Simulation of co-sinusoidally modulated dielectric medium by artificial dielectric at microwave frequencies

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Abstract

Theory of *s*-, *b*- and *a*-profiles which simulates a co-sinusoidally modulated dielectric medium by a non-uniformly corrugated circular cylindrical metallic structure excited in E_{01}-mode is developed; the spacing, and radius of the discs and the radius of the central rod being denoted by *s*, *b* and *a* respectively.

Key words: Microwave artificial dielectric, Modulation, Effects in dielectric constant, Simulation.

1. Introduction

The pioneering work of Lewin[1] on electrical constants of spherical conducting particles in a dielectric, and Kock[2] on artificial dielectric followed by the development of artificial microwave optics[3] in Germany, extensive work on metallic delay media by Cohn,[4-6] application of Lorentz static field theory by Brown and Jackson,[7] electrostatic solution applied to a simple artificial anisotropic dielectric medium by Collin,[8] transmission-line approach for metallic-disc medium by Macfarlane,[9] Brown[10] and by Chatterjee *et al*,[11-13] conformal mapping solution for strip artificial dielectric medium by Kolettis,[14] study of reflection and phase-shift properties of H_{01}-wave on transmission through a parallel-plate medium by Carlson and Heins[15] by applying Wiener Hopff technique and experimental studies of parallel-plate medium by Chatterjee *et al*[16] using interferometric method, optical approach in the case of an uniformly corrugated dielectric rod by Chatterjee *et al*[17-20] have contributed significantly to our knowledge in the field of microwave analogue of dielectric medium. Various types of artificial dielectrics in the form of two-dimensional array of rods and strips, three-dimensional array of spheres and discs, etc., have found useful applications for microwave work. It appears that so far no attempt has been made to study metal discs on rod structure as an artificial dielectric medium, though surface wave and radiation characteristics of a uniformly corrugated metal structure have been studied by Chatterjee *et al*.[21-25] The present work has been motivated by recent studies[26-30] on surface-wave characteristics of a conductor coated with multilayer dielectrics having various types of dielectric constant profiles in the direction transverse to the direction of propagation.

The paper presents a report of the theoretical study made to derive (i) an equivalent dielectric constant ε° of a uniformly corrugated circular cylindrical metallic rod excited in E_{01}-mode as a function of the spacing (s) and radius (b) of discs and radius (a) of the central rod, and (ii) the profiles $s(z)$, $b(z)$ and $a(z)$ of a non-uniformly corrugated circular cylindrical metallic rod corresponding to co-sinusoidal modulation $\varepsilon(z) = \mathring{\varepsilon}(1-\delta\cos[2\pi z]/L)$ of the dielectric constant $\mathring{\varepsilon}$ in the direction of propagation z where δ and L denote the modulation index and the period of modulation respectively.

2. Formulation of the Problem

The field components outside the uniformly corrugated rod (Fig. 1) excited in E_{01}-wave are given by

$$E_z^{(1)} = \sum_{m=-\infty}^{\infty} C_m H_0^{(1)}(j\gamma_m\rho)\exp(-j\beta_m z)$$

$$E_\rho^{(1)} = \sum_{m=-\infty}^{\infty} C_m \frac{\beta_m}{k_m} H_1^{(1)}(j\gamma_m\rho)\exp(-j\beta_m z)$$

$$H_\phi^{(1)} = \sum_{m=-\infty}^{\infty} C_m \frac{k_0^2}{\omega\mu_0\gamma_m} H_1^{(1)}(j\gamma_m\rho)\exp(-j\beta_m z) \qquad \rho \geq b \qquad (1)$$

which take into account the existence of spatial harmonics appearing due to the periodic nature of the structure.

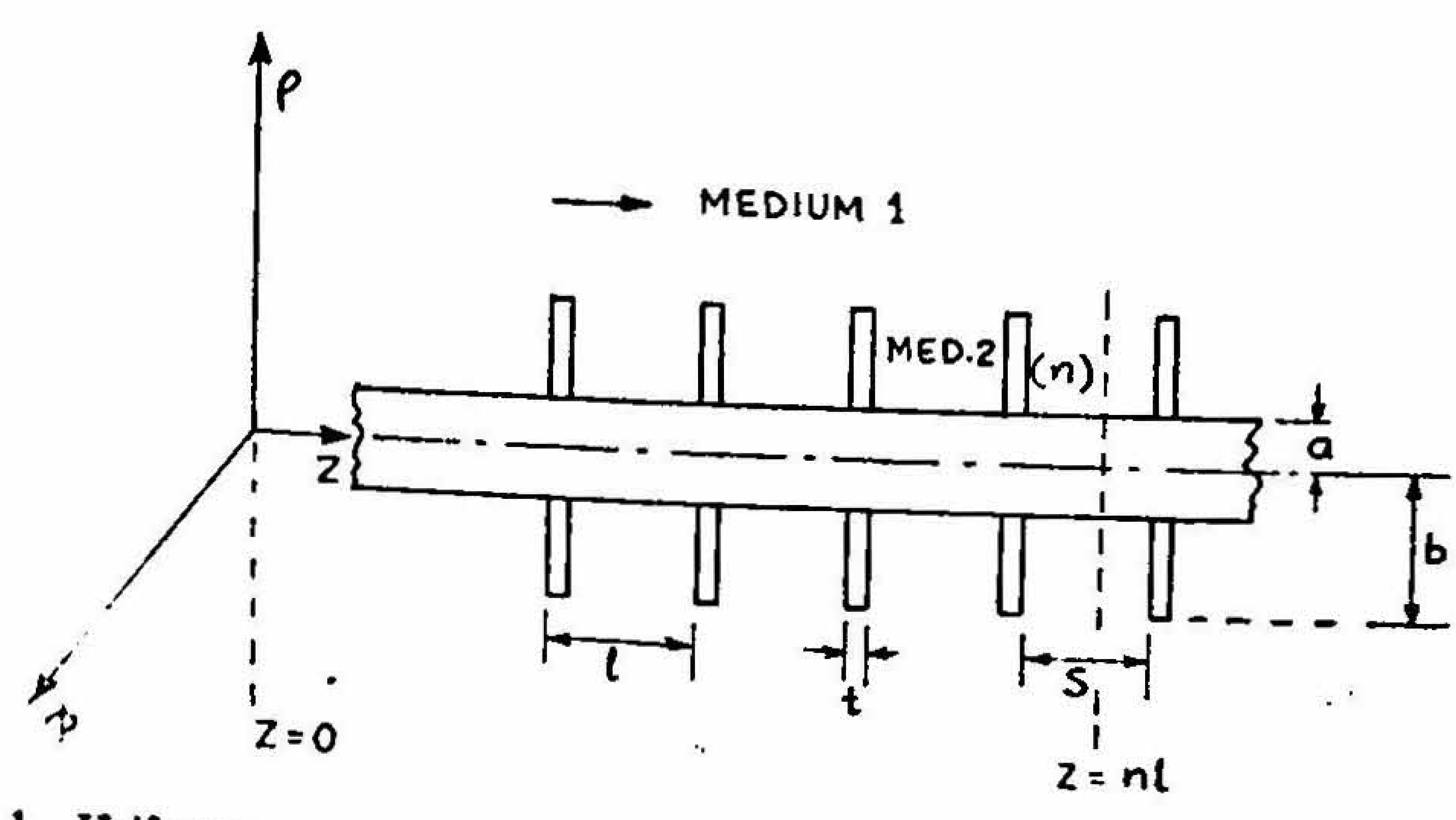

FIG. 1. Uniformly corrugated metal rod excited in E_{01}-mode.

The axial phase constant β_m and radial propagation constant γ_m for different spatial harmonics are related to each other

$$\beta_m^2 = \gamma_m^2 + k_0^2 \tag{2}$$

where k_0 denotes the free-space wave number for plane waves. The axial phase constant β_m for backward ($m = -1, -2, \ldots$) and forward ($m = +1, +2 \ldots$) spatial harmonics is related to the propagation constant β_m ($m = 0$) for the fundamental harmonic by

$$\beta_{m \atop (m\neq 0)} = \beta_0 + \frac{2\pi m}{l} \tag{3}$$

If the spacing s of the discs is small, each groove can be considered as a radial short-circuited (at $\rho = a$) transmission line. Hence, the field components in each grooved region ($\rho \leqslant b$, $\rho > a$) are given by

$$E_z^{(2)} = \left(\frac{-j\mu_0\omega}{k_0}\right)[A_1 J_0(k_0\rho) + A_2 y_0(k_0\rho)]\exp(-j\beta_0 nl)$$

$$H_\phi^{(2)} = [A_1 J_1(k_0\rho) + A_2 y_1(k_0\rho)]\exp(-j\beta_0 nl) \tag{4}$$

same as the components of T-wave in a radial transmission line.

The axial field component E_z which varies over the mouth of the groove at $\rho = b$ should satisfy the boundary conditions as well as the singularity condition at the edge of the discs and is assumed to have the form for $|z_n| < s/2$ at the n^{th} groove.

$$E_z = \frac{B}{[1-(2z_n/s)^2]^{1/2}}\exp(-j\beta_0 nl) \tag{5}$$

where $Z_n = Z - nl$ and equal to zero for

$$\frac{l}{2} \geq |z_n| \geqslant \frac{s}{2}.$$

Equating the two expressions for E_z at $\rho = b$ in equations (1) and (5), multiplying both sides by $\exp(j\beta_m z_n)$ and integrating with respect to z_n from $\frac{-l}{2}$ to $+\frac{l}{2}$, the following equation is obtained

$$C_m H_0^{(1)}(j\gamma_m b)\exp[j\beta_m(z_n - z)]\,dz_n = B\int_{-s/2}^{+s/2}\frac{\exp(-j\beta_0 l)\exp(j\beta_m z_n)}{\left[1-\left(\frac{2z_n}{s}\right)^2\right]^{1/2}}dz_n$$

which yields the relation between C_m and B as follows:

$$C_m = \frac{\pi s B}{2 l} \frac{J_0(\beta_m s/2)}{H_0^{(1)}(j\gamma_m b)} \tag{6}$$

Matching the average value of $E_z^{(1)}$ at $\rho = b$

$$E_{z_{av}}^{(1)}(\rho = b) = \frac{B}{s} \int_{-s/2}^{+s/2} \frac{\exp(-j\beta_0 nl)}{\left[1 - \left(\frac{2z_n}{s}\right)^2\right]^{1/2}} dz_n$$

with $E_z^{(2)}$ at $\rho = b$ and simplifying, the relation between the amplitude constants A and B is obtained as follows:

$$A = \frac{\pi B}{2} \frac{1}{F_0(k_0 b)} \tag{7}$$

where

$$F_0(k_0 b) = J_0(k_0 a)\, Y_0(k_0 b) - Y_0(k_0 a)\, J_0(k_0 b)\,.$$

The problem for simulating the corrugated structure as an equivalent artificial dielectric medium can be formulated by matching the average value of the component $H_\phi^{(1)}$ at $\rho = b$

$$H_{\phi_{av}}^{(1)}(\rho = b) = \frac{1}{s} \sum_{m=-\infty}^{\infty} C_m \frac{k_0^2}{\omega \mu_0 k_m} H_1^{(1)}(j\gamma_m b) \int_{-s/2}^{+s/2} \exp(-j\beta_m z)\, dz$$

with

$$H_\phi^{(2)} = j A \left(\frac{\mu_0}{\varepsilon_0}\right)^{1/2} F_1(k_0 b) \exp(-j\beta_0 nl)$$

in the groove at $\rho = b$, where

$$F_1(k_0 b) = J_0(k_0 a)\, Y_1(k_0 b) - Y_0(k_0 a)\, J_1(k_0 b).$$

The matching of the azimuthal components of the magnetic field at the mouth of the groove ($\rho = b$) yields the following relation[21]:

$$-\frac{F_1(k_0 b)}{F_0(k_0 b)} = \sum_{m=-\infty}^{+\infty} \frac{2k_0}{l} \frac{J_0(\beta_m s/2) \sin(\beta_m s/2)}{\beta_m \gamma_m} \frac{K_1(\gamma_m b)}{K_0'(\gamma_m b)} \tag{8}$$

where the Hankel functions have been transformed to second kind modified Bessel functions in order to make the arguments real by using the following relations:

$$H_0^{(1)}(j\gamma_m b) = -jK_0(\gamma_m b)\frac{2}{\pi}$$

$$H_1^{(1)}(j\gamma_m b) = -K_1(\gamma_m b)\frac{2}{\pi}$$

The propagation characteristics of E_{01}-wave in the corrugated rod are determined by solving eq. (8) for β_m or γ_m which are inter-related and involve the phase consant β_0 for the fundamental harmonic. Equation (8) can be simplified if the contribution of the fundamental harmonic is more predominant than that of higher order spatial harmonics. Or in other words if the relative amplitudes of higher order spatial harmonics with respect to the amplitude of the fundamental harmonic are very small, then the problem can be formulated only in terms of the fundamental harmonic.

The relative amplitudes of the spatial harmonics with respect to the fundamental harmonic amplitude A_0 are given by the following relation:

$$\frac{A_m}{A_0} = \left|\frac{J_0(\beta_0 s/2)}{J_0(\beta_m s/2)}\right| \frac{K_0(\gamma_m b)}{K_0(\gamma_0 b)} \frac{k_0(\gamma_0 \rho)}{k_0(\gamma_m \rho)} \qquad m \neq 0 \tag{9}$$

which is obtained from equations (1) and (6) and considering the axial component E_z and replacing the function $H_0^{(1)}$ by K_0. The ratio $A_m/A_0 \gtrless 1$ depending on the relative values of β_0 and β_m and hence of γ_m and γ_0, s being $\leqslant \lambda_0/2$. The factor

$$\frac{K_0(\gamma_m b)}{K_0(\gamma_0 b)} \frac{K_0(\gamma_0 b)}{K_0(\gamma_m \rho)} \lesseqgtr 1$$

according as $\gamma_m \gtrless \gamma_0$. Hence $A_m \lesseqgtr A_0$ according to $\beta_m \gtrless \beta_0$.
Since $r_0^2 = \beta_0^2 - k_0^2$, r_0 is real for $\beta_0 \geqslant k_0$. That is the phase velocity of the fundamental component $v_p < C$ and hence $\lambda_g < \lambda_0$ and the surface wave character of the wave can be maintained. But if $\beta_0 < k_0$, then $v_p > C$ and hence $\lambda_g > \lambda_0$ and γ_0 is imaginary which amounts to the surface wave being transformed to radiated wave. For a strongly bound surface wave, it is necessary that v_p should be low.

The inequality $A_m/A_0 \lesseqgtr 1$ according as $\beta_m/\beta_0 \gtrless 1$ signifies that the harmonic of highest amplitude has the highest phase velocity. Hence when spatial harmonic amplitudes are great.r than the fundamental amplitude, the spatial harmonics become more loosely bound and depending on the order of v_p may be lost by radiation. Further it can be shown that the attenuation suffered by higher order harmonics is much greater than the fundamental. Moreover, as the problem is concerned with establishing the equivalence or simulation of the corrugated structure as a uniform homogeneous dielectric, the effect of the spatial harmonics can be ignored and the problem can be formulated in terms of only the fundamental ($m = 0$) component of the wave as follows:

$$\frac{2k_0}{l}\frac{J_0(\beta_0 s/2)\sin(\beta_0 s/2)}{\beta_0\gamma_0}\frac{K_1(\gamma_0 b)}{K_0(\gamma_0 b)} = -\frac{F_1(k_0 b)}{F_0(k_0 b)} \tag{9 a}$$

3. Equivalent dielectric constant

The equivalent dielectric constant $\mathring{\epsilon}$ defined by $\mathring{\epsilon} = \frac{\beta_0^2}{k_0^2}$ can be found by solving eq. (9) for the fundamental phase constant β_0. Assuming s such that $(\beta_0 s/2) < 1$, eq. (9) can be transformed to the following quadratic equation in β_0^2:

$$s^5(\beta_0^2)^2 - 40\, s^3\beta_0^2 + C = 0 \tag{10}$$

which on solving yields the equivalent dielectric constant $\mathring{\epsilon}$ as

$$\mathring{\epsilon} = \frac{20\, s \pm (400\, s^2 - sC)^{1/2}}{s^3 k_0^2} \tag{11}$$

where

$$C = -768\left[\frac{s}{2} + \frac{lr_0}{2k_0}\frac{K_0(\gamma_0 b)}{K_1(\gamma_0 b)}\frac{F_1(k_0 b)}{F_0(k_0 b)}\right] \tag{11 a}$$

The following cases depending on the nature and magnitude of the arguments of the functions involved in the expression for $\mathring{\epsilon}$ may be of interest, the arguments being considered to be real

Case (i)	$k_0 a < 1$	$k_0 b > 1$	$\gamma_0 b > 1$
Case (ii)	$k_0 a < 1$	$k_0 b < 1$	$\gamma_0 b > 1$
Case (iii)	$k_0 a < 1$	$k_0 b < 1$	$\gamma_0 b < 1$
Case (iv)	$k_0 a < 1$	$k_0 b > 1$	$\gamma_0 b < 1$.

At X-band k_0 is of the order of 1·98 radians per cm and for practical surface wave structures, $a \not> 0\cdot25$ cm. Hence $k_0 a$ is always < 1. So far as the other inequalities are concerned, it depends on the values of γ_0 and b. For a very strongly bound surface wave $\gamma_0 b \gtrless 1$ depending only on the value of b. The inequality $k_0 b \gtrless 1$ at X-band depends also only on the value of b the disc radius.

Case (i).—The argument approximations $k_0 a < 1$, $k_0 b > 1$, $\gamma_0 b > 1$ lead to (Appendix A.1)

$$\frac{F_1(k_0 b)}{F_0(k_0 b)}\frac{K_0(\gamma_0 b)}{K_1(\gamma_0 b)} = -\frac{\frac{1}{k_0 b} + \frac{k_0 b}{2}\left(\ln\frac{k_0 a}{2} + 0\cdot577\right)}{\ln b/a}$$

Hence eq. (10) reduces to the following quadratic equation

$$\beta_0^4 - \frac{40}{s^2}\beta_0^2 - \frac{\chi}{k_0 s^5}\beta_0 - \frac{384}{s^4} = 0 \qquad (12)$$

which yields the following solutions for $\overset{\circ}{\epsilon}$

$$\overset{\circ}{\epsilon}_{1,2} = \left[\frac{e \pm \sqrt{e^2 - 4(\frac{1}{2}y - f)}}{2k_0}\right]^2$$

$$\overset{\circ}{\epsilon}_{3,4} = \left[\frac{-e \pm \sqrt{e^2 - 4(\frac{1}{2}y - f)}}{2k_0}\right]^2 \qquad (13)$$

where $y = \sqrt[3]{A} + \sqrt[3]{B} + \underline{b}/3$ which is obtained by considering only the real roots and ignoring the imaginary conjugate roots

$$e = \sqrt{A}, \quad f = \frac{B}{2\sqrt{A}}$$

$$A = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

$$B = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

$$p = -4\underline{d} - \underline{b}^2/3$$

$$q = \frac{8bd}{3} - \underline{C}^2 - \frac{2b^3}{27}$$

$$\underline{b} = -40/s^2 \qquad \underline{C} = -\chi/k_0 s^5, \qquad \underline{d} = -\frac{384}{s^4}$$

$$\chi = -384\, l\, \frac{\frac{1}{k_0 b} + \frac{k_0 b}{2}\left(\ln\frac{k_0 a}{2} + 0\cdot 577\right)}{\ln b/a}$$

Case (ii)—Similarly in this case

$$\frac{K_1(\gamma_0 b)}{K_0(\gamma_0 b)}\frac{F_1(k_0 b)}{F_0(k_0 b)} = \frac{\sin\left(k_0 b - \frac{3\pi}{4}\right) - \frac{2}{\pi}\cos\left(k_0 b - \frac{3\pi}{4}\right)\left(\ln\frac{k_0 a}{2} + 0\cdot 577\right)}{\sin\left(k_0 b - \frac{\pi}{4}\right) - \frac{2}{\pi}\cos\left((k_0 b - \frac{\pi}{4}\right)\left(\ln\frac{k_0 a}{2} + 0\cdot 577\right)}$$

Since

$$\frac{K_1(\gamma_0 b)}{K_0(\gamma_0 b)} \cong 1 \qquad \text{for } r_0 b > 1.$$

Hence eq. (10) reduces to

$$\beta_0^4 - \frac{40}{s^2}\beta_0^2 - \frac{\chi'}{k_0 s^6}\beta_0 - \frac{384}{s^4} = 0 \tag{14}$$

which yields the same type of solutions for $\overset{\circ}{\epsilon}_{1,2}$ and $\overset{\circ}{\epsilon}_{3,4}$ where all the symbols are the same except

$$\underline{C} = \frac{-\chi'}{k_0 S^6}$$

$$\chi' = 384\, l \frac{(\cos k_0 b + \sin k_0 b) + \frac{2}{\pi}(\cos k_0 b - \sin k_0 b)\left(\ln\frac{k_0 a}{2} + .577\right)}{(\sin k_0 b - \cos k_0 b) - \frac{2}{\pi}(\cos k_0 b + \sin k_0 b)\left(\ln\frac{k_0 a}{2} + .577\right)} \tag{14 a}$$

Case (iii)—in this case

$$\frac{K_0(\gamma_0 b)}{K_1(\gamma_0 b)}\,\frac{.F_1(k_0 b)}{F_0(k_0 b)} = \left[\frac{\frac{1}{k_0 b} + \frac{k_0 b}{2}\left(\ln\frac{k_0 a}{2} + 0.577\right)}{\ln b/a}\right]\gamma b \ln(0.89\,\gamma b)\,.$$

Hence eq. (10) reduces to

$$(\beta_0^2)^3 + \underline{a}(\beta_0^2)^2 + \underline{b}\,\beta_0^2 + \tau = 0 \tag{15}$$

which yields for the real root

$$\overset{\circ}{\epsilon} = \frac{1}{k_0^2}\left[\sqrt[3]{A} + \sqrt[3]{B} - \underline{a}/3\right] \tag{16}$$

where

$$\underline{a} = \frac{s^6 - 40 s^3 - s^6 k_0^2 - A'}{s^6}$$

$$\underline{b} = \frac{40 s^3 k_0^2 - 384 s - 40 s^3 + B'}{s^6}$$

$$\tau = \frac{384 s k_0^2 - 384 s - \underline{c}}{s^6}$$

$$A' = \alpha \log 0.89\, b + \alpha$$

$$B' = 2\alpha k_0^2 \log 0.89\, b + 2\alpha k_0^2 + \alpha - \alpha \log 0.89\, b$$

$$\underline{C} = \alpha k_0^4 \log 0.89\, b + \alpha k_0^4 + \alpha k_0^2 - \alpha k_0^2 - \log 0.89\, b$$

$$\alpha = \frac{lb}{2K_0 \log b/a}\left[\frac{1}{k_0 b} + \frac{k_0 b}{2}\left(\ln\frac{k_0 a}{2} + 0.577\right)\right]$$

$$A = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \, .$$

$$B = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

$$p = \underline{b} - \underline{a}^2/3$$

$$q = \tau - \frac{a\,b}{3} + \frac{2a^3}{27} \, .$$

Case (iv) :

$$\frac{\gamma_0 l}{2k_0} \frac{K_0(\gamma_0 b)}{K_1(\gamma_0 b)} \frac{F_1(k_0 b)}{F_0(k_0 b)} = -\frac{\chi''}{\beta_0^2 - k_0^2 + 1} [A''\beta_0^4 + B''\beta_0^2 + C''].$$

Hence, eq. (10) reduces to

$$(\beta_0^2)^3 + a'(\beta_0^2)^2 + b'(\beta_0)^2 + C' = 0 \tag{17}$$

which considering only the real root yields

$$\dot{\epsilon} = \frac{1}{k_0^2}\left[3\sqrt{A} + 3\sqrt{B} - \frac{a'}{3}\right] \tag{18}$$

where

$$A = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

$$B = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

$$p = -b' - \frac{(a')^2}{3}$$

$$q = C' - \frac{a'\,b'}{3} + \frac{2\,(a')^3}{27}$$

$$a' = \frac{s^5 - 40\,s^3 - s^5\,k_0^2 + 768\,\chi''\,A''}{s^5}$$

$$b' = \frac{40\,k_0^2\,s^3 - 384\,s - 40\,s^3 + 768\,\chi''\,B''}{s^5}$$

$$C' = \frac{384\,s\,k_0^2 - 384\,s + 768\,\chi''\,C''}{s^5}$$

$$A'' = 1 + \log 0{\cdot}89\,b$$

$$B'' = \log 0{\cdot}89\,b - 2k_0^2 \log 0{\cdot}89\,b - 2k_0^2 - 1$$

$$C'' = k_0^4 + k_0^2 + k_0^4 \log 0\cdot 89b - k_0^2 \log 0\cdot 89b$$

$$\chi'' = \frac{2\cdot 303\, bl}{2k_0} \frac{(\cos k_0 b + \sin k_0 b) + \frac{2}{\pi}(\cos k_0 b - \sin k_0 b)\left(\ln \frac{k_0 a}{2} + 0\cdot 577\right)}{(\sin k_0 b - \cos k_0 b) - \frac{2}{\pi}(\cos k_0 b + \sin k_0 b)\left(\ln \frac{k_0 a}{2} + 0\cdot 577\right)}$$

In all the four cases $\mathring{\varepsilon} = f(s, b, a)$. Hence positive real values of $\mathring{\varepsilon}$ can be found by a judicious selection of the three structure parameters s, b and a. Preliminary computations show that some of the roots are imaginary and some are negative. We will consider only the positive real roots of $\mathring{\varepsilon}$. It is, however, to be noted that there are forbidden bands for β as $f(s, b, a)$ and hence for $\mathring{\varepsilon}$.

4. Modulation of the dielectric constant of the artificial dielectric medium

Modulation of the artificial dielectric medium can be achieved by varying s, b or a, keeping the other two parameters constant in the direction (z) of propagation. The degree of modulation is controlled by the modulation index δ which can be defined by $(\delta = (\mathring{\varepsilon}_{max} - \mathring{\varepsilon}_{min})/(\mathring{\varepsilon}_{max} + \mathring{\varepsilon}_{min}) < 1$. The primary aim with which the present problem has been initiated is to study the propagation characteristics of E_{01}-wave in a co-sinusoidally modulated artificial dielectric medium which involves essentially the determination of $\beta = \beta(z)$ for different values of δ so as to gain an insight into the conditions of mode stability which is inherent in such problems. In order to verify the theory of propagation in a modulated artificial dielectric medium which will be published elsewhere it will be necessary to construct non-uniformly corrugated structures with $s = s(z)$, $b = b(z)$ or $a = a(z)$ which will be termed hereafter as s-, b- or a-profiles. So as to correspond to a structure with $\varepsilon(z) = \mathring{\varepsilon}\left(1 - \delta \cos \frac{2\pi z}{L}\right)$. The profile equations for the different cases under which $\mathring{\varepsilon}$ has been determined are given in the following sections.

5. The s-profiles

The s-profiles are derived in terms of $\varepsilon(z)$ from equations (12–17) corresponding to cases (i)-(iv) respectively and are given by

Case (i)

$$\varepsilon^2(z) . s^6 - \frac{40\, \varepsilon(z)}{k_0^2} s^3 - \frac{384}{k_0^4} s - \frac{\chi}{k_0^4} \sqrt{\varepsilon(z)} = 0 \tag{19}$$

Case (ii)

$$\varepsilon^2(z) . s^6 - \frac{40\, \varepsilon(z)}{k_0^2} s^3 - \frac{384}{k_0^4} s - \frac{\chi'}{k_0^4} \sqrt{\varepsilon(z)} = 0. \tag{20}$$

Case (iii)

$$\left[\varepsilon^3(z) + \frac{1}{k_0^2}\varepsilon^2(z) - \varepsilon^2(z)\right] . s^5 + \left[\frac{40}{k_0^2}\varepsilon^2(z) - \frac{40}{k_0^2}\varepsilon(z) + \frac{40}{k_0^4}\varepsilon(z)\right] . s^3$$

$$-\left[\frac{384}{k_0^4}\varepsilon(z) - \frac{384}{k_0^4} + \frac{384}{k_0^6}\right] . s - \left[\frac{A'}{k_0^2}\varepsilon^2(z) + \frac{B'}{k_0^4}\varepsilon(z) - \frac{C}{k_0^6}\right] = 0 \tag{21}$$

Case (iv)

$$\left[\varepsilon^3(z) + \frac{1}{k_0^2}\varepsilon^2(z) - \varepsilon^2(z)\right] . s^5 - \left[\frac{1}{k_0^2}\varepsilon^2(z) - \frac{40}{k_0^2}\varepsilon(z) + \frac{40}{k_0^4}\varepsilon(z)\right] . s^3$$

$$-\left[\frac{384}{k_0^4}\varepsilon(z) - \frac{384}{k_0^4} + \frac{384}{k_0^6}\right] . s$$

$$+\left[\frac{768\,\chi''\,A''}{k_0^2}\varepsilon^2(z) + \frac{768\,\chi''\,B''\,\varepsilon(z)}{k_0^4} + \frac{768\,\chi''\,C''}{k_0^6}\right] = 0 \tag{22}$$

where b and a are maintained constants.

6. The b-profiles

The b-profiles for the four cases have been similarly derived assuming $2 < b < 0$ and are given by

Case (i)

$$b(z) = \frac{-B' \pm \sqrt{(B')^2 - 4A'C'}}{2A'} \tag{23}$$

where

$$A' = 4A\,k_0 - k_0^3\,a - 0{\cdot}84b\,k_0^2$$
$$B' = -4A\,k_0 + 4A\,k_0 \ln a$$
$$C' = -4$$
$$A = \frac{1}{384\,l}\,\frac{k_0^4}{\sqrt{\varepsilon(z)}}\left[s\,\frac{384}{k_0^4} + \frac{s^3\,40\,\varepsilon(z)}{k_0^2} - s^5\,\varepsilon^2(z)\right]$$

Case (ii)

$$b(z) = \frac{1}{k_0}\,\frac{\tan^{-1}(D+1)\left\{1 + \frac{2}{\pi}\left(\ln\frac{k_0 a}{2} + 0{\cdot}577\right)\right\}}{D\left\{1 - \frac{2}{\pi}\left(\ln\frac{k_0 a}{2} + 0{\cdot}577\right)\right\} + \left\{1 + \frac{2}{\pi}\left(\ln\frac{k_0 a}{2} + 0.577\right)\right\}} \tag{24}$$

where

$$D = \frac{k_0^4}{\sqrt{\varepsilon(z)}}\left[s^5\,\varepsilon^2(z) - \frac{40}{k_0^4}\varepsilon(z)\,s^3 - \frac{384}{k_0^4}\,s^3\right]\frac{1}{384l}\,.$$

Case (iii)

$$b(z) = \sqrt[3]{\underline{A}} + \sqrt[3]{\underline{B}} - \frac{\alpha^3}{3} \tag{25}$$

where

$$\underline{A} = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

$$\underline{B} = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

$$p = \mathring{\beta} - \frac{\mathring{\alpha}^2}{3}$$

$$q = \mathring{\delta} - \mathring{\beta}\,\frac{\mathring{\alpha}}{3} + \frac{2\,\alpha^{\circ 3}}{27}$$

$$\mathring{\alpha} = \frac{\beta}{\alpha'}, \qquad \mathring{\beta} = -\frac{\gamma}{\alpha'}, \qquad \mathring{\delta} = \frac{\delta}{\alpha'}$$

$$\alpha' = l k_0 \psi \left(\ln \frac{k_0 a}{2} + 0\cdot577\right)$$

$$\beta = 4\,k_0^2\,\Theta - l\gamma k_0^2 \left(\ln \frac{k_0 a}{2} + 0\cdot 577\right)$$

$$\gamma = 4k_0^2\,\Theta' - 4k_0^2\,\Theta - 4k_0^2\,\Theta \log a - 2l\psi$$

$$\underline{\delta} = Hk_0^2\,\Theta' + \Theta' \log a + 2l\gamma$$

$$\psi = \frac{2\cdot67\,\varepsilon^2(z)}{k_0^2} + \frac{5\cdot34\,\varepsilon(z)}{k_0^3} - \frac{0\cdot89\,\varepsilon(z)}{k_0^4} - \frac{2\cdot67}{k_0^2} + \frac{0.89}{k_0^4}$$

$$\gamma = \frac{3\,\varepsilon(z)}{k_0^4} - \frac{\varepsilon^2(z)}{k_0^2} + \frac{1}{k_0^3} - \frac{3}{k_0^4}$$

$$\Theta = 0\cdot89\,(As^5 + Bs^3 - Cs)$$

$$\Theta' = \Theta\,/\,0\cdot89$$

$$A = \varepsilon^3(z) + \frac{\varepsilon^2(z)}{k_0^2} - \varepsilon^2(z)$$

$$B = \frac{40}{k_0^2}\,\varepsilon^2(z) - \frac{40}{k_0^3}\,\varepsilon(z) + \frac{40}{k_0^4}\,\varepsilon(z)$$

$$C = \frac{384}{k_0^4}\,\varepsilon(z) - \frac{384}{k_0^4} + \frac{384}{k_0^6}$$

Case (iv)

$$b(z) = \frac{1}{k_0} \text{ arc tan } \frac{Hk_0 b + G}{Jk_0 b + K} \tag{26}$$

where

$$H = \mathring{D}\mathring{b} - \mathring{b}\mathring{f}$$

$$G = k_0(\mathring{D}\mathring{b} + \mathring{g}\mathring{b})$$

$$J = \mathring{D}\mathring{\underline{C}} - \mathring{b}\mathring{f}$$

$$K = k_0(\mathring{E}\mathring{\underline{C}} - \mathring{g}\mathring{b})$$

$$\mathring{D} = [\mathring{A} s^5 - \mathring{B} s^3 - \mathring{C} s] \frac{1 \cdot 78\, k_0}{2 \cdot 303\, l}$$

$$\mathring{E} = [\mathring{A} s^5 - \mathring{B} s^3 - \mathring{C} s] \frac{2k_0}{2 \cdot 303\, l}$$

$$\mathring{b} = 1 + \frac{2}{\pi}\left(\ln \frac{k_0 a}{2} + 0 \cdot 577\right)$$

$$\mathring{\underline{c}} = 1 - \frac{2}{\pi}\left(\ln \frac{k_0 a}{2} + 0 \cdot 577\right)$$

$$\mathring{f} = 2 \cdot 67\, k_0^4\, \varepsilon^2(z) + 0 \cdot 89\, k_0^2\, \varepsilon(z) - 5 \cdot 34\, k_0^4\, \varepsilon(z) + 2 \cdot 67\, k_0^4 - 0 \cdot 89\, k_0^2$$

$$\mathring{g} = k_0^4\, \varepsilon^2(z) - 2k_0^4\, \varepsilon(z) + 3k_0^2\, \varepsilon(z) + k_0^4 - 3\, k_0^2$$

$$\mathring{A} = A\, \frac{k_0^6}{768}$$

$$\mathring{B} = B\, \frac{k_0^6}{768}$$

$$\mathring{C} = C\, \frac{k_0^6}{768}$$

$$A = \varepsilon^3(z) + \frac{\varepsilon^2(z)}{k_0^2} - \varepsilon^2(z)$$

$$B = \frac{40}{k_0^2}\, \varepsilon^2(z) - \frac{40}{k_0^2}\, \varepsilon(z) + \frac{40}{k_0^4}\, \varepsilon(z)$$

$$C = \frac{384}{k_0^4}\, \varepsilon(z) - \frac{384}{k_0^4} + \frac{384}{k_0^6}$$

where s and a are constant in all the cases.

7. The a-profiles

Case (i)

$$a(z) = \frac{-K_5 \pm \sqrt{K_5^2 - 4K_4 K_6}}{2K_4} \tag{27}$$

where

$$K_4 = \frac{(2 \cdot 303)^2 k_0 K_2}{2}$$

$$K_5 = -\left[\frac{2 \cdot 303\, k_0 l K_2}{2} \ln b - 2 \cdot 303\, K_1 + (2 \cdot 303)^2 K_2 + (2 \cdot 303)^2 \frac{k_0 K_2}{2}\right]$$

$$K_6 = -K_3$$

$$K_3 = K_1 \ln b + 2 \cdot 303\, K_1 - 2 \cdot 303\, K_2 \ln b - (2 \cdot 303)^2 K_2$$

$$K_2 = \frac{384}{k_0^4} \frac{k_0 b}{2} \left[\frac{\varepsilon(z) - 1}{2\sqrt{\varepsilon(z)}}\right]$$

$$K_1 = \varepsilon(z) s^5 - \frac{40}{k_0^2} \varepsilon(z) s^3 - \frac{384\, s}{k_0^4} + \frac{384\, l}{k_0^4} \left[\frac{\varepsilon(z) - 1}{2\sqrt{\varepsilon(z)}}\right] \left[\frac{1}{k_0 b} + \frac{0 \cdot 577\, k_0 b}{2}\right].$$

Case (ii)

$$a(z) = \frac{2}{k_0} \operatorname{antilog}\left[\frac{1}{2 \cdot 303} \frac{K_{11} K_9 - K_7}{K_{11} K_{10} + K_8}\right] \tag{28}$$

where

$$K_7 = \cos k_0 b \left(1 + \frac{1 \cdot 154}{\pi}\right) + \sin k_0 b \left(1 - \frac{1 \cdot 154}{\pi}\right)$$

$$K_8 = \frac{2}{\pi} \cos k_0 b - \frac{2}{\pi} . \sin k_0 b$$

$$K_9 = \sin k_0 b \left(1 - \frac{1 \cdot 154}{\pi}\right) - \cos k_0 b \left(1 + \frac{1 \cdot 154}{\pi}\right)$$

$$K_{10} = \frac{2}{\pi} \cos k_0 b + \frac{2}{\pi} \sin k_0 b$$

$$K_{11} = \frac{k_0^4}{\sqrt{\varepsilon(z)}} \frac{1}{384\, l} \left[\varepsilon^2(z) s^5 - \frac{40\, s^3}{k_0^2} \varepsilon(z) - \frac{768\, s}{2k_0^4}\right].$$

Case (iii)

$$a(z) = \frac{1}{K_{12} + \frac{k_0^2 b}{L_1}} \left\{\frac{1}{2 \cdot 303}\left(K_{12} \ln b - \frac{1}{k_0 b} - 0 \cdot 577 \frac{k_0 b}{2}\right) + K_{12} + \frac{k_0 b}{2}\right\}$$

where

$$K_{12} = \frac{2k_0^3 \left[\varepsilon^2(z)\, s^5 - \frac{40}{k_0^2}\, \varepsilon(z)\, s^3 - \frac{768\, s}{2k_0^4} \right]}{[\varepsilon(z) - 1]\, lb\, \{\ln(0\cdot 89\, b) + \frac{1}{2}\ln k_0^2 + \frac{1}{2}\ln[\varepsilon(z) - 1]\}}$$

Case (iv)

$$a(z) = \frac{2}{k_0} \text{antilog} \frac{1}{2\cdot 303} \frac{K'}{K''} \quad (30)$$

where

$$K' = \left[\left\{ \varepsilon(z)\, s^5 - \frac{40}{k_0^2}\, s^3\, \varepsilon(z) - \frac{768\, s}{2k_0^4} \right\} \frac{2k_0^3}{768\, lb\, [\varepsilon(z) - 1]} \right.$$

$$\times \left. \frac{1}{\ln(0\cdot 89\, b) + \frac{1}{2}\ln k_0^2 + \frac{1}{2}\ln[\varepsilon(z) - 1]} \right\} \times \left(\sin k_0 b - \cos k_0 b \right.$$

$$\left. \left. - \frac{1\cdot 154 \cos k_0 b}{\pi} - \frac{1\cdot 154 \sin k_0 b}{\pi} \right) \right]$$

$$+ \left[\cos k_0 b + \sin k_0 b + \frac{1\cdot 154 \cos k_0 b}{\pi} - \frac{1\cdot 154 \sin k_0 b}{\pi} \right]$$

$$K'' = \left[\left\{ \varepsilon(z)\, s^5 - \frac{40 s^3}{k_0^2}\, \varepsilon(z) - \frac{768\, s}{2k_0^4} \right\} \right.$$

$$\times \left\{ \frac{2k_0^3}{768\, lb\, [\varepsilon(z) - 1]} \times \frac{1}{\ln(0\cdot 89\, b) + \frac{\ln}{2}\, [\varepsilon(z) - 1]} \right\}$$

$$\left. - 1 \right] \times \left(\frac{2}{\pi} \cos k_0 b - \frac{2}{\pi} \sin k_0 b \right)$$

where in all the cases s and b are kept constant.

Numerical computations of the profiles for different degrees of modulation are in progress and will be reported later.

8. Appendix A.1

In deriving the relations for $\mathring{\epsilon}$ and the profiles the following argument approximations have been used appropriately for the four cases:

$x < 1$

$$K_0(x) \cong -\frac{2}{\pi} \ln(0\cdot 89\, x)$$

$$K_1(x) \cong \frac{2}{\pi x}$$

$$J_0(x) \cong 1 - \frac{x^2}{2^2} + \ldots \approx 1$$

$$J_1(x) \cong \frac{x}{2}$$

$$Y_1(x) \cong -\frac{2}{\pi x}$$

$$Y_0(x) \cong \frac{2}{\pi}\left(\text{in}\frac{x}{2} + 0{\cdot}577\right)$$

the general relations being

$$J_\gamma(x) \cong \frac{x^\nu}{2^\nu\ \gamma(\nu+1)}$$

$$\gamma(1) = 1$$

$$\gamma(2) = 1$$

$$Y_\nu(x) \cong -\frac{2^\nu\ \gamma(\nu)}{\pi\ x^\nu}$$

$$x > 1$$

$$J_0(x) \sim \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)$$

$$J_1(x) \sim \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\pi}{4}\right)$$

$$Y_0(x) \sim \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{\pi}{4}\right)$$

$$Y_1(x) \sim \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{3\pi}{4}\right)$$

$$\mathrm{K}_0(x)/\mathrm{K}_1(x) = 1$$

derived from the relations

$$J_n(x) \sim \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)$$

$$Y_n(x) \sim \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)$$

$$K_n(x) \sim \frac{\exp(-x)}{\sqrt{2\pi x}}.$$

Besides the above argument approximations, the following series expansions have also been used appropriately.

$$(a + x)^n = a^n + na^{n-1} x + \frac{n(n-1)}{2!} a^{n-2} x^2 + \dots$$

$$\ln x = 2 \cdot 302 \log x = 2 \cdot 302 \left[2 \left\{ \frac{x-1}{x+1} + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \dots \right] \quad x > 0$$

$$\ln x = 2 \cdot 302 \log x = 2 \cdot 302 \left[2(x-1) - \tfrac{1}{2}(x-1)^2 + \dots \right] \quad 2 > x > 0.$$

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References

1. Lewin, L. — *Jour. I.E.E.*, 1947, **94**, Part III, 65.
2. Kock, W. E. — *Bell Sys. Tech. Jour.*, 1948, **27**, 58.
3. Stuetzer, O. M. — *Proc. I.R.E.*, 1950, **38**, 1053.
4. Cohn, S. B. — *Jour. Appl. Phys.*, 1951, **22**, 628.
5. Cohn, S. B. — *Jour. Appl. Phys.*, 1949, **20**, 257.
6. Cohn, S. B. — Modern advances in microwave techniques, *Polytech. Inst. Brooklyn Symposium Proc.*, 1954, **4**, 465.
7. Brown John and Jackson Williams — *Proc. I.E.E..*, 1955, **102**, Part B, 37.
8. Collin, R. E. — *I.R.E. Trans.*, 1958, MTT-6, 206.
9. Mac Farlane, C. G. — *Jour. I.E.E.*, 1946, **93**, Part III A, 703.
10. John Brown — *Proc. I.E.E.*, 1950, **97**, Part III, 45.
11. Chatterjee, S. K., and Dhanalakshmi, C. — *Jour. I.T.E.*, 1960, **6**, 149.
12. Chatterjee, S. K. and Dhanalakshmi, C. — *Zeitschorift fur physik*, 1960, **158**, 196.
13. Chatterjee, S. K. and Dhanalakshmi, C. — *Jour. I.T.E.*, 1960, **6**, 83.
14. Kolettis, N. — *Case Inst. Technol. Sci. Rept.*, **2**, 1959, Contract AF-19 (604) 3887.
15. Carlson, J. F. and Heins, A. E. — *Quart. Appl. Math.*, 1947, **4**, 313; 1947, **5**, 82.

16. Chatterjee, S. K. and Vasudeva Rao, B. *Jour. Ind. Inst. Sci.*, 1955, **37**, Part B, 304.

17. Chatterjee, S. K. and Shankara, K. N. *Proc. Ind. Natl. Acad., Sci.*, 1973, **39**, Part A, 1.

18. Shankara, K. N. and Chatterjee, S. K. *Jour. Ind. Inst. Sci.*, 1972, **54**, 118.

19. Shankara, K. N. and Chatterjee, S. K. *Jour. Ind. Inst. Sci.*, 1972, **54**, 146.

20. Shankara, K. N. and Chatterjee, S. K. *Jour. Ind. Inst. Sci.*, 1972, **54**, 211.

21. Chatterjee, S. K. and Girija, H. M. *Jour. Ind. Inst. Sci.*, 1969, **51**, 38.

22. Girija, H. M. and Chatterjee, S. K. *Ind. Jour. Pure Appl. Phys.*, 1972, **10**, 794.

23. Girija, H. M. and Chatterjee, S. K. *Jour. Ind. Inst. Sci.*, 1971, **54**, 1.

24. Girija, H. M. and Chatterjee, S. K. *Jour. Ind. Inst. Sci.*, 1972, **54**, 1.

25. Glory John and Chatterjee, S. K. *Proc. I.E.E.—I.E.R.E. (India)*, 1972, **10**, 68.

26. Glory John, Chatterjee, R. and Chatterjee, S. K. *Jour. Ind. Inst. Sci.*, 1976, **58**, 123.

27. Glory John, Chatterjee, R. and Chatterjee, S. K. *Proc. I.E.E.—I.E.R.E. (India)*, 1974, **12**, 210.

28. Glory John, and Chatterjee, S. K. *Jour. Ind. Inst. Sci.*, 1974, **56**, 88.

29. Glory John, Chatterjee, R. and Chatterjee, S. K. *Jour. Ind. Inst. Sci.*, 1975, **57**, 113.

30. Glory John, Chatterjee, R. and Chatterjee, S. K. *Jour. Ind. Inst. Sci.*, 1977, **59** (1), 51.