Non-symmetrical bending of circular plates having small initial curvature under the combined action of lateral loads and forces in the middle plane of the plates

DEBABRATA SINHA* AND SUKUMAR BASULI
Department of Mathematics, Tripura Engineering College, Tripura

Received on January 31, 1976; Revised on August 20, 1977

Abstract

The note aims at solving the problem of bending of a circular plate in a non-symmetrical case where the plate had an initially given small deflection, and the plate bending subjected to the combined action of lateral loads and forces in the middle plane of the plate.

Key words: Circular plates, lateral loads and forces, non-symmetrical bending

1. Introduction

Several problems on the bending of circular plates (symmetrical as well as non-symmetrical cases) under the combined action of lateral loads and forces in the middle plane of the plate have been discussed by Timoshenko and Woinowsky-kriegar.[1] However we believe that the problems of non-symmetrical bending of circular plates having small initial curvature and under the action of lateral loads and forces have not been studied.

The object of this paper is to solve the problem of non-symmetrical bending of circular plates having a small initial curvature under the action of lateral loads and forces.

2. Problem

The general differential equation in polar co-ordinates for the bending of plates under the action of lateral loads and forces in the middle plane of the plate is given by

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)$$
$$=\frac{q}{D}-\frac{N}{D}\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) \tag{1.0}$$

* Udaipur Kirit Bikram Institution, P.O. Radhakishorepur, Tripura State, India.

where q = Lateral load, N = Uniform compressive force acting in the middle plane of the plate and D = Flexural rigidity of the plate.

If w_0 denotes the initial small deflection and w_1 is the additional deflection as a result of the applied load and force, we can write for a small total deflection.

$$w = w_0 + w_1. \tag{1.1}$$

Now since the l.h.s. of equation (1.0) was obtained from the expressions for the bending moments in the plate which were dependent only on the change in curvature, it follows that W on the l.h.s. of (1.0) must be replaced by W_1 only.[2] On the other hand, the effect of applied lateral load and force being dependent on the total curvature, W on the r.h.s. of equation (1.0) must be replaced by $W_0 + W_1$ in this problem. Thus for the initially curved circular plate equation (1.0) becomes:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 W_1}{\partial r^2} + \frac{1}{r}\frac{\partial W_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W_1}{\partial \theta^2}\right)$$
$$= \frac{q}{D} - \frac{N}{D}\left[\frac{\partial^2}{\partial r^2}(W_0 + W_1) + \frac{1}{r}\frac{\partial}{\partial r}(W_0 + W_1) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(W_0 + W_1)\right] \tag{1.2}$$

3. Solution of the problem

Let us now try to solve the equation (1.2) in the case where the initial curvature is defined by the relation:

$$W_0 = \sum_{m=0}^{\infty} A_{0m}(a^2 - r^2)^2 \cos m\theta + \sum_{m=1}^{\infty} B_{0m}(a^2 - r^2)^2 \sin m\theta. \tag{1.3}$$

Let the load be of the form

$$q = \sum_{m=0}^{\infty} f_m(r) \cos m\theta + \sum_{m=1}^{\infty} F_m(r) \sin m\theta \tag{1.4}$$

where $f_m(r)$ and $F_m(r)$ are functions of r only.

Substituting (1.3) and (1.4) in (1.2) we obtain

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 W_1}{\partial r^2} + \frac{1}{r}\frac{\partial W_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W_1}{\partial \theta^2}\right)$$
$$+ \frac{N}{D}\left(\frac{\partial^2 W_1}{\partial r^2} + \frac{1}{r}\frac{\partial W_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W_1}{\partial \theta^2}\right)$$
$$= \frac{1}{D}\left[\sum_{m=0}^{\infty} f_m(r) \cos m\theta + \sum_{m=1}^{\infty} F_m(r) \sin m\theta\right]$$

$$+\frac{N}{D}\left[\sum_{m=0}^{\infty} A_{om}\left\{(16-m^2)r^2+(2m^2-8)a^2-\frac{m^2a^4}{r^2}\right\}\cos m\theta\right.$$

$$\left.+\sum_{m=0}^{\infty} B_{om}\left\{(16-m^2)r^2+(2m^2-8)a^2-\frac{m^2a^4}{r^2}\right\}\sin m\theta\right]. \tag{1.5}$$

If we now assume for W_1 the form

$$W_1 = R_0 + \sum_{m=1}^{\infty} R_m\cos m\theta + \sum_{m=1}^{\infty} R_m'\sin m\theta \tag{1.6}$$

where R_0, R_m, R_m' are functions of r only, we get by (1.5) and (1.6) the following differential equations for R_m, R_m' and R_0,

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}\right)\left(\frac{d^2R_m}{dr^2}+\frac{1}{r}\frac{dR_m}{dr}-\frac{m^2R_m}{r^2}\right)$$

$$+\frac{N}{D}\left(\frac{d^2R_m}{dr^2}+\frac{1}{r}\frac{dR_m}{dr}-\frac{m^2R_m}{r^2}\right)$$

$$=\frac{1}{D}f_m(r)+\frac{N}{D}A_{om}\left\{(16-m^2)r^2+(2m^2-8)a^2-\frac{m^2a^4}{r^2}\right\} \tag{1.7}$$

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}\right)\left(\frac{d^2R_m'}{dr^2}+\frac{1}{r}\frac{dR_m'}{dr}-\frac{m^2R_m'}{r^2}\right)$$

$$+\frac{N}{D}\left(\frac{d^2R_m'}{dr^2}+\frac{1}{r}\frac{dR_m'}{dr}-\frac{m^2R_m'}{r^2}\right)$$

$$=\frac{1}{D}F_m(r)+\frac{N}{D}B_{om}\left\{(16-m^2)r^2+(2m^2-8)a^2-\frac{m^2a^4}{r^2}\right\} \tag{1.8}$$

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2R_0}{dr^2}+\frac{1}{r}\frac{dR_0}{dr}\right)+\frac{N}{D}\left(\frac{d^2R_0}{dr^2}+\frac{1}{r}\frac{dR_0}{dr}\right)$$

$$=\frac{f_0(r)}{D}-\frac{8NA_{oo}}{D}(a^2-2r^2). \tag{1.9}$$

To find the solution of (1.7) let us put

$$\frac{d^2R_m}{dr^2}+\frac{1}{r}\frac{dR_m}{dr}-\frac{m^2R_m}{r^2}+\frac{N}{D}R_m=\chi_m(r)$$

Then we get from (1.7), the following equation:

$$r^2\frac{d^2\chi_m}{dr^2}+r\frac{d\chi_m}{dr}-m^2\chi_m=\frac{r^2}{D}f_m(r)+\frac{NA_{0m}}{D}\{(16-m^2)r^4+(2m^2-8)r^2a^2-m^2a^4\}. \tag{2.0}$$

The solution for χ_m is:

$$\chi_m(r) = C_m r^m + D_m r^{-m} + \psi_m(r) + \frac{NA_{0m}}{D}(r^4 - 2a^2 r^2 + a^4)$$

where $\psi_m(r) =$ Particular integral due to the term $\frac{r^2 f_m(r)}{D}$ in (2.0). Replacing χ_m by this expression we get:

$$\frac{d^2 R_m}{dr^2} + \frac{1}{r}\frac{dR_m}{dr} + \left(\frac{N}{D} - \frac{m^2}{r^2}\right) R_m$$

$$= C_m r^m + D_m r^{-m} + \psi_m(r) + \frac{N}{D} A_{0m}(r^4 - 2a^2 r^2 + a^4). \tag{2.1}$$

Now putting

$$\frac{N}{D} = \frac{1}{l^2}, \quad r = lz, \quad \frac{d}{dr} = \frac{1}{l}\frac{d}{dz},$$

equation (2.1) takes the form:

$$Z^2 \frac{d^2 R_m}{dz^2} + z\frac{dR_m}{dz} + (z^2 - m^2) R_m$$

$$= C_m (lZ)^{m+2} + D_m (lz)^{-m+2} + l^2 z^2 \psi_m(lz)$$

$$+ A_{0m}(l^4 z^6 - 2a^2 l^2 z^4 + a^4 z^2) \tag{2.2}$$

Now let

$$R_m = A_m J_m(z) + B_m y_m(z) \tag{2.3}$$

be the general solution of (2.2). Then by the method of variation of parameters[3] we have

$$A_m' J_m(z) + B_m' y_m(z) = 0$$

$$A_m' J'_m(z) + B_m' y'_m(z)$$

$$= C_m (lz)^{m+2} + D_m (lz)^{-m+2} + l^2 z^2 \psi_m(lz) + A_{0m}(l^4 z^6 - 2a^2 l^2 z^4 + a^4 z^2)$$

Solving for A'_m and B'_m, we have

$$A_m' = \frac{[C_m (lz)^{m+2} + D_m (lz)^{-m+2} + l^2 z^2 \psi_m(lz) + A_{0m}(l^4 z^6 - 2a^2 l^2 z^4 + a^4 z^2)] y_m(z)}{J_m'(z) y_m(z) - J_m(z) y_m'(z)} \tag{2.4}$$

$$B_m' = \frac{[C_m (lz)^{m+2} + D_m (lz)^{-m+2} + l^2 z^2 \psi_m(lz) + A_{0m}(l^4 z^6 - 2a^2 l^2 z^4 + a^4 z^2)] J_m(z)}{J_m(z) y_m'(z) - J_m'(z) y_m(z)} \tag{2.5}$$

Integration of (2.4) and (2.5) gives:

$$A_m = - C_m \int l^{m+2} z^{m+3} y_m(z)\, dz - D_m l^{-m+2} \int z^{-m+3} y_m(z)\, dz - l^2 \int z^3 \psi_m(lz)\, y_m(z)\, dz + A_{0m} [- l^4 \int z^7 y_m(z)\, dz + 2a^2 l^2 \int z^5 y_m(z)\, dz - a^4 \int z^3 y_m(z)\, dz] + E_m \quad (2.6)$$

$$B_m = C_m l^{m+2} \int z^{m+3} J_m(z)\, dz + D_m l^{-m+2} \int z^{-m+3} J_m(z)\, dz + l^2 \int z^3 \psi_m(lz)\, J_m(z)\, dz + A_{0m} [l^4 \int z^7 J_m(z)\, dz - 2a^2 l^2 \int z^5 J_m(z)\, dz + a^4 \int z^3 J_m(z)\, dz] + F_m \quad (2.7)$$

So the general solution for R_m is of the form:

$$R_m = C_m l^{m+2} \phi_{m+3,\, m}(z) + D_m l^{-m+2} \phi_{-m+3, m}(z) + l^2 \theta_m(z) + A_{0m} [l^4 \phi_{7,\, m}(z) - 2a^2 l^2 \phi_{5,\, m}(z) + a^4 \phi_{3,\, m}(z)] + E_m J_m(z) + F_m y_m(z) \quad (2.8)$$

where

$$\phi_{n,\, m}(z) = y_m(z) \int z^n J_m(z)\, dz - J_m(z) \int z^n y_m(z)\, dz$$

$$= y_m(z) \left[z^{n+1} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{m+2k}}{\lfloor k . \sqrt{k+m+1} \;.\; (n+m+2k+1)} \right]$$

$$+ \frac{\pi}{2} J_m(z) \left[\frac{z^{n+1}}{2} \sum_{k=0}^{m-1} \frac{\lfloor m-k-1 \,.\, \left(\frac{z}{2}\right)^{-m+2k}}{\lfloor k \,.\, (-m+n\, 2k+1)} \right.$$

$$+ z^{n+1} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{m+2k}}{\lfloor k \,.\, \lfloor m+k \,.\, (m+n+2k+1)^2}$$

$$- z^{n+1} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{m+2k}}{\lfloor k \,.\, \lfloor m+k \,.\, (m+n+2k+1)}$$

$$\left. \times \left\{ \log \frac{z}{2} - \frac{1}{2} \psi(k+1) - \frac{1}{2} \psi(m+k+1) \right\} \right] \quad (2.9)$$

in which

$$\psi(m+1) = \sum_{k=1}^{\infty} k^{-1} + \psi(1)$$

$$\psi(1) = -0{\cdot}57722 \tag{3.0}$$

for $n = m + 3,\ -m + 3,\ 3,\ 5$ and 7.

Also

$$\theta_m(z) = y_m(z) \int z^3 \psi_m(lz) J_m(z)\, dz - J_m(z) \int z^3 \psi_m(lz) y_m(z)\, dz. \tag{3.1}$$

similar expressions can be obtained for R_m' by solving the equation (1.8).

By solving the equation (1.9), the expression for R_0 can be obtained in the form (Sinha).[4]

$$R_0 = \sum_{n=1}^{\infty} \left[\frac{2 \int_0^a f_0(r)\, r\, J_0(\alpha_n r)}{a^2 \alpha_n^2 (\alpha_n^2 - k^2) J_0^2(\alpha_n a)} + \frac{64 k^2 A_{00} J_2(\alpha_n a)}{\alpha_n^2 (\alpha_n^2 - k^2) J_0^2(\alpha_n a)} \right] \times [J_0(\alpha_n r) - J_0(\alpha_n a)] \tag{3.2}$$

in which

$$k^2 = \frac{N}{D}$$

and α_n is the n-th root of the equation

$$J_1(\alpha a) = 0.$$

Substituting these expressions for R_0, R_m and R_m' in (1.6) we obtain the additional deflection W_1. With this value of W_1 and taking W_0 from (1.3) we can get the total deflection W at any point (r, θ) of the circular plate.

The constants of integration C_m, D_m, E_m and F_m in each particular case must be determined so as to satisfy the given boundary conditions.

A particular case : Let us consider the case of a circular plate with free boundary. Such a condition is found in the case of a circular foundation slab, supporting a chimney. We assume here that the moment M produced due to the wind pressure produces reactions in the slab following a linear law and thus we obtain similar type of loading as discussed in the foregoing.

The boundary conditions at the outer boundary of the plate, which is free, are

$$[M_r]_{r=a} = 0 \tag{3.3}$$

$$[V_r]_{r=a} = \left(Q_r - \frac{1}{r}\frac{\partial}{\partial r} M_{rt}\right)_{r=a} = 0. \tag{3.4}$$

We also assume here that the inner portion of the plate of radius b is considered rigid and the edge of the plate along the circle of radius b is clamped. For this the following boundary conditions must be satisfied.

$$\left[\frac{\partial W}{\partial r}\right]_{r=b} = \left[\frac{W}{r}\right]_{r=b} \tag{3.5}$$

Let the load be of the form

$$f_1(r) = q = q_{10} \cdot r \tag{3.6}$$

As the solution of the equation (1.2) we take only the term of the series (1.6) that contains the function R_1. We assume

$$W = \Big[A_{01}(a^2 - r^2)^2 + c_1 l^3 \phi_{4,1}\left(\frac{r}{l}\right) + D_1 l \phi_{2,1}\left(\frac{r}{l}\right) + \frac{q_{10} l^5}{8D}\phi_{6,1}\left(\frac{r}{l}\right)$$

$$+ A_{01}\left\{l^4 \phi_{7,1}\left(\frac{r}{l}\right) - 2a^2 l^2 \phi_{5,1}\left(\frac{r}{l}\right) + a^4 \phi_{3,1}\left(\frac{r}{l}\right)\right\}$$

$$+ E_1 J_1\left(\frac{r}{l}\right) + F_1 y_1\left(\frac{r}{l}\right)\Big] \cos\theta \tag{3.7}$$

where

$$l^2\, 0_m(z) = \frac{q_{10}\, l^5}{8\, D}\phi_{6,1}\left(\frac{r}{l}\right)$$

and

$$\phi_{n,1}\left(\frac{r}{l}\right) = \frac{1}{l^{n+1}} y_1\left(\frac{r}{l}\right) \int r^n J_1\left(\frac{r}{l}\right) dr$$

$$- \frac{1}{l^{n+1}} J_1\left(\frac{r}{l}\right) \int r^n y_1\left(\frac{r}{l}\right) dr \tag{3.8}$$

for $n = 2, 3, 4, 5, 6$ and 7 only.

Let us now take $a = 10$ units, $b = 1$ unit, $l = 10$ units. Then for $m = 1$, we take $F_1 = 0$ so as to eliminate an infinitely large bending moment at the central portion of the plate.

Solving the equations obtained by using the boundary conditions, we obtain the values of the Constants C_1, D_1 and E_1:

$$\left.\begin{aligned} C_1 &= 17\cdot537\frac{q_{10}}{D} + 3541\cdot2313\,A_{01} \\ D_1 &= -3113\cdot207\frac{q_{10}}{D} + 277645\cdot6\,A_{01} \\ E_1 &= 0. \end{aligned}\right\} \tag{3.9}$$

Thus in this particular example, we have from (3.5), the following expression for the deflection :

$$\begin{aligned} W = \Big[& A_{01}\,(10^2 - r^2)^2 + \left(17\cdot537\,\frac{q_{10}}{D} + 3541\cdot2313\,A_{01}\right) 10^3\,\phi_{4,1}\left(\frac{r}{10}\right) \\ & + \left(277645\cdot6\,A_{01} - 3113\cdot207\,\frac{q_{10}}{D}\right)\phi_{2,1}\left(\frac{r}{10}\right) + 10^5\,.\,\frac{q_{10}}{D}\,\phi_{6,1}\left(\frac{r}{10}\right) \\ & + A_{01}\,10^4\left\{\phi_{7,1}\left(\frac{r}{10}\right) - 2\phi_{5,1}\left(\frac{r}{10}\right) + \phi_{3,1}\left(\frac{r}{10}\right)\right\}\Big]\cos\theta\,. \end{aligned} \tag{4.0}$$

We can now tabulate the deflection in any direction, say for $\theta = 60°$ and for the different values of the distances from the centre of the plate.

Table I

Deflections at different points

Here we have chosen

$$A_{01} = \frac{q_{10}}{100\ D}\,.$$

Deflection (W)	$r = 2$ Units $\theta = 60°$	$r = 4$ Units $\theta = 60°$	$r = 6$ Units $\theta = 60°$	$r = 8$ Units $\theta = 60°$	$r = 10$ Units $\theta = 60°$
$\frac{WD}{100q_{10}}$	0·4676	1·034	2·375	3·670	5·178

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