A METHOD FOR SEPARATION OF STRESSES IN TWO- AND THREE-DIMENSIONAL PHOTOELASTICITY

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ABSTRACT

A method for separation of stresses in two and three-dimensional photo elasticity using the harmonisation of first stress invariant along a straight section is developed. For two-dimensions, the equations of equilibrium are reformulated in terms of sum and difference of normal stresses and relations are obtained which can be used for harmonisation of the first invariant of stress along a straight section.

A similar procedure is adopted for three-dimensions by making use of the Beltrami-Michell equations. The new relations are used in finite difference form to evaluate the sum of normal stresses along straight sections in a three-dimensional body. The method requires photoelastic data along the section as well as adjacent sections. This method could be used as an alternative to the shear difference method for separation of stresses in photoelasticity. The accuracy and reliability of the method is verified by applying the method to problems whose solutions are known.

Key words : Photoelasticity, separation of stresses, harmonisation.

1. INTRODUCTION

It is well known that the pure photoelastic method cannot solve completely a general stress problem. The conventional photoelastic method provides two independent equations for a two-dimensional problem and five independent equations for a three-dimensional problem. Since there are three unknown stress components for a two-dimensional problem and six unknown stress components for a three-dimensional problem, the equations obtained by using photoelasticity are not sufficient to determine the complete state of stress at any interior point. Hence with a view to obtain additional relation, several auxiliary methods have been developed both for two- and three-dimensional problems, which in conjunction with photoelastic analysis provides the complete information for solving a general stress problem. The description of these auxiliary methods, which are also referred as separation methods, can be found in several text books [1–3]. Though there are several separation methods available for two-dimensional problems, efforts are still being made to evolve better and simpler techniques. In this respect recently [4, 5] some efforts have been made to develop methods for separation of stresses in two-dimensional problems. The situation is quite different as far as three-dimensional problems are concerned and only the shear difference method is widely used to determine all the six stress components at any interior point. However it is known that the shear difference method gives results which may have appreciable error and hence used with some reservation. Recently the authors [6] have developed an experimental-numerical hybrid technique for the complete solution of three-dimensional problem using experimentally determined surface stresses. This method can be used when the stress distribution is required throughout the body. If the stress determination is required only along some important sections, then it is preferable to use a method which confines to the determination of stresses along the required section. It is only with this point in view the method described in this paper has been developed.

In the suggested method of separation here, the sum of normal stresses is computed along straight sections by making use of the concept of harmonisation [4, 7]. The method is first developed for two-dimensional problems and then extended to three-dimensional problems. The method makes use of the photoeleastic data along the interior sections and can be used as an alternative to shear difference method. The application of the method suggested here has been explained through examples in which an estimate of the error in the results compared to the shear difference method has also been made.

2. PROCEDURE FOR TWO-DIMENSIONS

The equilibrium equations can be written for two-dimensional problem as (when body forces are absent)

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \tag{1}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \tag{2}$$

and the compatability equation as

$$\nabla_1^2 (\sigma_x + \sigma_y) = \nabla_1^2 S_1 = 0. \tag{3}$$

Using the concept of stress difference elasticity [5, 8] and operating on the equations (1) to (3) the following relations can be obtained.

$$2\frac{\partial^2 S_1}{\partial x^2} = -\frac{\partial^2 D}{\partial x^2} - \frac{\partial^2 D}{\partial y^2} \tag{4}$$

$$2\frac{\partial^2 S_1}{\partial y^2} = \frac{\partial^2 D}{\partial x^2} + \frac{\partial^2 D}{\partial y^2}. \tag{5}$$

Equations (4) and (5) can be now rewritten as

$$\frac{\partial^2 P}{\partial x^2} = -\frac{\partial^2 D}{\partial y^2} \tag{6}$$

$$\frac{\partial^2 Q}{\partial y^2} = \frac{\partial^2 D}{\partial x^2} \tag{7}$$

where $P = 2S_1 + D$ and $Q = 2S_1 - D$.

Equations (6) and (7) can be used to determine the values of S_1 along sections parallel to x or y axis respectively.

The values of D can be obtained along the section of integration and the adjacent sections ($j-1$, j and $j+1$) (Fig. 1) from the photoelastic data as

$$D = \sigma_x - \sigma_y = NF\cos 2\theta. \tag{8}$$

Now the right hand side of equation (6) can be written in finite difference form for any point 0 ($i.j$) as

$$-\left(\frac{\partial^2 D}{\partial y^2}\right)_{i,j} = (f_1)_{i,j} = -(D_{i,j+1} - 2D_{i,j} + D_{i,j-1})/h^2. \tag{9}$$

Equation (6) can now be written as

$$\frac{\partial^2 P}{\partial x^2} = f_1 \tag{10}$$

or in finite difference form as

$$(P_{i-1,j} - 2P_{i,j} + P_{i+1,j})/h_1^2 = (f_1)_{i,j}. \tag{11}$$

Thus the recurrence formula for P at any point 0 (i,j) becomes

$$P_{i,j} = [(P_{i-1,j} + P_{i+1,j}) - h_1^2 (f_1)_{i,j}]/2. \tag{12}$$

The value of f_1 can be calculated for mesh points along the line of integration $i = 1$ to $n - 1$ (Fig. 1) using relations (8) and (9). The values of P along AB can then be calculated using the recurrence formula (12)

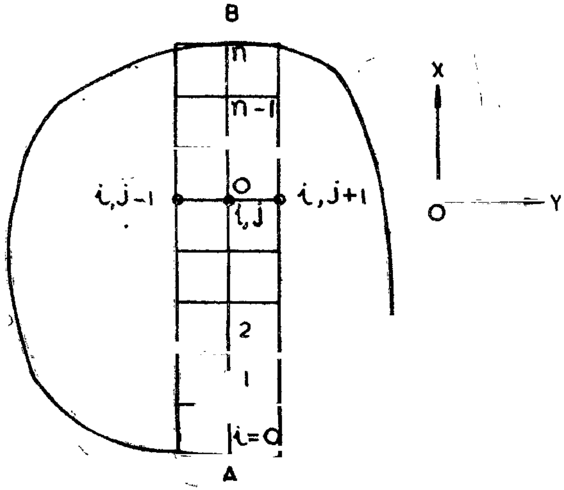

FIG. 1. Finite difference mesh for harmonisation of S_1 along a section parallel to x-axis.

and an iteration procedure. The values of P at the end points A and B of the section should be known apriori. After determining P values at all interior points, S_1 can be computed at any point on the section from

$$(S_1)_{i,j} = [P_{i,j} - D_{i,j}]/2. \tag{13}$$

In case the section is choosen parallel to y-axis then equation (7) can be made use of to compute the values of S_1. The recurrence relation for Q at any point 0 (i, j) becomes

$$Q_{i,j} = [(Q_{i,j-1} + Q_{i,j+1}) - h^2 (f_2)_{i,j}]/2 \tag{14}$$

where

$$(f_2)_{i,j} = (D_{i+1,j} + D_{i-1,j} - 2D_{i,j})/h_1^2. \tag{15}$$

Now S_1 can be computed from Q by

$$(S_1)_{i,j} = (Q_{i,j} + D_{i,j})/2. \tag{16}$$

3. Procedure for Three-Dimensions

The first three of the Beltrami-Michell equations [9] can be written as

$$(1 + \nu) \nabla^2 \sigma_x + \frac{\partial^2 S}{\partial x^2} = 0 \tag{17}$$

$$(1 + \nu) \nabla^2 \sigma_y + \frac{\partial^2 S}{\partial y^2} = 0 \tag{18}$$

$$(1 + \nu) \nabla^2 \sigma_z + \frac{\partial^2 S}{\partial z^2} = 0. \tag{19}$$

Adding equations (17) to (19), we get

$$\nabla^2 (\sigma_x + \sigma_y + \sigma_z) = \nabla^2 S = 0. \tag{20}$$

The photoelastic data from slices taken normal to x, y and z axes give

$$\begin{aligned} \sigma_y - \sigma_z &= N_x F \cos 2\theta_x = D_x \\ \sigma_z - \sigma_x &= N_y F \cos 2\theta_y = D_y \\ \sigma_x - \sigma_y &= N_z F \cos 2\theta_z = D_z \end{aligned} \tag{21}$$

respectively. The normal stresses σ_x, σ_y and σ_z may be obtained in terms of S, D_x, D_y and D_z as follows

$$\sigma_x = (S + D_z - D_y)/3 \tag{22}$$

$$\sigma_y = (S + D_x - D_z)/3 \tag{23}$$

$$\sigma_z = (S + D_y - D_x)/3. \tag{24}$$

Now equation (17) can be written in the form

$$\frac{\partial^2}{\partial x^2} [S + \sigma_x (1 + \nu)] = (1 + \nu) \left[- \frac{\partial^2 \sigma_x}{\partial y^2} - \frac{\partial^2 \sigma_x}{\partial z^2} \right]. \tag{25}$$

Substituting for σ_x in terms of S, D_y and D_z we get after simplification

$$\frac{\partial^2}{\partial x^2} (3S + C_1) = - \frac{\partial^2 C_1}{\partial y^2} - \frac{\partial^2 C_1}{\partial z^2} \tag{26}$$

where

$$C_1 = (1 + \nu) (D_z - D_y).$$

Equation (26) can be used to compute values of S along a section parallel to x axis. Equation (26) can also be written as

$$\frac{\partial^2}{\partial x^2}(R_1) = g_1 \tag{27}$$

where

$$R_1 = 3S + C_1$$

and

$$g_1 = -\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) C_1.$$

At any point 0 (i, j, k) the value of g_1 can be calculated from the photo-elastic data using a finite difference expansion as (Fig. 2)

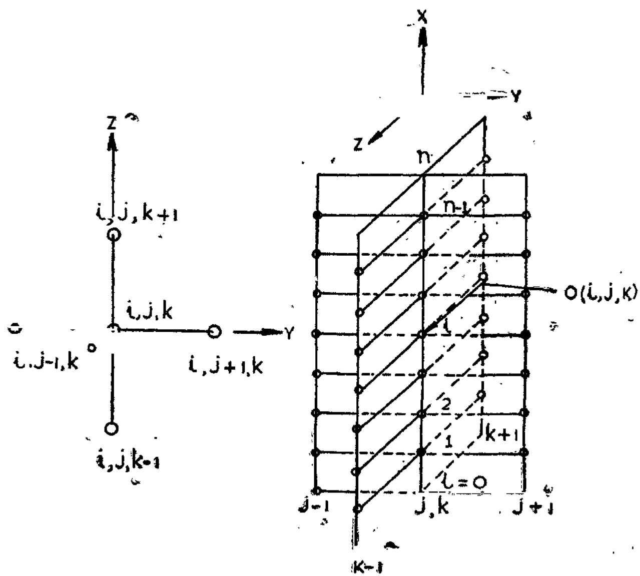

FIG. 2. Finite difference lattice for harmonisation of S along a section parallel to x-axis.

$$(g_1)_{i,j,k} = [4(C_1)_{i,j,k} - (C_1)_{i,j-1,k} - (C_1)_{i,j+1,k} - (C_1)_{i,j,k-1} - (C_1)_{i,j,k+1}]/l^2. \tag{28}$$

Thus the value of g_1 can be computed at all points along the section of interest namely $i = 1$ to $n - 1$. (Fig. 2). Hence equation (27) in finite difference will be

$$[(R)_{i-1,j,k} - 2(R_1)_{i,j,k} + (R_1)_{i+1,j,k}]/l^2 = (g_1)_{i,j,k}$$

or the recurrence relation for R_1 is

$$(R_1)_{i,j,k} = \{[(R_1)_{i-1,j,k} + (R_1)_{i+1,j,k}] - l^2 (g_1)_{i,j,k}\}/2. \tag{29}$$

Equation (29) can be used to calculate the values of R_1 at all points along a section parallel to x-axis. An iteration procedure can be used for determining R_1 values at all interior points, while the values of R_1 should be known at $i = 0$ and $i = n$. Once R_1 is calculated S can be determined from

$$S_{i,j,k} = [(R_1)_{i,j,k} - (C_1)_{i,j,k}]/3. \tag{30}$$

In case the integration is done along sections parallel to y or z axis the other relations similar to (27) can be used. These relations are

$$\frac{\partial^2}{\partial y^2}(R_2) = g_2 \tag{31}$$

$$\frac{\partial^2}{\partial z^2}(R_3) = g_3 \tag{32}$$

where

$$g_2 = -\frac{\partial^2 C_2}{\partial z^2} - \frac{\partial^2 C_2}{\partial x^2}$$

$$g_3 = -\frac{\partial^2 C_3}{\partial x^2} - \frac{\partial^2 C_3}{\partial y^2} \tag{33}$$

and

$$R_2 = 3S + C_2, \quad R_3 = 3S + C_3. \tag{34}$$

The values of C_2 and C_3 can be computed from

$$C_2 = (D_x - D_z)(1 + \nu)$$

$$C_3 = (D_x - D_y)(1 + \nu). \tag{35}$$

4. EXAMPLES

To illustrate the application of the method suggested here and to verify the accuracy of the method, first a two-dimensional problem for which the exact theoretical solution known is considered. A section parallel to x-axis (Fig. 3) is selected with in a semiinfinite plane subjected to a concenttrated load. The section CD is divided into discrete points and the values of f_1 are calculated at these points using equation (9). The values of D along

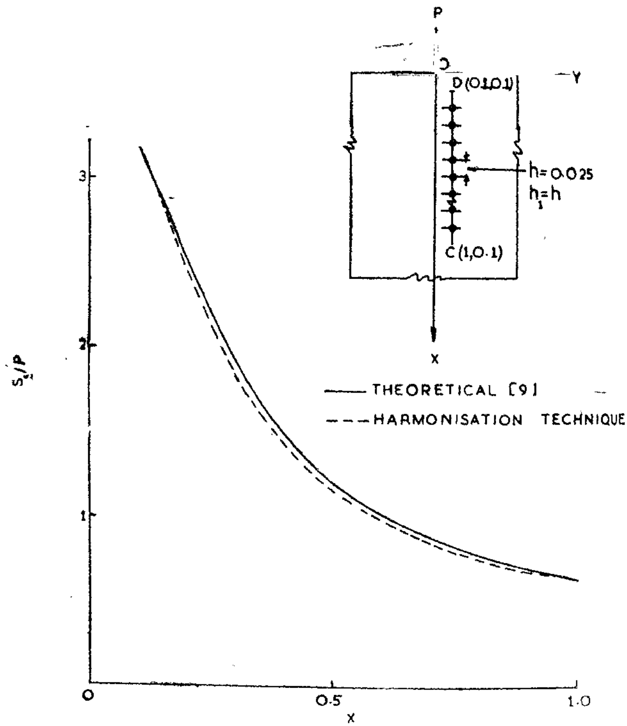

FIG. 3. Distribution of S_1 along CD obtained using the harmonisation technique and comparison with theoretical values.

CD and the adjacent sections as well as that of P at the end points are obtained from the Flamant's solution [9]. Using the recurrence relation given in equation (12) and an iteration procedure, the values of P are computed at interior points. The S_1 values are then calculated using equation (15). The variation of S_1 along section CD is shown in Fig. 3 in which the exact theoretical variation is also shown for comparison.

A second example, that of a three-dimensional problem is considered to illustrate the application of the proposed method. For this a section is taken parallel to x-axis within a semiinfinite elastic medium subjected to a concentrated load (Fig. 4). The values of C_1 and g_1 are calculated for

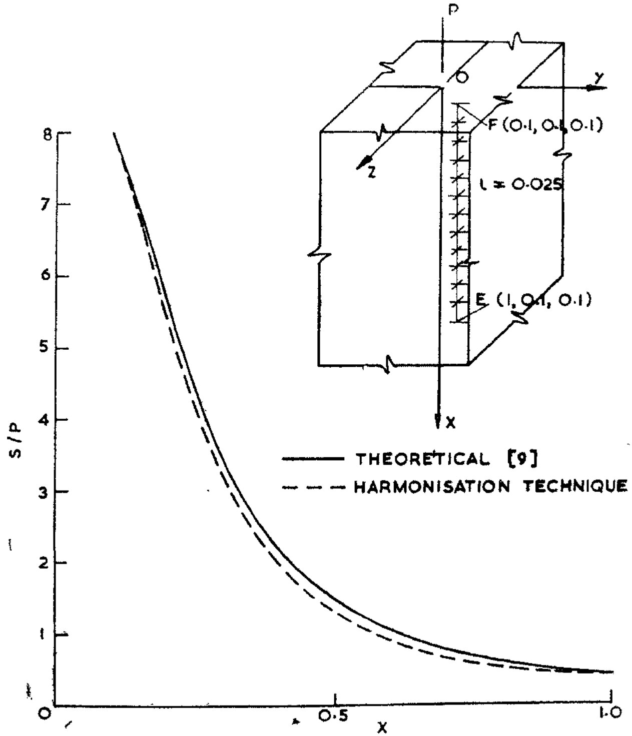

FIG. 4. Distribution of S along EF obtained using the harmonisation technique and comparison with theoretical results.

every point on the section EF as well as R_1 values at end points are determined using the Boussinesq solution [9]. Using recurrence relation given in equation (29) and an iteration procedure, the values of R_1 are determined at all interior points. The S values are then computed using equation (30). The variation of S along the section EF so obtained is shown in Fig. 4 in which the theoretical variation is also shown for comparison.

To illustrate the use of the method as an auxiliary one for separation of stresses in two-and-three dimensional photoleasticity the following two examples have been considered.

(i) Finite strip subjected to partially distributed load.

A finite strip was cut from a sheet of birefringent material Araldite cast using the following composition:

Araldite CY 230	..	100 pbw
Hardener Hy 951	..	10 pbw.

The material fringe value as well as the elastic constants at room temperature were determined using clabration specimens. The following values were obtained for the material:

Modulus of Elasticity	..	21,000 kg/cm^2
Poisson's ratio	..	0·35
Material fringe value	..	12·45 kg/cm/fringe.

The model was loaded at room temperature and the isoclinics and isochromatic data were obtained at a number of points along the central and the two adjacent sections. The procedure described in section 2 was used to calculate the values of S along the central section. The individual values of normal stresses are determined from:

$$\sigma_x = (S + D)/2 \tag{36}$$

$$\sigma_y = (S - D)/2.$$

The values of σ_x and σ_y are also computed using the shear difference method. The values thus obtained are shown in Fig. 5 in which the theoretical results of Iyengar [10] are also incorporated for comparison.

(ii) Finite prism subjected to partially distributed load.

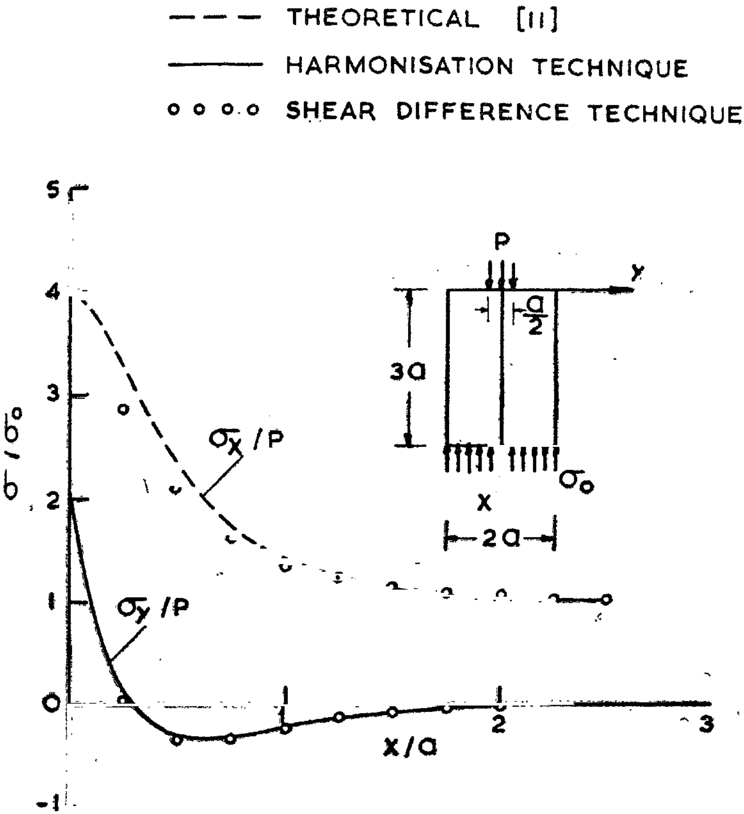

FIG. 5. Distribution of normal stresses along central section of a finite strip subjected to partially distributed load.

A finite prism was made using casting resin Araldite having the following composition:

Araldite CY 230	..	100 pbw
Pthalic Anhydride	..	30 pbw.

The casting was done at an elevated temperature (110° C) [11]. The stresses were frozen in the prism at a temperature of 120° C. The properties of this material at the stress freezing temperature was determined using calibration specimens. The properties of the material were as follows:

Modulus of Elasticity (120° C)	..	126 kg/cm²
Poisson's ratio (120° C)	..	0·45
Material fringe value (120° C)	..	0·302 kg/cm/fringe.

Slices were then cut from the stress frozen model and the isochromatic and isoclinic data were obtained along the central axis and the four adjacent sections. The procedure given under section 3 was used to compute values of S along the central axis. From the values of S so determined and the photoleastic data, the individual stresses are computed from:

$$\sigma_x = [S + D_z - D_y]/3$$
$$\sigma_y = (\sigma_x - D_z). \tag{37}$$

The stresses thus obtained along the central axis are shown in Fig. 6 The stresses were also obtained using shear difference method for the same problem along the central axis and these results are also shown in Fig. 6

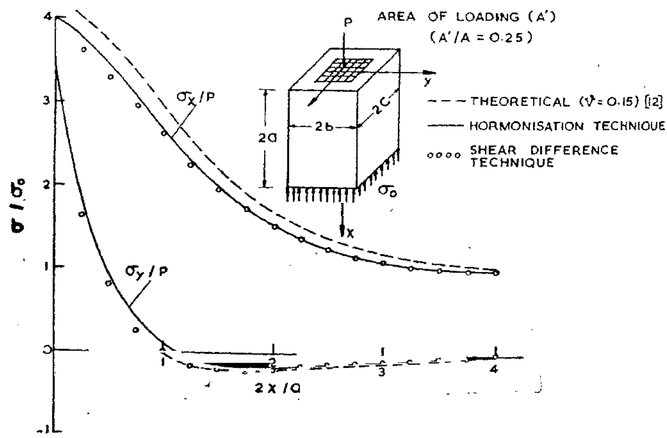

FIG. 6. Distribution of normal stresses along the central section of a finite prism subjected to partially distributed load.

for comparison. Also some available theoretical results for the same problem [12] is included for comparison.

5. Discussion and Conclusion

An examination of equations (12) and (29) show that the grid or lattice spacing (*h* or *l*) present in these equations are virtual and will be eliminated when finite difference form for f_1 or g_1 is substituted. However, if the grid or lattice spacing is different in different directions, then a ratio of this will occur in these equations. An error analyses indicates that the truncation errors in omitting the higher order terms in the finite difference expansion is less in the present method compared to the error involved in step by step integration of the equilibrium equation in shear difference method.

The first two examples show that the method suggested here can be employed to compute the sum of normal stresses along straight sections with reasonable accuracy for both two-and three-dimensional problems. The examples of finite strip and prism illustrate the application of the method for separation of stresses in two-and three-dimensional photoelasticity. It is interesting to note from Figs. 5 and 6 that the present method eliminates the cumulative error which is inherent in shear difference method. In shear difference method the cumulative error increases with the number of points taken for integration along a section where as in the harmonisation technique such a procedure only needs more number of iterations for convergence. It may be concluded that the method presented here can be used as an alternative to shear difference method for separation of stresses in two-and three-dimensional photoelasticity.

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LIST OF SYMBOLS

D	=	$\sigma_x - \sigma_y$
F	=	Material fringe value
h	=	Grid spacing in y direction
h_1	=	Grid spacing in x direction
i, j, k	=	Integers denoting the position of a point
l	=	Lattice spacing
N	=	Isochromatic fringe order
n	=	Any integer number
S	=	$\sigma_x + \sigma_y + \sigma_z$
S_1	=	$\sigma_x + \sigma_y$
x, y, z	=	Cartesian coordinates
∇_1^2	=	$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$
∇^2	=	$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$
ν	=	Poisson's ratio
$\sigma_x, \sigma_y, \sigma_z$	=	Normal stress components
τ_{xy}	=	Cartesian shear stress

(Other symbols are defined in the text whereever they occur first).