Spectrum of a Schrödinger Operator with a class of damped oscillating potentials

ANNAPURNA SARKAR

Bethune College, Calcutta

Received on July 12, 1977

Abstract

The nature of the spectrum of the operator

$$M \equiv -\frac{d^2}{dx^2} + V_1(x), \quad V_1(x) \equiv \frac{A \sin px}{1+x},$$

A, p real positive constants, as well as that of the perturbed operator

$$L \equiv -\frac{d^2}{dx^2} + V(x), \quad V(x) = V_1(x) + l(x),$$

$l(x)$ continuous and $l(x) \in L(0, \infty)$ under a Sturmian boundary condition is investigated. It is found that L has the same spectrum as M; M having a continuous spectrum over the whole of the positive half of the real axis with points of the point spectium embedded in it; the negative half of the real axis however does not belong to the spectrum.

Key words: Spectrum, damped oscillating potential, deficiency index, limit point, Prüfer transformation, iteration, Wronskian, perturbed/unperturbed operator.

1. Introduction

Consider the differential equation

$$\frac{d^2 y}{dx^2} + \{\lambda - q(x)\} y = 0, \tag{1.1}$$

λ complex: $\lambda = \mu + i\nu$, $\nu \neq 0$ and $q(x)$ real valued and continuous in x in $0 \leqslant x < \infty$.

The homogeneous boundary condition at $x = 0$ is

$$y(0) \cos \alpha + y'(0) \sin \alpha = 0 \tag{1.2}$$

where α is a real parameter.

If $\phi(x, \lambda)$ and $\theta(x, \lambda)$ be two solutions of (1.1) which at $x=0$ take real initial conditions

$$\left.\begin{aligned} \phi(0, \lambda) &= -\sin \alpha, & \phi'(0, \lambda) &= \cos \alpha \\ \theta(0, \lambda) &= -\cos \alpha, & \theta'(0, \lambda) &= -\sin \alpha \end{aligned}\right\} \tag{1.3}$$

then there exists an analytic function $m(\lambda)$, regular for $v = im\,\lambda \gtrless 0$ such that for $0 \leqslant x < \infty$,

$$\psi(x, \lambda) = \theta(x, \lambda) + m(\lambda)\,\phi(x, \lambda) \tag{1.4}$$

is a solution of (1.1), satisfies (1.2) and belongs to $L^2[0, \infty)$; $m(\lambda)$ depends on the choice of the real parameter α. (Titchmarsh,[7] p. 25 or Coddington and Levinson,[2] p. 228).

The spectrum as usual is the set of values of λ which contributes to the expansion formula and the spectrum of the differential operator $T: -d^2/dx^2 + q(x)$, represented by $\sigma(T)$, may be regarded as the complement of the set $\rho(T)$ of all points λ for which $m(\lambda)$ is regular. The spectrum $\sigma(T)$ is obviously given by

$$\sigma(T) = P\sigma(T) \cup C\sigma(T) \cup PC\sigma(T) \tag{1.5}$$

where $P\sigma(T)$ is the point spectrum, $C\sigma(T)$ the continuous spectrum and $PC\sigma(T)$ the point continuous spectrum of T, at least one of the set being non-empty. The point λ is an eigenvalue if and only if $\lambda \in P\sigma(T) \cup PC\sigma(T)$.

The spectral properties of the operator T are characterised by the properties of $m(\lambda)$ on the real λ-axis and it is easy to deduce from § 4 and the theorem of § 5 in Chaudhuri and Everitt[3], that

(i) $\mu \notin \sigma(T)$, if and only if $m(\lambda)$ is regular at μ, *i.e.*, $\lim\limits_{\nu\to 0}[im\,m(\lambda)] = 0$.

(ii) $\mu \in P\sigma(T)$, if and only if $m(\lambda)$ has a pole at μ, but $m(\lambda)$ is regular in any neighbourhood excluding μ: or if and only if $m(\lambda)$ tends to infinity as λ tends to μ and $im\{m(\lambda)\}$ tends to zero as λ tends to any point in a neighbourhood excluding μ.

(iii) $\mu \in C\sigma(T)$, if and only if $m(\lambda)$ is not regular in any neighbourhood including μ and $\lim\limits_{\nu\to 0}\{\nu\, m(\mu + i\nu)\} = 0$; *i.e.*, $\lim\limits_{\nu\to 0}[im\,m(\lambda)]$ is a continuous and non-vanishing function bounded for all $\mu \in (\mu_1, \mu_2)$ on the real λ-axis.

(iv) $\mu \in PC\sigma(T)$, if and only if $\lim\limits_{\nu\to 0} i\nu\,\{m(\mu + i\nu)\} = s \neq 0$, s real and $\{[m(\lambda) - s]/(\lambda - \mu)\}$ is not regular at μ; *i.e.*, $m(\lambda)$ tends to infinity as λ tends to μ and $\lim\limits_{\nu\to 0}[im\,m(\lambda)]$ is a continuous non-vanishing function of μ in any neighbourhood excluding μ.

(v) The spectrum is a pure point spectrum or a discrete spectrum if and only if $m(\lambda)$ is meromorphic.

Our object in the present paper is to discuss the nature of the spectrum of (1.1) when $q(x)$ is of the form

$$q(x) = \frac{A \sin px}{x + 1},$$

where A and p are real positive constants.

The differential equation (1.1) now takes the form

$$M[y] = \lambda y : M \equiv -\frac{d^2}{dx^2} + V_1(x) \tag{1.6}$$

with

$$V_1(x) = \frac{A \sin px}{x+1},$$

the damped oscillating potential: $V_1^{(n)}(x) \notin L[0, \infty)$, but

$$V_1^{(n)}(x) \in L^2[0, \infty), \quad n = 0, 1, 2, \ldots .$$

Since $V_1 \in L^2[0, \infty)$, it follows that the deficiency index of the operator M is (1, 1) and we are in the limit point case at infinity (*vide* Naimark,[4] p. 305 or Putnam[5]).

Since

$$\frac{A \sin px}{1+x} > \frac{-A}{1+x},$$

it is possible to choose a positive constant K such that

$$\frac{A \sin px}{1+x} > -K \text{ for all } x,\ 0 \leqslant x < \infty.$$

Hence also by Cor. 1, (p. 231), Coddington and Levinson,[2] M is in the limit point case at infinity.

The paper ends with a discussion on the nature of the spectrum of the differential operator

$$L : L[y] = \lambda y, \qquad L \equiv -\frac{d^2}{dx^2} + V(x) \tag{1.7}$$

the perturbed form of (1.6), where $V(x) = V_1(x) + l(x)$, $l(x)$ being a continuous function of x which belongs to $L\ [0, \infty)$.

2. Asymptotic values of a pair of solutions of (1.6)

Let $y_1(x) = y_1(x, \lambda)$ be a solution of (1.6). Applying the Prüfer transformation (Rosenfeld[6] p. 397)

$$y_1(x) = \rho(x, \lambda) \sin[\Theta(x, \lambda)], \quad \lambda^{-\frac{1}{2}} y'_1(x) = \rho(x, \lambda) \cos[\Theta(x, \lambda)],$$

it follows from (1.6) on slight reduction and necessary subsequent integration that

$$\rho(x, \lambda) = C_1 \exp\{\tfrac{1}{2} A\lambda^{-\frac{1}{2}} \int_0^x (t+1)^{-1} \sin pt \sin[2\,\Theta(t, \lambda)]\, dt \tag{2.1}$$

$$C_1 = \rho(0, \lambda),$$

and

$$\Theta(x, \lambda) = \Theta_0 + \lambda^{\frac{1}{2}} x - A\lambda^{-\frac{1}{2}} \int_0^x (t+1)^{-1} \sin pt \sin^2[\Theta(t, \lambda)]\, dt \tag{2.2}$$

$$\Theta_0 = \Theta(0, \lambda).$$

Since $Si(z) = \int_0^z t^{-1} \sin t\, dt$, $Ci(z) = \gamma + \log z + \int_0^z t^{-1} (\cos t - 1)\, dt$, γ, Euler's constant (Abramowitz,[1] p. 231), where

$Si(z) \sim z^{-1} + K_1$, $Ci(z) \sim z^{-1} + K_2$, as $|z|$ tends to infinity, K_j, $j = 1, 2$, being small constants (Abramowitz[1] pp. 232–33), it follows by iteration from (2.2), substitution of $\omega = p(t+1)$ and subsequent reduction that, as x tends to infinity,

$$\int_0^x (t+1)^{-1} \sin pt\, dt = o\{(x+1)^{-1}\} + o(1) \tag{2.3}$$

and

$$\int_0^x (t+1)^{-1} \begin{matrix}\sin \\ \cos\end{matrix} \{(p \pm 2\lambda^{\frac{1}{2}})\, t\}\, dt = o[(x+1)^{-1}] + o(1). \tag{2.4}$$

Hence making use of (2.4) in the iterated result obtained from (2.2), we obtain

$$\Theta(x, \lambda) \sim \lambda^{\frac{1}{2}} x + o(1) \tag{2.5}$$

leading to

$$\begin{matrix}\sin \\ \cos\end{matrix} [\Theta(x, \lambda)] \sim \begin{matrix}\sin \\ \cos\end{matrix} \lambda^{\frac{1}{2}} x + o(1), \tag{2.6}$$

as x tends to infinity.

Thus for sufficiently large x and non-zero λ,

$$\rho(x, \lambda) = C_1 \exp[f_1(x, \lambda)] \tag{2.7}$$

and

$$y_1(x, \lambda) = C_1 \exp[f_1(x, \lambda)]\, [\sin \lambda^{\frac{1}{2}} x + o(1)] \tag{2.8}$$

$$= C_1 \exp[f_1(x, \lambda)] \sin \Theta(x, \lambda), \tag{2.9}$$

by (2.6), where $f_1(x, \lambda) = \frac{1}{2} A\lambda^{-\frac{1}{2}} \int_0^x (t+1)^{-1} \sin pt \sin 2\lambda^{\frac{1}{2}}\, dt.$

Similarly, if $y_2(x, \lambda)$ be the other solution of (1.6), where

$$y_2(x, \lambda) = \lambda^{-\frac{1}{2}} [\rho(x, \lambda)]^{-1} \cos [\Theta(x, \lambda)],$$

$$y'_2(x, \lambda) = -[\rho(x, \lambda)]^{-1} \sin [\Theta(x, \lambda)],$$

we have

$$y_2(x, \lambda) = C_2 \lambda^{-\frac{1}{2}} \exp[-f_1(x, \lambda)] [\cos \lambda^{\frac{1}{2}} x + o(1)] \tag{2.10}$$

$$= C_2 \lambda^{-\frac{1}{2}} \exp[-f_1(x, \lambda)] [\cos \Theta(x, \lambda)], \tag{2.11}$$

by (2.6), for sufficiently large x and $\lambda \neq 0$.

Since the Wronskian $W(y_1, y_2) = -1$, it follows that $y_1(x, \lambda)$, $y_2(x, \lambda)$ form a fundamental set of solutions of (1.6). Hence choosing

$$y_1(0, \lambda) = 0, \qquad y_2(0, \lambda) = 1, \qquad y'_1(0, \lambda) = 1, \qquad y'_2(0, \lambda) = 0,$$

it follows that the constants C_1, C_2 in (2.9) and (2.11) are given respectively by $\lambda^{-\frac{1}{2}}$ and $\lambda^{\frac{1}{2}}$.

3. Nature of the spectrum of (1.6)

We now establish the following theorem giving the nature of the spectrum of (1.6).

Theorem: The self-adjoint differential operator M defined in (1.6) under the boundary condition (1.2) has a spectrum $\sigma(M)$ which extends over the positive half of the real λ-axis and consists of the two subsets $C\sigma(M)$ and $PC\sigma(M)$, where $PC\sigma(M)$ consists of the point $\mu = p^2/4$, $0 < \mu < \infty$ and $C\sigma(M)$, the whole of the right half line $(0, \infty)$ of the real λ-axis with the exclusion of the point $\mu = p^2/\mu$.

As already pointed out in the introduction, we are in the limit point case at infinity and therefore [by Coddington and Levinson,[2] formula (2.13)], we have

$$m(\lambda) = -\lim_{b\to\infty} y_1(b, \lambda)/y_2(b, \lambda) \tag{3.1}$$

$$= -\lim_{b\to\infty} \lambda^{-\frac{1}{2}} \frac{[\exp(f_1(b, \lambda))]\ \sin[\Theta(x, \lambda)]}{[\exp(-f_1(b, \lambda))]\ \cos[\Theta(x, \lambda)]}$$

$$= \lim_{b\to\infty} \left[-i\lambda^{\frac{1}{2}} \{\exp(2f_1(b, \lambda))\} \frac{\exp(-i\lambda^{\frac{1}{2}} b) - \exp(i\lambda^{\frac{1}{2}} b)}{\exp(-i\lambda^{\frac{1}{2}} b) + \exp(i\lambda^{\frac{1}{2}} b)} \right], \text{ by (2.6)} \tag{3.2}$$

Putting $\lambda = \sigma + i\tau$ it follows that as b tends to infinity, the second factor on the right side of (3.2) tends to a constant k, where $k = +1$ or -1 according as $\tau > 0$ or < 0. Hence finally,

$$m(\lambda) = k \lim_{b\to\infty} [-i\lambda^{\frac{1}{2}} \exp\{2 f_1(b, \lambda)\}]. \tag{3.3}$$

Two cases are to be distinguished.

Case I. When λ tends to a positive real value, say $\lambda \to \sigma^2$, $\sigma \neq 0$. Then from (3.3),

$$\lim_{\lambda \to \sigma^2} im \{m(\lambda)\} = -k \sigma^{-1} \exp \left\{ A \sigma^{-1} \int_0^\infty \frac{\sin pt}{t+1} \sin 2\sigma t \, dt \right\} \tag{3.4}$$

The integral on the right of (3.4) is convergent when $p \neq 2\sigma$ and divergent when $p = 2\sigma$. Therefore, when $\sigma^2 \neq p^2/4$, the left hand side of (3.4) does not tend to zero or infinity but is a continuous function, bounded for all non-zero real values of λ. But the left hand side of (3.4) tends to infinity when $\sigma^2 = p^2/4$.

It follows therefore from characterisation of the spectrum in § 1 that there exists a point of the point spectrum on the right half line of the real λ-axis at $\lambda = p^2/4$ which is embedded in the continuous spectrum extending throughout $(0, \infty)$, with the exclusion of the point $\lambda = p^2/4$.

Case II. When λ tends to a negative real value, say $\lambda \to -\tau^2$, $\tau > 0$. It follows from (3.3) that

$$\lim_{\lambda \to -\tau^2} m(\lambda) = -k \cdot \frac{1}{\tau} \exp \left[-\frac{A}{\tau} \int_0^\infty \frac{\sin pt}{1+t} \sinh 2\tau t \, dt \right]. \tag{3.5}$$

The integral on the right being obviously divergent, it follows that $m(\lambda)$ tends to zero as $\lambda \to -\tau^2$. Thus $m(\lambda)$ is regular for $\lambda \in (-\infty, 0)$. The left hand side of the real λ-axis therefore does not belong to the spectrum. When $\tau < 0$, $\lambda^{\frac{1}{2}}$ is replaced by $-i\tau$ and the same result follows. The theorem is therefore completely proved.

4. Spectrum of the perturbed operator L

If $Y_j(x, \lambda) \equiv Y_j(x)$, $j = 1, 2$, be the solutions of the perturbed equation (1.7), $Y_j(x)$ are derived in terms of the solutions $y_j(x)$, $j = 1, 2$, of the unperturbed equation (1.6) by

$$Y_1(x) = y_1(x) + \int_x^\infty G(x, t, \lambda) \, l(t) \, Y_1(t) \, dt, \quad 0 \leqslant x \leqslant t < \infty \tag{4.1}$$

$$Y_2(x) = y_2(x) + \int_b^x G(x, t, \lambda) \, l(t) \, Y_2(t) \, dt, \quad 0 < b \leqslant t \leqslant x < \infty \tag{4.2}$$

where

$$G(x, y, \lambda) = \frac{1}{W(\lambda)} [y_1(x) \, y_2(t) - y_1(t) \, y_2(x)] \tag{4.3}$$

$W(\lambda)$ being the Wronskian of $y_1(x)$ and $y_2(x)$.

It is noted that the boundary condition at $x = 0$, given by (1.2), remains the same for both the perturbed and the unperturbed cases.

On some simple calculations it follows that

$$Y_j(x, \lambda) = y_j(x, \lambda)\,[1 + o(1)], \qquad j = 1, 2,$$

as x tends to infinity, uniformly for all λ belonging to the domain of definition of $y_j(x, \lambda)$, $j = 1, 2$. Thus the operator L has the same spectrum as the unperturbed operator M.

We therefore conclude that the spectrum of the operator

$$L \equiv -\frac{d^2}{dx^2} + \frac{A \sin px}{1 + x} + l(x), \qquad 0 \leqslant x < \infty,$$

$A, p > 0$ and $l(x) \in L\,[0, \infty)$ and the boundary condition (1.2) shows penetration of a point of the point spectrum into a continuous spectrum extending over the whole of the right half of the real λ-axis, the left half remaining empty.

The author expresses her sincere thanks to Dr. N. K. Chakravarty for his keen interest and helpful criticism during the preparation of the paper.

References

1. ABRAMOWITZ, M. AND STEGUN, I. A. — *Handbook of Mathematical Functions*, 1965, Dover, NY.
2. GODDINGTON, E. A. AND LEVINSON, N. — *Theory of Ordinary Differential Equations*, 1955, 228, McGraw-Hill, NY.
3. CHAUDHURI, J. AND EVERITT, W. N. — On the spectrum of ordinary second order differential operators. *Proc. Roy. Soc. Edinburg* (A) 1968, **68** (II) (7), 95–119.
4. NAIMARK, M. A. — Linear differential operator, 1963, Akademie-Verlag, Berlin.
5. PUTNAM, C. R. — Integrable potentials and half line spectra. *Proc. Amer. Math. Soc.* 1955, **6**, 243–246.
6. ROSENFELD, N. S. — The eigenvalues of singular differential operators. *Com. Pure and Appl. Maths*, 1960, 13, 395–405.
7. TITCHMARSH, E. C. — Eigenfunction expansions associated with second order differential equations, 1962, Part I, 25 Oxford University Press (2nd Edn.).
8. TITCHMARSH, E. C. — Eigenfunction expansions associated with second order differential equations, 1958, Part II, Oxford University Press.