SOME NEW APPLICATIONS OF THE SOLUTION OF THE EQUATION $AX + XB = -Q$

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ABSTRACT

Studied in this paper are some control problems which can conveniently utilize the solution of the linear algebraic equation $AX + XB = -Q$. These problems include, in addition to the already known ones,

(i) *Evaluation of ISE (Integral of the squared error) for a linear time invariant system with a single input of the polynomial type.*

(ii) *Determination of ISE of a model following system.*

(iii) *Simplification of large dynamic systems.*

Key words: Time invariant system, Integral square error, Model following, Simplification of large systems.

1. INTRODUCTION

It is known that the solution of the linear algebraic equation

$$AX + XB = -Q \tag{1}$$

where A and B are square matrices is useful in

(*a*) the study of stability of a dynamical system [1] (with $B = A'$)

(*b*) the construction of observers for linear time invariant multivariable systems [2, 3]

(*c*) studying suboptimal problems by aggregation [4, 5]

(*d*) the pole assignment of multi-input systems [6].

References [3] and [5] give methods of obtaining the explicit solution for (1). A survey of various methods of solving (1) can be found in [7].

In this paper we are concerned with some new applications of the solution of (1). It is well known [11] that for a unique solution X to exist when

Q is not a null matrix, A and $-B$ should not have common eigenvalues. Further if the eigenvalues of A and B are stipulated to have negative real parts, then X is given by

$$X = \int_0^\infty e^{At} Q e^{Bt} \, dt. \tag{2}$$

But evaluation of X *via* (2) is a formidable task. Computationally it would be simpler to solve for X using one of the methods suggested in [3], [5] and [7]. This advantage is availed in discovering new applications for X in this paper.

This paper is organised as follows. In section 2.1, we are concerned with the determination of ISE for polynomial type of inputs, more specifically for step, ramp and parabolic inputs. In section 2.2, we consider the evaluation of an improved error criterion essentially for the same inputs considered in section 2.1. In section 3, the evaluation of ISE for a model following system is considered. In section 4, the application of (2) to the simplification of large dynamical systems is dealt with.

2.1. *Determination of ISE-polynomial input.*—Consider the system described by

$$\begin{aligned} dx(t)/dt &= Ax(t) + bu(t) \\ y(t) &= hx(t) \end{aligned} \tag{3}$$

where x is $(nX1)$ state vector, u and y are scalar input and output respectively. A, b and h are matrices of appropriate dimensions. In what follows, we are concerned only with inputs which are polynomials in time t.

Let

$$e(t) = u(t) - y(t). \tag{4}$$

Then for a stable system (eigenvalues with negative real parts) ISE $= \int_0^\infty e^2(t)\, dt$ is a measure of the quality of the transient response of the system. For the system (3), for zero initial state, it is known that

$$y(t) = \int_0^t h e^{A(t-\tau)} b u(\tau) \, d\tau, \tag{5}$$

From (5) it is easy to derive explicit expressions when $u(t)$ is a unit step, ramp or parabolic inputs. If $u(t)$ is a unit step input, then (5) yields

$$y(t) = -hA^{-1}b + he^{At}A^{-1}b \tag{6}$$

when

$$u(t) = t,$$

(5) takes the form

$$y(t) = hA^{-2}e^{At}b - hA^{-2}b - hA^{-1}bt \tag{7}$$

when

$$u(t) = t^2/2!,$$

we get

$$y(t) = hA^{-3}e^{At}b - hA^{-3}b - hA^{-2}bt - hA^{-1}b\,t^2/2! \tag{8}$$

Equations (6), (7) and (8) are derived under the assumption that A^{-1} exists. This is true since A is a strictly stable matrix and does not have any poles at the origin or on the imaginary axis.

If the input is a polynomial, input of the type $u(t) = 1 + t + (t^2/2!)$ then by superposition

$$\begin{aligned} y(t) = {} & (hA^{-3}e^{At}b + hA^{-2}e^{At}b + hA^{-1}e^{At}b) \\ & - hA^{-3}b - hA^{-2}b(1+t) - hA^{-1}b(1+t+t^2/2!). \end{aligned} \tag{9}$$

From (4) and (6) to (9), it is easy to derive expressions for $e(t)$ when inputs are as described above. From these expressions it is clear that for the ISE to be finite we require that

$$\begin{aligned} hA^{-1}b &= -1 \\ hA^{-2}b &= 0 \\ hA^{-3}b &= 0. \end{aligned} \tag{10}$$

Under these restrictions we get the following expressions for a step, ramp, parabolic and a polynomial input respectively.

$$e(t) = -hA^{-1}e^{At}b \tag{11}$$

$$e(t) = -hA^{-2}e^{At}b \tag{12}$$

$$e(t) = -hA^{-3}e^{At}b \tag{13}$$

$$e(t) = -(hA^{-3} + hA^{-2} + hA^{-1})e^{At}b, \tag{14}$$

Under these conditions A. A. Krasovskii [8] developed a formula for calculating ISE for step inputs. Krasovskii's method involves expansion of $E(s)$ as

$$E(s) = \frac{1}{s} \frac{\sum_{k=1}^{M+1} U_k s^{k-1}}{\sum_{k=1}^{N+1} V_k s^{k-1}}$$

and evaluation of N-th order determinats formed from V_i, $i = 1, 2, \ldots N$. We present here an alternate method utilising the solution of the equation (1).

From (11), it is true that for step inputs

$$\begin{aligned} \text{ISE} &= \int_0^\infty e^2(t)\, dt = \int_0^\infty (hA^{-1} e^{At}\, bb'\, e^{A't} A^{-1'} h')\, dt \\ &= hA^{-1} X A^{-1'} h' \end{aligned} \tag{15}$$

where $X = \int_0^\infty e^{At}\, bb'\, e^{A't}\, dt$ is the well-known solution of

$$AX + XA' = -bb'. \tag{16}$$

In a similar way when the inputs are ramp, parabolic and polynomial inputs the ISE can be expressed utilising X, the solution of (16). More explicitly

$$\text{ISE} = hA^{-2} XA^{-2'} h', \quad (u(t) = t) \tag{17}$$

$$\text{ISE} = hA^{-3} XA^{-3'} h', \quad (u(t) = t^2/2!) \tag{18}$$

$$\text{ISE} = (hA^{-3} + hA^{-2} + hA^{-1})\, X\, (A^{-3'} h' + A^{-2'} h' + A^{-1'} h') \quad (u(t) = 1 + t + t^2/2!). \tag{19}$$

2.2. *Improved error criterion.*—In many cases, the minimisation of the ISE may result in systems with excessively strong oscillations. In such cases, it is desirable to consider the criterion

$$I_k = \int_0^\infty \left[e^2(t) + T\left(\frac{de}{dt}\right)^2 \right] dt \tag{20}$$

where T is a specified constant [8].

The procedure described in -2.1 may be extended to evaluate I_k when the inputs are of the polynomial type. Here we shall consider only step and ramp inputs. For other types of polynomial inputs the application is straightforward.

To evaluate I_k, we need find only $\int_0^\infty (de/dt)^2\, dt$ since $\int_0^\infty e^2(t)\, dt$ is evaluated already in §2.1. When $u(t)$ is a *unit step*, for $t > 0$, using (4) we get

$$\frac{de}{dt} = -\, dy/dt = -\, hAx - hb = -\, he^{At}\, b. \tag{21}$$

When $u(t)$ is a unit ramp, for $t > 0$, we get

$$\frac{de}{dt} = 1 - (dy/dt) = 1 - hAx - hbt = -\, hA^{-1}\, e^{At}\, b \text{ using (10).} \tag{22}$$

Along the same lines of §2.1 it is easy to see that

$$I_k = hA^{-1}\, XA^{-1'}\, h' + hXh' \; [u(t) = \text{step-input}] \tag{23}$$

and

$$I_k = hA^{-2}\, XA^{-2'}\, h' + hA^{-1}\, XA^{-1'}\, h' \; [u(t) = t] \tag{24}$$

where X is the solution of (16).

From §2·1 and 2.2 it is interesting to note that the evaluation of the ISE or I_k involves only the solution X of (16) for polynomial type of inputs.

3. Model Following System

There are situations when we stipulate that the output of the system (3) should follow the output of another system called the Model described by

$$\begin{aligned} dx_d/dt &= A_d x_d + b_d u \\ y_d &= h_d x_d \end{aligned} \tag{25}$$

when both (3) and (25) are excited by the same input. In such cases we need to evaluate

$$I_d = \int_0^\infty [y(t) - y_d(t)]\, dt. \tag{26}$$

Augmenting (3) with (25) we get

$$\begin{aligned} \begin{bmatrix} dx/dt \\ dx_d/dt \end{bmatrix} &= \begin{bmatrix} A & 0 \\ 0 & A_d \end{bmatrix} \begin{bmatrix} x(t) \\ x_d(t) \end{bmatrix} + \begin{bmatrix} b \\ b_d \end{bmatrix} u(t) \\ &= A_a x_a + b_a u \end{aligned} \tag{27}$$

$$e(t) = y - y_d = [h - h_d] \begin{bmatrix} x \\ x_d \end{bmatrix} = h_a x_a$$

when $u(t)$ is a step-input, using (6), we get

$$e(t) = - h_a A_a^{-1} b_a + h_a A_a^{-1} e^{A_a t} b_a'.$$

For I_d to be finite we require that $h_a A_a^{-1} b_a = 0$. Under this condition

$$I_d = h_a A_a^{-1} X A_a^{-1'} h_a'$$

where X is the solution of

$$A_a X + X A_a' = - b_a b_a'.$$

Similar expressions can be derived in an analogous way when the input is any other polynomial type.

4.1. *A brief summary of a method of simplification of large dynamic systems.*—The problem of simplifying linear dynamical systems was the subject of a number of papers by Chidambara [9] and Davison [10]. The problem can be stated as follows [9]:

Given an exact p-input, q-output n-th order linear time invariant system

$$dx/dt = Jx + Gu$$
$$y = Kx \tag{28}$$

where J is the Jordan matrix; to find a simplified model of order $l \leqslant n$ given by

$$\dot{z} = Fz + Du$$
$$y = Ez \tag{29}$$

such that

(i) the simplified model (29) retains l dominant eigenvalues of the system (28),

(ii) the model amplitudes should be such that the integral of the squared error between the exact and simplified models is minimum for a step input.

(iii) the initial and final values of the transient response of the simplified model under the influence of a polynomial input upto second degree in time shall show no error when compared with the corresponding exact response.

To solve the above problem, rewrite (28) [10]

$$\begin{bmatrix} dx_1/dt \\ dx_2/dt \end{bmatrix} = \begin{bmatrix} J_p & 0 \\ 0 & J_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} u \tag{30}$$

$$y(t) = [K_1 K_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where J_p contains l predominant eigenvalues and J_n the remaining $(n - l)$ eigenvalues. Following [9], assume the simplified model as

$$\dot{z} = J_p\, z + G_1 u$$
$$y^* = (K_1 + Q)\, z. \tag{31}$$

In (31) Q has to be selected such that for $i = 1, 2, \ldots, q$; $\int_0^\infty (\epsilon'_i\, \epsilon_i)\, dt$ is minimum where

$$\epsilon_i = \begin{bmatrix} \epsilon_{i1} \\ \epsilon_{i2} \\ \cdot \\ \cdot \\ \cdot \\ \epsilon_{ip} \end{bmatrix}$$

where ϵ_{ij} is the error in the i-th output for a step input at the j-th input node

The complete solution to this problem was given by Chidambara [9]. Let q_1' be the i-th row of Q. Then Q is determined from

$$\begin{bmatrix} M & N' \\ N & 0 \end{bmatrix} \begin{bmatrix} q_i \\ \gamma_i \end{bmatrix} = \begin{bmatrix} v_i \\ w_i \end{bmatrix} \tag{32}$$

where γ_i are Lagrange multipliers, and

$$M = \int_0^\infty (J_p^{-1}\, e^{J_p t}\, G_1 G_1'\, e^{J'_p t}\, J_p^{-1'})\, dt \tag{33}$$

$$N = G_1' \begin{bmatrix} J_p^{-1'} \\ J_p^{-2'} \\ J_p^{-3'} \end{bmatrix}$$

$$v_i = \int_0^\infty (J_p^{-1}\, e^{J_p t}\, G_1 G_2'\, e^{J'_n t}\, J_n^{-1'}\, K'_{2,i})\, dt \tag{34}$$

$$w_i = G_2' \begin{bmatrix} J_n^{-1} \\ J_n^{-2} \\ J_n^{-3} \end{bmatrix} K'_{2,i}$$

where $K'_{2,i}$ is the i-th column of K_2'.

Also if

$$\mu_i = \text{Min} \int_0^\infty \epsilon_i' \epsilon_i \, dt$$

then

$$\mu_i = K_{2,i} M_2 K'_{2,i} + q_i' M q_i - 2V_1' q_i$$

where

$$M_2 = \int_0^\infty (J_n^{-1} e^{J_n t} G_2 G_2' e^{-J_n' t} J_n^{-1'}) \, dt$$

$$\mu = \underset{i=1,2,\ldots,q}{\text{Max}} \mu_i$$

determines the goodness of the model.

4.2. *Application of the solution of* (1).—From (32), one can evaluate for q_i's only if M, v_i are known. To determine the goodness of the simplified model one needs M_2. As was done in the previous section, one can express M, v_i and M_2 as solutions of the algebraic equation (1) more specifically

$$M = J_p^{-1} X J_p^{-1'}$$

where X is the solution of

$$J_p X + X J_p' = - G_1 G_1' \quad (35)$$

$$v_i = J_p^{-1} Y J_n^{-1'} K_{2,i}$$

where Y is the solution of

$$J_p Y + Y J_n' = - G_1 G_2' \quad (36)$$

and finally

$$M_2 = J_n^{-1} Z J_n^{-1'}$$

where Z is the solution of

$$J_n Z + Z J_n' = - G_2 G_2'. \quad (37)$$

Combining (35), (36) and (37), one can write

$$Ja + aJ' = GG' \quad (38)$$

where

$$J = \text{diag}\ \{J_p, \quad J_n\}$$

$$a = \begin{bmatrix} X & Y \\ Y' & Z \end{bmatrix}$$

and

$$G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}$$

Thus the calculation of the matrices M, v_i and M_2 by evaluating the integrals is converted to solving the algebraic equation (38).

5. Conclusions

Some problems which were being solved by involved and cumbersome computations are shown to be capable of being solved conveniently by the application of the solution of linear algebraic equation $AX + XB = -Q$.

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