17) leads to

$$\frac{g(\pm x)}{\pm x} > b + B\,[\omega^2 + (b-a)^2]^{\frac{1}{2}}\,(x/A)^{b-a/b} - \frac{(b-a)\,B}{\omega^2\,(b-a)^2} \tag{19}$$

REFERENCES

[1] Hale, J. .. Functional Differential Equations. *Appl. Math. Sc.*, Springer-Verlag, 1971, 3, 50–60.

[2] Seifert, G. .. Liapunov-Razumikhin Conditions for Asymptotic Stability in Functional Differential Equations of Volterra Type. *J. of Differential Eqns.*, 1974, **16** (2), 289-297.

[3] Goursat, E. .. *A Course in Mathematical Analysis*, Vol. 1, Dover Publ., 1959, pp. 19, 20.