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## THE SOLUTION OF SOME TRANSCENDENTAL EQUATIONS


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ABSTRACT

While discussing the problem of Transverse Plasma oscillations excited by interpenetrating beams of positively and negatively charged particles, we had to solve a large number of transcendental equations. The following four are the typical of them:

$$x \tanh x+(x^2-\alpha^2)^{1/2}=0, \qquad (1)$$

$$x \coth x+(x^2-\alpha^2)^{1/2}=0, \qquad (2)$$

$$x \tan x=C \qquad (3)$$

and

$$x \cot x+C=0. \qquad (4)$$

After discussing the existence of real, pure imaginary and complex roots using the well known theorems of algebra and complex variables we have solved them by graphical methods and the first few roots in each case are tabulated.


INTRODUCTION

1. While discussing the problem of transverse plasma oscillations excited by interpenetrating beams of positively charged particles we had to solve a large number of transcendental equations. In this short note we shall discuss only four typical types of them, namely,

$$y \tanh y+c=0, \qquad [1.1]$$

$$y \coth y+c=0, \qquad [1.2]$$

$$x \tanh x+(x^2-\alpha^2)^{1/2}=0, \qquad [1.3]$$

$$x \coth x+(x^2-\alpha^2)^{1/2}=0, \qquad [1.4]$$

where $c$ and $\alpha$ are real parameters.

It is interesting to note that such equations may occur in a number of different contexts where we have to solve the boundary value problems associated with second order differential equations for example problems dealing with stability, propagation of waves with infinitesimal amplitudes, heat conduction and so on.

We may note that in the preparation of the tables for the roots of the equations we have obtained the zeroth order approximation to the roots with the help of the graphs and sharpened the roots by using Newton's method or its generalisation to deal with simultaneous equation.

2. We shall first consider the equation

$$y \tanh y + c = 0 \tag{2.1}$$

for positive and negative values of $c$.

*Case (i)*: $c>0$.

It is obvious that this equation does not possess any real root when $c > 0$.

Putting $y = ix$, it reduces to $x \tan x = c$. [2.2]

Carslaw and Jaeger[1] have discussed the equation [2.2] for $c > 0$. They have shown that this equation does not possess any complex roots but has real roots which form an infinite sequence $(x_i)$ such that

$$x_1 < x_2 < x_3 < x_4 < \cdots < x_n < \cdots$$

with $x_n \to \infty$ as $n \to \infty$. They have also tabulated the first six roots for a number of values of $c$ lying between 0 and 100.

We can easily prove that

$$x_{n+1} \simeq n\pi\left(1 + \frac{c}{n^2\pi^2}\right), \text{ when } c \to 0 \tag{2.3}$$

and

$$x_{n+1} \simeq (2n+1)\frac{\pi}{2}\left(1 + \frac{1}{1+c}\right), \text{ when } c \to +\infty. \tag{2.4}$$

These formulae can be used to find out the higher root for small and large values of $c$. Further, we can utilise these to find out the higher roots and the roots for values of $c$ as large as we please. It is interesting to note that the expression [2.3] gives roots accurate to four decimal places for $0 \leq c \leq 0.1$, as compared with the tables given in [1]. We can extend these tables by using the expression [2.4] for values of $c$ greater than 100.

*Case* (*ii*) $c < 0$ :

When $c$ is negative, the equation [2.1] can be written as

$$y \tanh y - c = 0, \qquad [2.5]$$

with $c$ positive.

It can be easily proved that this equation does not possess any pure imaginary or complex roots and possesses only one real root for each value of $c$.

When $c \to 0$,

$$y \simeq c^{1/2} + (1/6)\, c^{3/2}. \qquad [2.6]$$

When $c > 4$, the equation [2.5] has roots very close to $c$. In fact, from Newton's approximation formula the first order correction is given by

$$\delta y = \frac{c \tanh c - c}{\tanh c + c \operatorname{sech}^2 c}.$$

The magnitude of this correction is small when $c$ is large. Table I gives the roots for $0 \leq c \leq 4$. It is interesting to note that the roots given by the expression [2.6] for $0 \leq c < 0.5$ are accurate upto four decimal places as compared with the table I.

TABLE I

The roots of the equation $y \tanh y + c = 0$ for $-4 \leq c \leq 0$.
The roots of this equation for all negative values of $c$ are real.

| $c$ | $y$ |
|---|---|
| 0 | 0 |
| − 0.5 | 0.7717 |
| − 1.0 | 1.1997 |
| − 1.5 | 1.6218 |
| − 2.0 | 2.0653 |
| − 2.5 | 2.5319 |
| − 3.0 | 3.0144 |
| − 4.0 | 4 0028 |

3. The discussion of the equation

$$y \coth y + c = 0$$

proceeds on the same lines as in § 2.

We record our results briefly below :

(i) When $c > 0$, it does not possess any real root.

(ii) When $c \geqslant -1$ it possesses an infinite sequence of pure imaginary roots $\{i x_n\}$, where $x_1 < x_2 \cdots < x_n < \cdots$ and $x_n \to \infty$ as $n \to \infty$.

(iii) When $c \to 0$, $x_n \simeq (2n-1)\frac{\pi}{2}\left[1 + \frac{4c}{(2n-1)^2 \pi^2}\right]$.

This formula gives roots correct to four places of decimals when $0 \leqslant c \leqslant 0{\cdot}1$.

(iv) When $c \to \infty$, $x_n \simeq n\pi \left(1 - \frac{1}{1+c}\right)$

(v) When $c < -1$, it admits only one real root for each value of $c$. Table II gives roots for some selected values of $c$ between $-4$ and $-1$.

(vi) When $c < -4$, the root is close to $c$ and we can improve this root by using Newton's formula.

TABLE II

The roots of the equation $y \coth y + c = 0$ for $-3 \leqslant c \leqslant -1$.
The roots of this equation for all negative values of $c < -1$ are real.

| $c$ | $y$ |
|---|---|
| $-1$ | 0 |
| $-1.1$ | 0.5808 |
| $-1.4$ | 1.1425 |
| $-1.6$ | 1.4251 |
| $-1.8$ | 1.6792 |
| $-2.0$ | 1.9152 |
| $-2.5$ | 2.4641 |
| $-3.0$ | 2.9850 |

4. In this Section, we shall discuss the equation

$$x \tanh x + (x^2 - \alpha^2)^{1/2} = 0. \qquad [4.1]$$

When $x$ is real and $x^2 > \alpha^2$, both the terms of [4.1] are positive, hence this equation does not admit any real root greater than $\alpha$ and less than $-\alpha$. Again, if $x^2 < \alpha^2$ the second term will become pure imaginary so that the equation [4.1] does not admit real roots between $-\alpha$ and $\alpha$. Thus the equation does not admit real root.

We can easily show, by putting $ix$ for $x$ in equation [4.1], that it does not possess pure imaginary roots.

We shall now examine the existence of complex roots. We follow the graphical method as it also leads to the determination of the roots approximately. We write the equation [4 1] in the form

$$\alpha \cosh x = \pm x. \tag{4.2}$$

Putting $x = u + iv$ in [4.2] and separating the real and imaginary parts we get the following equations determining $u$ and $v$:

$$\alpha \cosh u \cos v = \pm u, \qquad [i]$$

$$\alpha \sinh u \sin v = \pm v. \qquad [ii]$$

We write these in the following form:

$$v = 2n\pi \pm \cos^{-1}\left(\pm \frac{u}{\alpha \cosh u}\right), \quad n = 0, \pm 1, \pm 2, \ldots, \qquad [iii]$$

$$u = \pm \sinh^{-1}\left(\frac{v}{\alpha \sin v}\right), \qquad [iv]$$

where we have to consider only the principal values of the inverse functions on the right hand sides.

We first consider the curves

$$v = 2n\pi \pm \cos^{-1}\left(\frac{u}{\alpha \cosh u}\right) \qquad [v]$$

and

$$u = \sin h^{-1}\left(\frac{v}{\alpha \sin v}\right) \qquad [vi]$$

as from the solution of these equations we can make definite statement about the roots of [iii] and [iv].

When $u = 0$, (v) gives $v = \pi/2$ and as $u \to \pm \infty$, $v$ tends asymptotically to $(\pi/2)$ as shown by continuous curves $A$ drawn schematically in Fig. I. The curve [vi] consists of infinite number of branches of the type of dashed curves $A$ shown in Fig. I. With $u \to \infty$ as $v \to \pm (2n\pi + 0)$, $n = 1, 2, \cdots$ or $v \to \pm (2n+1)\pi - 0$, $n = 0, 1, 2,$ so that the branches of the curves for which $u > 0$ lie in the ranges $\cdots, (-2\pi, -3\pi), (-\pi, \pi), (2\pi, 3\pi), (4\pi, 5\pi), \cdots$ for $v$. Similarly, the branches of the curves for which $u < 0$ lie in the ranges $\cdots, (-2\pi, -\pi), (\pi, 2\pi), (3\pi, 4\pi), \cdots$. Thus the curves [v] and [vi] have points of intersection for which $u$ is both positive and negative. Considering the entire family of curves [iii] and [iv] by plotting also the set of curves marked $B$ in Fig. I for the equations

$$v = 2n\pi \pm \cos^{-1}\left(-\frac{1}{\alpha} \cdot \frac{u}{\cosh u}\right)$$

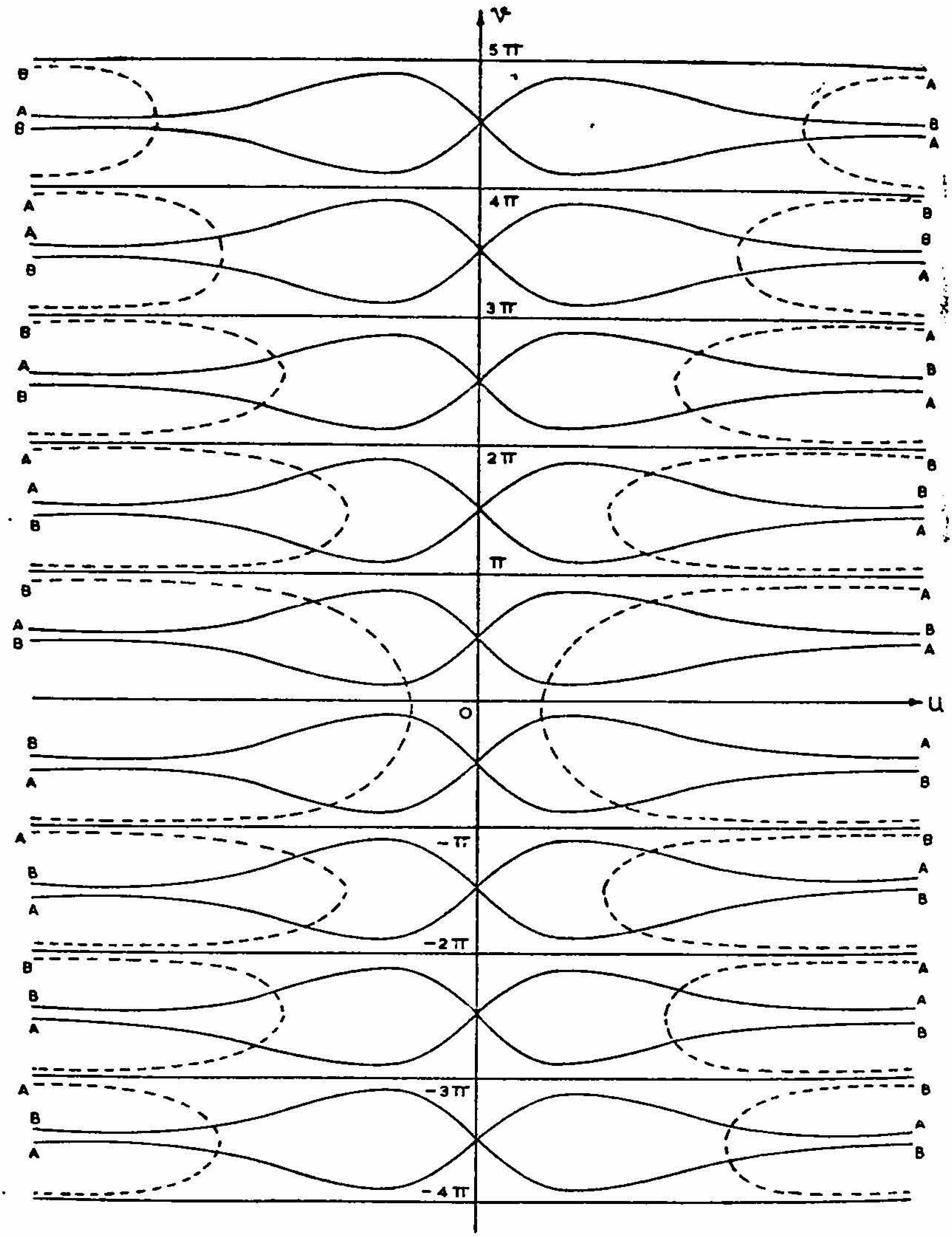


FIG. I

The schematic representation of the equations (iii) (Continuous curves) and (Dashed curves) (iv)

$$u = -\sinh^{-1}\left(\frac{1}{\alpha}\cdot\frac{v}{\sin v}\right),$$

we can easily show that they admit infinite number of roots of the type $\pm u \pm iv$. The main emphasis here is on the fact that the roots will occur in complex conjugate pairs.

It is possible for us to specify the ranges of $\alpha$ for which the equation [4.2] admits the complex roots. From equation [vi], it is clear that $v$ is real if

$$-1 < \frac{u}{\alpha \cosh u} < 1.$$

Taking $u$ first to be positive the inequality reduces to $\alpha > (u/\cosh u)$ or $\alpha > (u/\cosh u)_{max} = 0.66$, attained when $u = 1.20$. The same statement is true even when $u < 0$. Thus we find that the equation [4.2] admits complex conjugate roots only when $\alpha > 0.66$.

Hence we conclude that when $\alpha > 0.66$, the equation [4.1] admits infinite number of complex conjugate roots.

Table III gives the first five roots for few physically interesting values of $c$. They were obtained numerically on an Elliot 803 of Hindustan Aeronautics Ltd. They are correct upto six decimal places.

TABLE III

The first five roots $x_n = u_n + i\,v_n$ of $x \tanh x + (x^2 - \alpha^2)^{1/2} = 0$, for $\alpha = 0.7,\ 0.9,\ 1,\ 2,\ 10,\ 100$

| $\alpha$ | | $u_1$ | $v_1$ | $u_2$ | $v_2$ | $u_3$ |
|---|---|---|---|---|---|---|
| 0.7 | ...... | 1.169363 | 0.329099 | 2 645484 | 4.149363 | 3.141937 |
| 0.9 | ...... | 1.032276 | 0.759622 | 2.378062 | 4.205040 | 2.883332 |
| 1.0 | ...... | 0.9760545 | 0.8709514 | 2.267556 | 4.229099 | 2.775439 |
| 2.0 | ...... | 0.6332211 | 1.305441 | 1.575510 | 4.394466 | 2.077241 |
| 10 | ...... | 0.1549460 | 1.555485 | 0.4519877 | 4.671433 | 0.7177541 |
| 100 | ...... | 0.015706 | 1.570639 | 0.047102 | 4.711918 | 0.078452 |

| $\alpha$ | | $v_3$ | $u_4$ | $v_4$ | $u_5$ | $v_5$ |
|---|---|---|---|---|---|---|
| 0.7 | ...... | 7.456526 | 3.469318 | 10.682110 | 3.715102 | 13.87586 |
| 0.9 | ...... | 7.488539 | 3.213700 | 10.704800 | 3.460901 | 13.89350 |
| 1.0 | ...... | 7.502165 | 3.106798 | 10.714420 | 3.354490 | 13.90094 |
| 2.0 | ...... | 7.594886 | 2.409337 | 10.779080 | 2.657942 | 13 95070 |
| 10 | ...... | 7.797384 | 0.947251 | 10.93167 | 1.143888 | 14.07095 |
| 100 | ...... | 7.853200 | 0.109725 | 10.994480 | 0.171923 | 17.27707 |

From the above table we note that the equation [4.1] has infinite number of complex roots with modulii forming an infinite sequence $\{r_i\}$ such that

$$r_1 < r_2 < r_3 \cdots < r_n < \cdots$$

with $r_n \to \infty$ as $n \to \infty$. Also we note that as $\alpha$ increases the value of $u$ decreases, whereas $v$ increases. These results are important from the point of view of physical problems especially dealing with stability and wave propagation.

For finding out an approximate value for the first root of [4.2], we employ the following theorem valid for the analytic functions: The abscissæ of the points at which $f(x)$ attains a positive minima or a negative maxima give approximately the real parts of the complex zeros of the function. We thus find that the real part $u$ is approximately given as

$$u \simeq \sin h^{-1}(1/\alpha)$$

and the corresponding value of $v$ will be given approximately by

$$v \simeq \cos^{-1}\left[\frac{\sinh^{-1}(1/\alpha)}{\sqrt{(1+\alpha^2)}}\right].$$

These expressions also predict the decrease and increase in the values of $u$ and $v$ respectively with the corresponding increase in $\alpha$. These approximate formulae are useful in locating roughly the first root of the equation, for different values of $\alpha$.

5. We shall now consider the equation

$$x \coth x + (x^2 - \alpha^2)^{1/2} = 0, \qquad [5.1]$$

which can be rewritten as

$$\alpha \sinh x = \pm ix. \qquad [5.2]$$

It can be easily shown that [5.2] does not admit any real or pure imaginary roots. For considering the existence of the complex roots, following the same scheme as in § 4 we have the following equations determining $u$ and $v$:

$$u = \mp \sinh^{-1}\left(\frac{v}{\alpha \cos v}\right) \qquad [\text{vii}]$$

$$v = n\pi \pm (-1)^n \sin^{-1}\left(\frac{u}{\alpha \cosh u}\right), \qquad [\text{viii}]$$

$n = 0, \pm 1, \pm 2, \cdots,$

where now $\sin^{-1}[u/(\alpha \cosh u)]$ lies between $-(\pi/2)$ and $(\pi/2)$. Fig. 2 represents these families of curves schematically. From this figure, it is clear

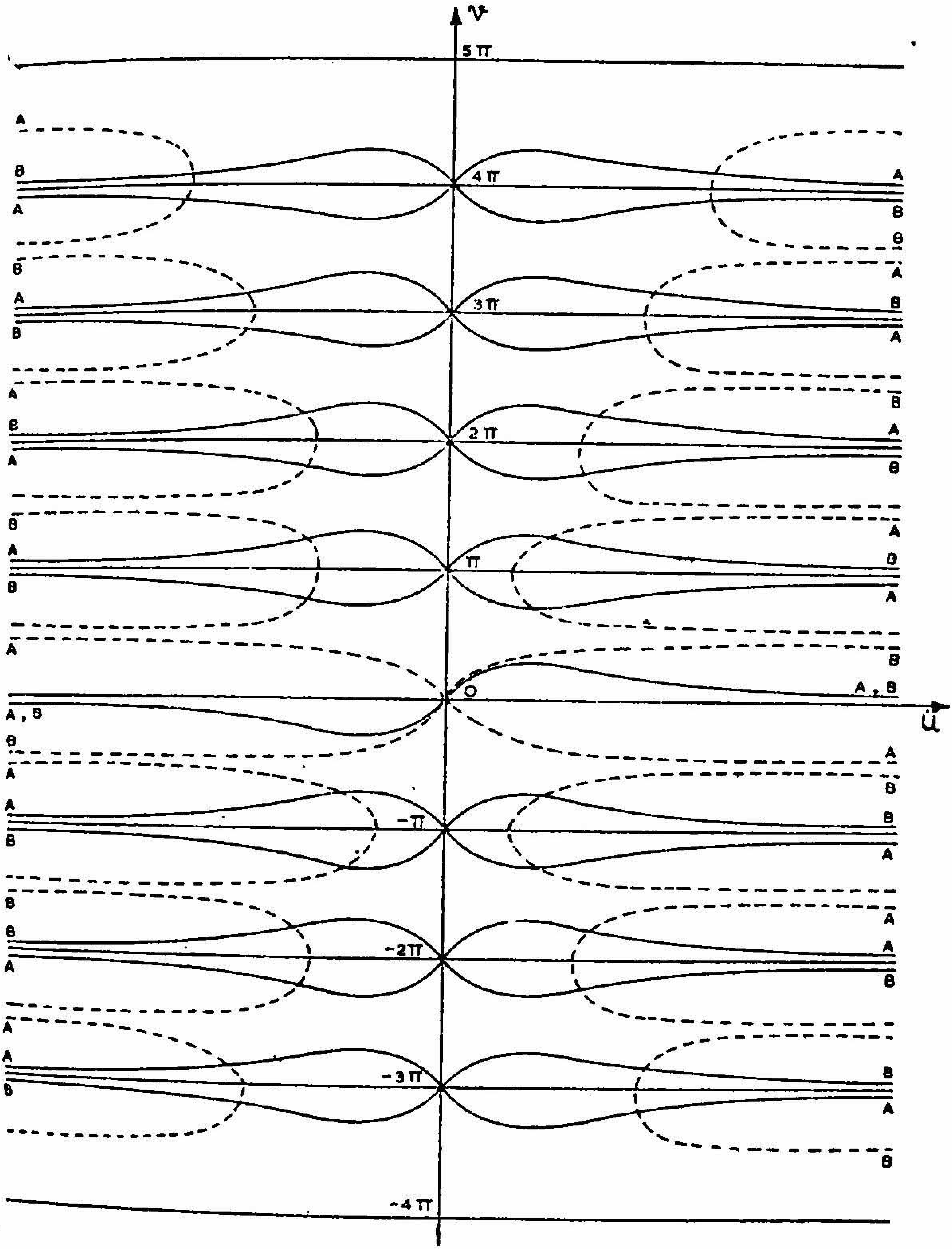


FIG. II
Schematic representation of the equations (vii) (Dashed curves) and (viii) (Continuous curves)

that the equations [vii] and [viii] admit infinite number of roots of the type $\pm u \pm iv$. Once again we emphasise that in this case also the roots will occur in complex coujugate pairs.

As in the previous section, we can show from the equation [viii] that the equation [5.1] admits complex roots only for $\alpha > 0.66$. For $\alpha \leq 0.66$ it does not possess any root. Hence this equation just as the equation [4.1] admits infinite number of complex conjugate roots for $\alpha > 0.66$.

The Table IV gives the first five roots for the values of $c$, which we have considered in § 4. These roots were also obtained correct upto six decimal places with the help of Elliot 803.

TABLE IV

The first five roots $x_n = u_n + i\,v_n$ of $x \coth x + (x^2 - \alpha^2)^{1/2} = 0$ for $\alpha = 0.7, 0.9, 1, 2, 10, 100$

| $\alpha$ | | $u_1$ | $v_1$ | $u_2$ | $v_2$ | $u_3$ |
|---|---|---|---|---|---|---|
| 0.7 | ... | 2.240106 | 2.402441 | 2.925442 | 5.819702 | 3 319120 |
| 0.9 | ... | 1.958202 | 2.495504 | 2.663822 | 5.860191 | 3.062320 |
| 1.0 | ... | 1.844749 | 2.536346 | 2.554980 | 5.877520 | 2.955020 |
| 2.0 | ... | 1.186899 | 2.803864 | 1.857530 | 5.996275 | 2.257085 |
| 10 | ... | 0.3065419 | 3.112320 | 0.5893779 | 6.233165 | 0.8369336 |
| 100 | ... | 0.0314076 | 3.141279 | 0.0627843 | 6.282559 | 0.094099 |

| $\alpha$ | | $v_3$ | $u_4$ | $v_4$ | $u_5$ | $v_5$ |
|---|---|---|---|---|---|---|
| 0.7 | ... | 9.074991 | 3.599767 | 12.281658 | 3.818487 | 15.46613 |
| 0.9 | ... | 9.101535 | 3.344969 | 12.301503 | 3.564736 | 15.48201 |
| 1.0 | ... | 9.112793 | 3.238349 | 12.309886 | 3.458483 | 15.48870 |
| 2.0 | ... | 9.188942 | 2.541369 | 12.366114 | 2.762309 | 15.53334 |
| 10 | ... | 9.363701 | 1.049324 | 12.500862 | 1.231691 | 15.64167 |
| 100 | ... | 9.423841 | 0.125323 | 12.565127 | 0.156426 | 15.70642 |

From the Table IV we note that the equation [5.1] has inflnite number of complex roots with modulii forming an infinite sequence $\{r_i\}$ such that

$$r_1 < r_2 < r_3 \cdots < r_n \cdots$$

with $r_n \rightarrow \infty$ as $n \rightarrow \infty$. Also it is interesting to note that as in the previous section the value of $u$ and $v$ decrease and increase respectively with the increasing values of $\alpha$.

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## REFERENCE

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