

THE INFLUENCE OF END FRICTION ON ELASTIC PARAMETERS OBTAINED IN TRIAXIAL TESTS

M. NAYAK*

(Senior Research Scholar, Department of Civil and Hydr. Engg., Indian Institute of Science, Bangalore 560012, India)

Received on January 28, 1974

ABSTRACT

The effect of end restraint on elastic parameters, particularly Young's modulus and Poisson's ratio, determined from triaxial tests, is investigated. These elastic constants are not the same as those appearing in the Hooke's law. However, for a test specimen of height to diameter ratio more than or equal to 2, the expression for lateral deformation at the exterior of the middle of the test specimen without considering end restraint, is shown to be almost exact. Hence knowing one of the two elastic parameters the other one can be determined accurately using this expression. When both the parameters are to be determined they can be obtained from a single test with the help of the graphs presented herein, provided the radial deformations at the top edge and at the exterior of the middle of the test specimen are measured. For cases where the ends can be assumed to be fully restrained, as in tests using porous stones at both ends of the sample, the true Young's modulus can be obtained directly from the results presented in this paper.

Key words. Elastic parameters, end restraint, Poisson's ratio, triaxial test, Young's modulus.
IGC: D5.

INTRODUCTION

The elastic parameters, namely, Young's modulus and Poisson's ratio, bulk and shear moduli and the coefficient of earth pressure at rest, are usually determined from triaxial tests. The usual assumption made is that during testing the sample deforms uniformly in both the vertical and lateral directions. But actually deformation is not fully uniform as shear stress is introduced due to friction at the ends of the sample [2]. The effect of end friction on shear strength and pore pressure has been studied by several investigators [3, 4, 11]. Here an attempt has been made to study the influence of end friction on the values of elastic parameters obtained from triaxial tests.

* Present Address: Foundation Engineer, Engicon Engineering Consultants, 64, B. B. Ganguly Street, Calcutta 700012.

The problem of fully end-restrained cylinder was first solved by [8] and later on by [10] and [6] [1] gave a solution for a more general case, by introducing a friction factor, ϕ , ranging from 0 to 1 which corresponds to no restraint and full restraint, respectively. More exact solutions have been obtained by [5] and [9] Both of them use radial deformation at top of the sample as one of the boundary conditions and hence use of their solution requires exact knowledge of radial deformation at top surface.

The difference between apparent Young's modulus as obtained from laboratory tests, and the true Young's modulus, has been pointed out by some investigators. [7] reported that a cylinder with height to a diameter ratio equal to 1 and a Poisson's ratio of 1/3 would have a measured modulus approximately 5% larger than true modulus. For a Poisson's ratio of 1/4, [6] noted an error of approximately the same magnitude.

The present analysis is based on Balla's solution. A thorough investigation on the elastic constants, and in particular, Young's modulus obtained from triaxial tests, has been made. This has been accomplished by introducing two non-dimensional parameters, which directly indicate the magnitude of the error associated with the conventional method of finding the elastic parameters. From the graphs presented, it is possible to determine the true values of the elastic parameters, E and μ from a single test, provided lateral deformations at the top edge and the exterior of the middle of the test specimen are known. For the cases where slippage at the ends is negligible, the true values of Young's modulus can be obtained directly by assuming the lateral strain at the top edge to be zero.

THEORY

For a homogeneous isotropic elastic material the stress-strain relationships, in cylindrical co-ordinates, for axi-symmetric condition are

$$\left. \begin{aligned} \epsilon_r &= \frac{1}{E} [\sigma_r - \mu (\sigma_\theta + \sigma_z)] \\ \epsilon_\theta &= \frac{1}{E} [\sigma_\theta - \mu (\sigma_z + \sigma_r)] \\ \epsilon_z &= \frac{1}{E} [\sigma_z - \mu (\sigma_r + \sigma_\theta)] \\ \gamma_{rz} &= \frac{1}{G} \tau_{rz} \end{aligned} \right\} \quad (1)$$

where σ_r , σ_θ and σ_z are normal stresses in radial, tangential and vertical directions, ϵ_r , ϵ_θ and ϵ_z are the corresponding strain components; τ_{rz} is the shear stress along rz plane and γ_{rz} is the corresponding shear strain. E , μ and G are the elastic constants, namely, Young's modulus, Poisson's ratio and shear modulus, respectively.

For taking the end friction into account Balla introduced a friction factor ϕ defined as

$$\phi = \left[1 - \frac{u_{H,R}}{u_{(H,R) \max}} \right] \tag{2}$$

where $u_{H,R}$ is the radial displacement at point B (Fig. 1) in the presence of friction and $u_{(H,R) \max}$ is the corresponding value in the complete absence of friction. Taking origin at the centre of the sample and denoting the height and diameter as $2H$ and $2R$, Balla's expressions for vertical deformation, w^* at top and radial deformation u at point A become

$$E \cdot \frac{w}{H} = \sigma_1 \left[1 - 2\mu\gamma - (1 + \mu) \left(\frac{\mu}{1 + \mu} - \gamma \right) \right. \\ \times \frac{\phi}{3(1 - \mu) + 8(1 + \mu) \left(1 - \frac{3K}{\pi^2} \right) a^2} \\ \left. \times \left(6\mu \frac{1 - \mu}{1 + \mu} - \frac{2}{45} a^2 \right) \right] = \sigma_1 \cdot f_1 \tag{3}$$

$$E \cdot \frac{u}{R} = \sigma_1 \left[-\mu + (1 - \mu)\gamma - \left(\frac{\mu}{1 - \mu} - \gamma \right) \right. \\ \times \frac{\phi}{3(1 - \mu) + 8(1 + \mu) \left(1 - \frac{3K}{\pi^2} \right) a^2} \\ \times \left(-\frac{3}{2}(3 - 5\mu) + 4(1 - \mu^2)a^2 + \frac{3}{2}(1 - \mu - 2\mu^2) \right. \\ \left. - \frac{24}{\pi}(1 - \mu^2)a^2 \sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2 \pi V_n} \frac{1}{I_1(n\pi/a)} \right. \\ \left. \times [U_n I_0(n\pi/a) - I_1(n\pi/a)] \right] = \sigma_1 \cdot f_2 \tag{4}$$

* In the expression given in Balla (1960) the second term in brackets on the R.H.S. of Eq. (3) reads as $-4\mu\gamma$, whereas this should be as given here.

where, σ_1 and $\gamma\sigma_1 (= \sigma_3)$ are applied vertical and radial stresses, n is a positive integer and I_0 and I_1 are Bessel functions of imaginary arguments. The other constants appearing in equations (3) and (4) are as follows:

$$U_n = \frac{(1 - \mu) \frac{I_0(n\pi/a)}{I_1(n\pi/a)} - (3 - \mu) \frac{a}{n\pi}}{(1 - \mu) \left(\frac{4a^2}{n^2 \pi^2} - 1 \right) + (1 + 3\mu - 2\mu^2) \frac{a}{n\pi} \frac{I_0(n\pi/a)}{I_1(n\pi/a)}} \quad (5)$$

$$V_n = 1 - U_n \left[2(1 - \mu) \frac{a}{n\pi} + \frac{I_0(n\pi/a)}{I_1(n\pi/a)} \right] \quad (6)$$

$$K = \sum_{n=1}^{\infty} \frac{\cos^2 n\pi}{n^2 V_n} \left[1 - U_n \frac{I_0(n\pi/a)}{I_1(n\pi/a)} \right] \quad (7)$$

$$a = H/R. \quad (8)$$

It can be seen from equations (3) through (7) that u and w are independent of the absolute values of H and R and depend on a only.

In the absence of end friction the sample deforms uniformly retaining its perfect cylindrical shape and everywhere within the sample $\sigma_r = \sigma_\theta = \sigma_3$, the applied radial stress and $\sigma_z = \sigma_1$, the applied vertical stress. For this condition, σ_1 and σ_3 also become the principal stresses. Using equation (1) the vertical deformation, w at the top surface, and radial deformation, u at point, A , are obtained as

$$E \cdot \frac{w}{H} = \sigma_1 - 2\mu\sigma_3 = \sigma_1 (1 - 2\mu\gamma) \quad (9)$$

$$E \cdot \frac{u}{R} = \sigma_3 - \mu(\sigma_1 + \sigma_3) = \sigma_1 [\gamma - \mu(1 + \gamma)] \quad (10)$$

In the presence of end friction, equations (9) and (10) are not exact and the estimation of the error in the use of these equations can be obtained by comparing equations (3) and (4) with equations (9) and (10). Here two non-dimensional parameters β and η are introduced, which are defined as follows:

$$\beta = E(u/R)/[\sigma_3 - \mu(\sigma_1 + \sigma_3)] = f_2/[\gamma - \mu(1 + \gamma)] \quad (11)$$

$$\eta = E(w/H)/(\sigma_1 - 2\mu\sigma_3) = f_1/(1 - 2\mu\gamma). \quad (12)$$

It can be seen from the above equations that the deviations of the non-dimensional parameters, β and η from unity give a direct indication of the errors involved in the use of equations (9) and (10). Also it can be noted from equations (3), (4), (11) and (12) that β and η are independent of the absolute values of σ_1 and σ_3 and depend on the ratio γ only. The other factors influencing β and η are ϕ , μ and α .

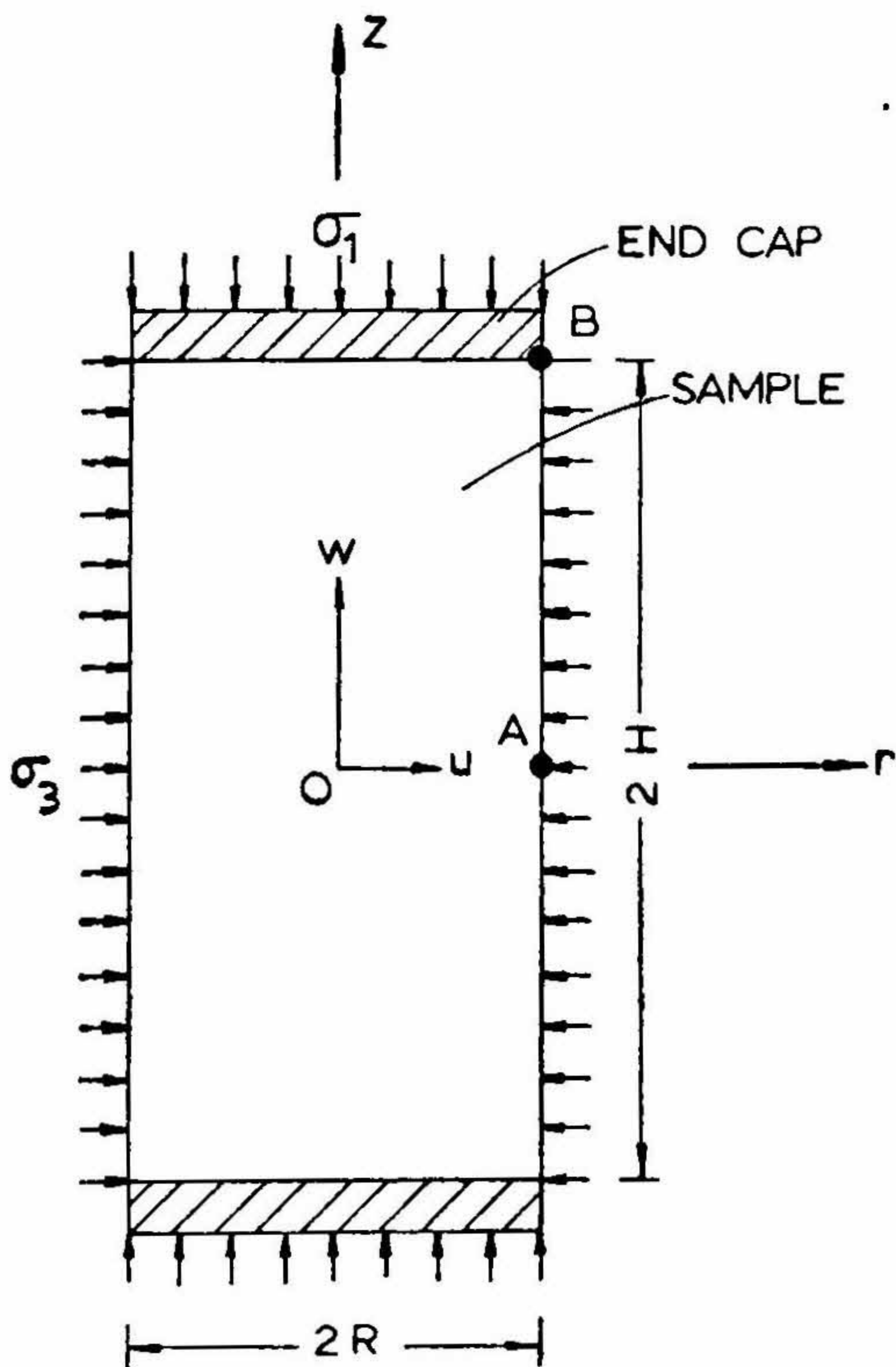


FIG. 1. The triaxial test specimen.

RESULTS AND DISCUSSIONS

For the computations of the values of β and η , a value of $\alpha = 2$, which is usually the case with triaxial tests, is taken. The values of Poisson's

ratio, μ , have been taken as 0.3, 0.4 and 0.5. The applied stress ratio, γ , has been varied from 0 to 2 in increments of 0.2, and ϕ values used are from 0 to 1.0 in increments of 0.25. The values of β and η obtained using equations (11) and (12) for various μ , γ and ϕ values, have been presented in Table I and these results are discussed below.

Variation of β

Following Saint-Venant's principle, it can be expected that the effect of end restraint should be minimum at the farthest cross section from ends, *i.e.*, at the centre of the specimen. This implies that in the central zone stresses and strains should be independent of friction factor, ϕ . This can be verified from Table I which shows further that the parameter β is almost independent of any of the parameters μ , γ and ϕ , the deviation of β from unity lying within 1%. This implies that equation (10) is correct for all practical purposes, irrespective of ϕ , as long as $a \geq 2$ and can be used for determining elastic parameters when it is possible to measure radial deformation at the point A . Therefore if Poisson's ratio is known, an exact value of Young's modulus can be determined in terms of u , the radial deformation at A , from the equation

$$E = (R/u) \cdot [\sigma_3 - \mu(\sigma_1 + \sigma_3)]. \quad (13)$$

The coefficient of earth pressure at rest, K_0 , is obtained from triaxial tests for no lateral strain at the point A [2] For this condition equation (10) gives

$$\mu = \sigma_3 / (\sigma_1 + \sigma_3) \quad (14)$$

and

$$K_0 = \mu / (1 - \mu) \quad (15)$$

Equation (10) being almost exact these values will be very accurate.

Variation of η :

It can be seen from Table I that η varies significantly with μ , γ and ϕ . For all the values of μ and ϕ considered herein, the variation of η with γ has been shown in Fig. 2, for the case of triaxial compression (*i.e.*, $\gamma < 1$). It can be seen that η is always less than 1 and the deviation of η from unity increases with μ , γ as well as with ϕ . For any particular set of values of

ϕ and μ , the error is minimum for $\gamma = 0$ and this error increases as γ increases. For all the values of μ , deviation of η from unity increases with the friction factor, ϕ . Also it is seen that higher values of μ result in larger errors.

TABLE I

γ	ϕ	$\mu = 0.3$		$\mu = 0.4$		$\mu = 0.5$	
		β	η	β	η	β	η
0.0	1.0	1.01	0.90	1.01	0.86	1.01	0.80
	0.25	1.00	0.98	1.00	0.96	1.00	0.95
0.2	1.0	1.01	0.90	1.01	0.84	1.01	0.77
	0.25	1.00	0.98	1.00	0.96	1.00	0.94
0.4	1.0	1.01	0.90	1.01	0.83	1.01	0.72
	0.25	1.00	0.98	1.00	0.96	1.00	0.93
0.6	1.0	1.01	0.90	1.01	0.80	1.01	0.62
	0.25	1.00	0.98	1.00	0.95	1.00	0.91
0.8	0.1	1.01	0.90	1.01	0.75	1.01	0.32
	0.25	1.00	0.98	1.00	0.94	1.00	0.83
1.0	1.0	1.01	0.90	1.01	0.63	*	*
	0.25	1.00	0.97	1.00	0.91		
1.4	1.0	1.01	0.89	1.01	1.39	1.01	1.22
	0.25	1.00	0.97	1.00	1.10	1.00	1.05
1.8	1.0	1.01	0.95	1.01	1.04	1.01	1.07
	0.25	1.00	0.99	1.00	1.01	1.00	1.02

* Cannot be computed as the denominators of the right hand side of equations (11) and (12) become zero.

The values of Young's modulus and Poisson's ratio are required for the calculation of elastic settlement. The Young's modulus has much greater influence on elastic settlement compared to the Poisson's ratio and hence

an exact value of the Young's modulus is necessary. Since Young's modulus is generally determined from triaxial compression tests by using equation (9), this equation needs special attention. On the other hand the exact value of Poisson's ratio can be found from equation (14), as shown earlier.

Equations (3), (9) and (12) reveal that the parameter η is nothing but the ratio of the true Young's modulus, E , to the apparent Young's modulus, $E_{obs.}$, as obtained from equation (9), *i.e.*,

$$\eta = E/E_{obs.} \quad (16)$$

Since for compression tests η is always less than unity (Fig. 2), $E_{obs.}$ is always more than the actual value. Since the elastic settlement is inversely proportional to Young's modulus, the predicted elastic settlement obtained using $E_{obs.}$ will always be less than the actual value. For example, for very rough ends ($\phi = 1.0$), $\mu = 0.5$, and $\gamma = 0$, $E_{obs.}$ will be $1/0.8$ times the actual value and hence the predicted settlement will be 0.8 times the actual settlement, *i.e.*, 20 per cent less than the actual settlement.

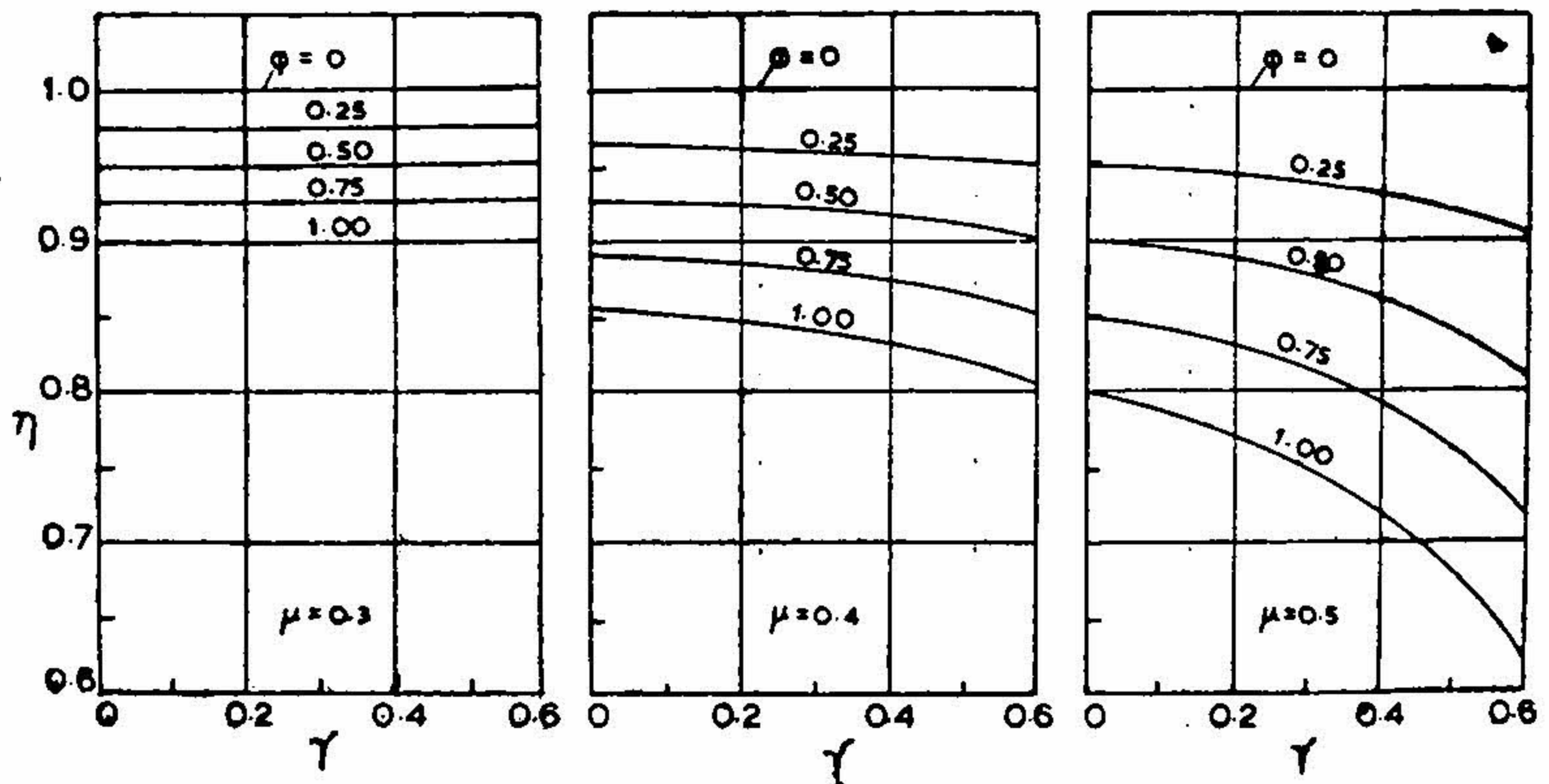


FIG. 2. Variation of η with γ for various μ and ϕ values.

The error in the modulus value so determined increases with end friction and for reasonably smooth ends ($\phi = 0.25$), this error is not more than 5 and 9 per cent for $\gamma = 0$ and 0.6 respectively, for the range of Poisson's ratios studied. If the radial deformations at the points A and B can be measured during the test, the exact values of Young's modulus as well as Poisson's ratio can be calculated from a single test, using Fig. 2 and equations (2), (9) and (10) as follows;

1. Obtain E and μ from equations (9) and (10). This value of E is E_{obs} .

2. Calculate ϕ from equation (2), since $u_{H, R}$ has been measured, and equation (10) being almost exact $u_{(H, R) max}$ becomes radial deformation at the point A , which is also measured.

3. Using the values of γ , μ and ϕ , get η from Fig. 2 and hence E from equation (16). With this value of E obtain a more exact value of μ from equation (10). This completes the first cycle.

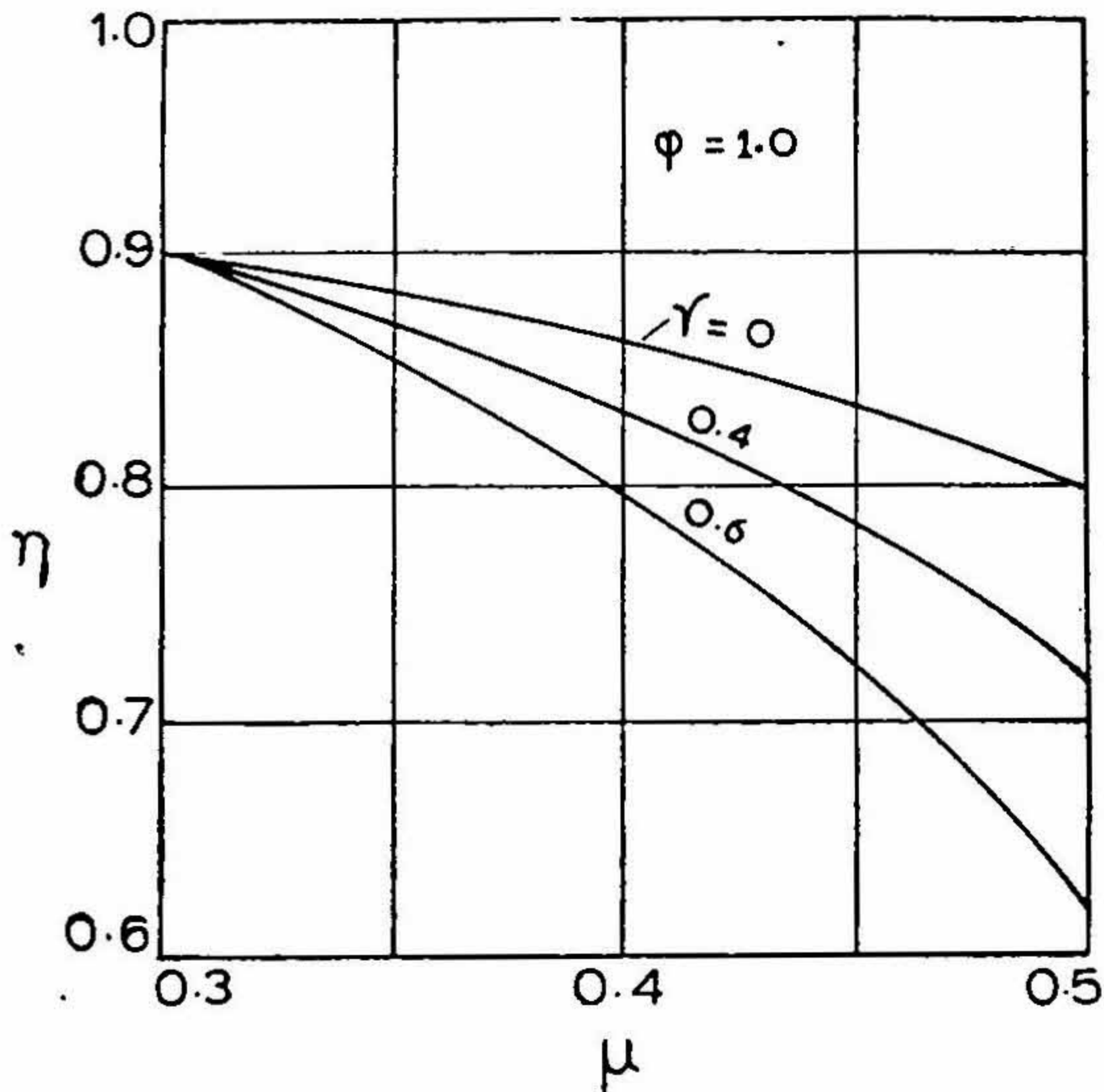


FIG. 3. Variation of η with μ for the case of full restraint, for various γ values.

4. Better accuracy in the values of E and μ may be achieved by calculating a more exact value of E_{obs} from equation (9) with the new value of μ and repeating step 3.

Measuring lateral deformation at the point B (Fig. 1) is quite difficult for ordinary triaxial tests and some approximations regarding the friction factor, ϕ , is necessary. The use of porous stones for the measurement of the pore pressure in standard triaxial tests, however, restricts the slippage at the ends to a very low value and hence the friction factor, ϕ , can be

approximated to 1 in such cases. For this condition, the parameter η is plotted in Fig. 3, for various applied stress ratios. The figure shows that the deviation of the observed Young's modulus, $E_{\text{obs.}}$ from true Young's modulus, E , increases with μ as well as with γ . For the cases where porous stones are introduced for measuring pore pressure, the Young's modulus can be obtained from Fig. 3, with reasonable accuracy.

CONCLUSIONS

For determining the true Young's modulus from triaxial tests, end friction should be minimised as otherwise it can introduce a significant error in the modulus value and hence in the elastic settlement obtained from it. The expression for lateral deformation at the exterior of the centre of the specimen [*i.e.*, equation (10), which does not consider end friction] is almost exact for $\alpha \geq 2.0$ and the elastic parameters determined from lateral strain measurement using this equation, *i.e.*, Poisson's ratio, μ , and coefficient of earth pressure at rest, K_0 , are exact. It is possible to determine the true values of the elastic parameters E and μ from a single test, provided lateral deformations at the top edge and the exterior of the middle of the sample and vertical deformation at top are measured. In cases of triaxial tests using porous stones at either ends of the sample, the Young's modulus can be obtained directly from Fig. 3.

NOTATION

E	= Young's modulus;
$E_{\text{obs.}}$	= Young's modulus obtained from triaxial tests;
f_1, f_2	= functions of μ, ϕ, α and γ , defined by equations (3) and (4);
G	= shear modulus;
H	= half height of the test specimen;
I_0, I_1	= Bessel functions of imaginary arguments of zero and first order;
K	= constant, defined by equation (7);
K_0	= coefficient of earth pressure at rest;
	= positive integer;
nr, θ, z	= cylindrical co-ordinate system;

Influence of End Friction on Elastic Parameters Obtained in

- R = radius of the specimen;
- u = radial displacement at the point A (Fig. 1);
- $u_{H, R}$ = radial displacement at the point B (Fig. 1);
- $u_{(H, R) \max.}$ = radial displacement at the point B (Fig. 1), in the complete absence of friction;
- U_n, V_n = constants, defined by equations (5) and (6);
- w = vertical displacement at the top surface;
- α = height to diameter ratio of the test specimen;
- β = a non-dimensional parameter, defined by equation (11);
- γ = ratio of radial to vertical stress in a triaxial test;
- γ_{rz} = shear strain along rz plane;
- $\epsilon_r, \epsilon_\theta, \epsilon_z$ = normal strains along radial tangential and vertical directions;
- η = a non-dimensional parameter, defined by equation (12);
- $\sigma_r, \sigma_\theta, \sigma_z$ = normal stresses along radial, tangential and vertical directions;
- σ_1, σ_3 = applied vertical and radial stresses in a triaxial test;
- τ_{rz} = shear stress along rz plane;
- ϕ = friction factor, defined by equation (2).

REFERENCES

- [1] Balla, A. .. Stress conditions in triaxial compression. *Jnl. Soil Mech. Fnds. Div. Soc. Civ. Engrs.*, 1960, 86, No. SM6, pp. 57-84.
- [2] Bishop, A. W. and Henkel, D. J. The measurement of soil properties in the triaxial test, London. The English Languages Book Society and Edward Arnold Ltd., 1969, pp. 28-30, 140-144.
- [3] Blight, G. E. .. The effect of nonuniform pore pressures on laboratory measurements of the shear strength of soils. *A.S.T.M. Spl. Tech. Pub.*, 1963 *a*, No. 361, pp. 173-184.
- [4] Blight, G. E. .. Discussion on Pore pressures within soil specimens in triaxial compression by Crawford, Carl B. *A.S.T.M. Spl. Tech. Pub.*, 1963 *b*, No. 361, pp. 199-204.

- [5] Brady, B. T. .. An exact solution to the radially end-constrained circular cylinder under triaxial loading. *Int. Jnl. Rock. Mech. and Mining Sci.*, Pergamon Press, 1971, 8, No. 2, pp. 165-178.
- [6] D'Appolonia, E. and Newmark, N. M. A method for solution of the restrained cylinder under axial compression. *Proc. First U.S. Natl. Conf. App. Mech. Am. Soc. Mech. Engrs.*, 1957, pp. 217-226.
- [7] Edelman, F. .. On the compression of a short cylinder between rough and blocks. *Quarterly of Appl. Math. Brown Univ.*, 1949, 7, No. 3.
- [8] Filon, L. N. G. .. The elastic equilibrium of circular cylinders under certain practical systems of load. *Trans. Roy. Soc. London*, 1902, Ser. A, 198, pp. 147-233.
- [9] Peng, S. D. .. Stresses within elastic circular cylinders loaded uniaxially end triaxially. *Int. Jnl. Rock. Mech. Mining Sci.*, Pergamon Press, 1971, No. 5, pp. 399-432.
- [10] Pickett .. Application of the fourier method to the solution of certain boundary problems in the theory of elasticity. *Jnl. App. Mech. Am. Soc. Mech. Engrs.*, 1944, II, pp. A 176-189.
- [11] Rowe, P. W. and Barden, L. Importance of free ends in triaxial testing. *Jnl. Soil Mech. Fnds. Div. Am. Soc. Civ. Engrs.*, 1964, 90, No. SMI. pp, 1-27.