EFFECT OF CONCENTRATED FORCE ON A CRACK IN DISSIMILAR MEDIA

S. C. GUPTA

(*Department of Applied Mathematics, Indian Institute of Science, Bangalore 560012*)

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ABSTRACT

Effect of concentrated force or edge dislocation with Burger's vector on a line crack in dissimilar media has been studied in this paper. Crack surfaces may be subjected to surface loads or opened by rigid inclusions. Complex variable methods have been employed to study the distribution of stresses and displacements everywhere and in particular at the tips of the crack.

Key words: Concentrated force, Crack problem, dissimilar media, transform technique.

INTRODUCTION

Crack problems have been solved mainly by employing either transform techniques [1] or complex variable methods suggested by Muskhelishvili [2]. Tranform techniques employed by Snedden [3, 4, 5], Stallybrass [6, 7], Shrivastava [8, 9] and others for cracks in elastic media assume the additional knowledge of some of the stress and displacement components on certain lines of symmetry, whereas, the complex variable techniques adopted by Muskhelishvili [2], England [10, 11] and others [12] require the knowledge of resultant vector and resultant moment of external forces and stresses and rotation at infinity. In the analysis that follows it has been shown that the two approaches can make difference in solutions.

A moderate bibliography of problems concerning cracks in dissimilar media can be found in England's paper [10, 11]. The present paper deals with the study of first and second fundamental problems of two-dimensional elasticity for a crack at the interface of two-bonded dissimilar half planes in the presence of a concentrated force

1. *Concentrated Force in Dissimilar Media*

Consider two dissimilar elastic half planes bonded together along the x-axis. The upper half plane is the region $y > 0$ and lower half plane $y < 0$.

The line joining these two regions will be denoted by L. Let a concentrated force $X + iY$ act at some arbitrary point $z = z_0$ $(z_0 = x_0 + iy_0)$ of the plane. Throughout the paper the elastic constants of the upper half plane will be denoted by the subscript 1 and those of the lower half plane by the subscript 2. The upper and lower half planes will be denoted by S^+ and S^- respectively. We shall follow the notations of Muskhelishvili [2] for stresses, displacements and stress functions.

Stresses and displacements can be expressed in terms of complex potentials $\Phi(z)$ and $\Psi(z)$ as follows

$$\begin{aligned} X_x + Y_y &= 2\,[\Phi(z) + \overline{\Phi(z)}] \\ Y_y - X_x + 2i\,X_y &= 2\,[\bar{z}\,\Phi'(z) + \Psi(z)] \\ 2\mu\,(u + iv) &= k\,\phi(z) - z\,\overline{\Phi(z)} - \overline{\Psi(z)} \end{aligned} \tag{1}$$

where $\Phi(z) = \phi'(z)$ and $\Psi(z) = \psi'(z)$, μ is the shear modulus of elasticity and $k = 3 - 4\sigma$ for plane strain and $k = (3 - \sigma)/(1 + \sigma)$ for generalized plane stress; σ being Poisson's ratio. By $\Phi(z)$ and $\Psi(z)$ we shall always mean the stress functions in (1).

Introduce the function

$$\Omega(z) = \bar{\Phi}(z) + z\bar{\Phi}'(z) + \bar{\Psi}(z) \tag{2}$$

so that

$$\Psi(z) = \bar{\Omega}(z) - \Phi(z) - z\,\Phi'(z) \tag{3}$$

where $\bar{\Phi}(z) = \overline{\Phi(\bar{z})}$ and similarly for other functions. For large $|z|$, $\Phi(z)$, $\Psi(z)$ and $\Omega(z)$ have the following forms

$$\Phi(z) = -\frac{X + iY}{2\pi(1 + k)}\,\frac{1}{z} + 0\left(\frac{1}{z^2}\right), \Psi(z) = \frac{k\,(X - iY)}{2\pi(1 + k)}\frac{1}{z} + 0\left(\frac{1}{z^2}\right)$$

and

$$\Omega(z) = \frac{k\,(X + iY)}{2\pi\,(1 + k)} \times \frac{1}{z} + 0\left(\frac{1}{z^2}\right) \tag{4}$$

where (X, Y) is the resultant vector of external forces.

Following conditions should be satisfied:

$$(1\,a)\quad Y_y^+ - iX_y^+ = Y_y^- - iX_y^- \quad \text{on} \quad L \tag{5 i}$$

$$u^+ + iv^+ = u^- + iv^- \qquad \text{on} \quad L, \tag{5 ii}$$

$+$ and $-$ superscripts stand for stresses and displacements in the upper and lower half planes, respectively. The condition (5 *ii*) is equivalent to the condition

$$\frac{\partial}{\partial x}(u^{+} + iv^{+}) = \frac{\partial}{\partial x}(u^{-} + iv^{-}) \qquad (5\,iii)$$

provided the displacements are single valued. We shall use (5 *iii*) in place of (5 *ii*) to avoid logarithmic singularities.

(2 *a*) The functions $\Phi(z)$ and $\Psi(z)$ have required singularities which arise due to concentrated force at z_0.

(3 *a*) For large $|z|$, $\Phi(z)$, $\Omega(z)$ and $\Psi(z)$ are $0(z^{-1})$.

It may be noted that

$$Y_y^{+} - iX_y^{+} = \Phi^{+}(x) + \Omega^{-}(x) \quad \text{and} \quad Y_y^{-} - iX_y^{-} = \Phi^{-}(x) + \Omega^{+}(x) \quad \text{on} \quad L \qquad (6)$$

where $\Phi^{+}(x)$ and $\Omega^{+}(x)$; $\Phi^{-}(x)$ and $\Omega^{-}(x)$ are the values of $\Phi(z)$ and $\Omega(z)$ as $y \to 0+$ and $y \to 0-$ respectively.

From (5 *i*) and (6)

$$[\Phi(x) - \Omega(x)]^{+} = [\Phi(x) - \Omega(x)]^{-} \quad \text{on L}$$

and hence $\{\Phi(z) - \Omega(z)\}$ is holomorphic throughout the plane except possibly at some finite number of points where it has poles. Noting the singularities of $\Phi(z)$ and $\Omega(z)$, we find

$$\Phi(z) - \Omega(z) = -\frac{C}{z - z_0} - \frac{k\,C}{z - \bar{z}_0} + \frac{2iy_0\bar{C}}{(z - \bar{z}_0)^2} = g(z) \ \text{(say)}$$

where

$$C = (X + iY)/2\pi(1 + k). \qquad (7)$$

From (5 *iii*) we get

$$\mu_2\{k_1\Phi^{+}(x) - \Omega^{-}(x)\} = \mu_1\{k_2\,\Phi^{-}(x) - \Omega^{+}(x)\} \quad \text{on} \quad L. \qquad (8)$$

Substituting the value of $\Omega(z)$ from (7) into (8), it can be seen that

$$\Phi^{+}(x) - \alpha\Phi^{-}(x) = \{(\mu_1 - \mu_2)/(\mu_1 + \mu_2 k_1)\}\,g(x) \quad \text{on} \quad L, \qquad (9)$$

where

$$\alpha = (\mu_2 + \mu_1 k_2)/(\mu_1 + \mu_2 k_1).$$

From (9), $\Phi(z)$ can be obtained as the sum of a Cauchy integral taken along the line L and a function which accounts for the concentrated force at $z=z_0$. Let $\Phi(z)=\Phi_1(z)$ and $\Omega(z)=\Omega_1(z)$ for $z \in S^+$, and $\Phi(z)=\Phi_2(z)$ and $\Omega(z)=\Omega_2(z)$ for $z \in S^-$. If z_0 belongs to the interior of S^+, then

$$\Phi_1(z)=\frac{(\mu_1-\mu_2)}{(\mu_1+\mu_2k_1)}\left\{-\frac{k_1C_1}{z-\bar{z}_0}+\frac{2iy_0\bar{C}_1}{(z-\bar{z}_0)^2}\right\}-\frac{C_1}{z-z_0}, \quad z \in S^+$$

$$\Phi_2(z)=-\frac{\mu_2(1+k_1)}{\mu_2+\mu_1k_2}\frac{C_1}{z-\bar{z}_0}, \quad z \in S^- \tag{10}$$

$$\Omega_1(z)=\frac{\mu_2(1+k_1)}{\mu_1+\mu_2k_1}\left\{\frac{k_1C_1}{z-\bar{z}_0}-\frac{2iy_0\bar{C}_1}{(z-\bar{z}_0)^2}\right\}, \quad z \in S^+ \tag{11}$$

$$\Omega_2(z)=\frac{(\mu_1k_2-\mu_2k_1)}{(\mu_2+\mu_1k_2)}\frac{C_1}{z-z_0}+\frac{k_1C_1}{z-\bar{z}_0}-\frac{2iy_0\bar{C}_1}{(z-z_0)^2}, \quad z \in S^-$$

where

$$C_1=(X+iY)/2\pi(1+k_1).$$

If $\Psi(z)=\Psi_1(z)$ for $z \in S^+$ and $\Psi(z)=\Psi_2(z)$ for $z \in S^-$, then

$$\Psi_1(z)=\overline{\Omega_2(\bar{z})}-\Phi_1(z)-z\,\Phi_1'(z)$$

$$=\frac{(\mu_1k_2-\mu_2k_1)}{(\mu_2+\mu_1k_2)}\frac{\bar{C}_1}{(z-\bar{z}_0)}+\frac{k_1\bar{C}_1}{z-z_0}-\frac{\bar{z}_0C_1}{(z-z_0)^2}$$

$$-\frac{(\mu_1-\mu_2)\bar{z}_0k_1C_1}{(\mu_1+\mu_2k_1)(z-\bar{z}_0)^2}+\frac{2(\mu_1-\mu_2)iy_0(z+\bar{z}_0)\bar{C}_1}{(\mu_1+\mu_2k_1)(z-\bar{z}_0)^2}, \quad z \in S^+ \tag{12}$$

and

$$\Psi_2(z)=\overline{\Omega_1(\bar{z})}-\Phi_2(z)-z\Phi_2'(z)$$

$$=\frac{\mu_2(1+k_1)}{\mu_1+\mu_2k_1}\left\{\frac{k_1\bar{C}_1}{z-z_0}+\frac{2iy_0C_1}{(z-z_0)^2}\right\}-\frac{\mu_2(1+k_1)z_0C_1}{(\mu_2+\mu_1k_2)(z-z_0)^2}, \quad z \in S^-.$$

For an edge dislocation with Burger's vector the analysis remains the same except some changes in the constants occurring in various functions.

It can be verified that normal and shearing stresses and displacements are continuous across L and are given by the following expressions

$$Y_y^+-iX_y^+=Y_y^--iX_y^- \quad \text{on } L$$

$$=-\frac{\mu_2(1+k_1)}{\mu_2+\mu_1k_2}\frac{C_1}{x-z_0}+\frac{\mu_2(1+k_1)}{\mu_1+\mu_2k_1}\left\{\frac{k_1C_1}{x-\bar{z}_0}-\frac{2iy_0\bar{C}_1}{(x-\bar{z}_0)^2}\right\}, \tag{13}$$

$$u^+ + iv^+ = u^- + iv^- \quad \text{on} \quad L$$

$$= -\frac{k_2(1+k_1)}{2(\mu_2+\mu_1 k_2)} C_1 \log(x-z_0) - \frac{(1+k_1)}{2(\mu_1+\mu_2 k_1)}$$

$$\times \left\{ k_1 C_1 \log(x-\bar{z}_0) + \frac{2iy_0\bar{C}_1}{(x-\bar{z}_0)} \right\}. \tag{14}$$

When $\mu_1 = \mu_2$ and $k_1 = k_2$, $\Phi(z)$ and $\Psi(z)$ are reduced to well known results for a concentrated force in an infinite medium [2]. Similar analysis holds good even if there are more than one concentrated forces in the medium.

If z_0 belongs to the interior of S^- then $\Phi(z)$ and $\Psi(z)$ cannot be derived from (10) and (12) but can be obtained similarly.

When z_0 tends to a point on L from the interior of S^+, say $z_0 = b$, b real, then $\Phi(z)$, and $\Omega(z)$ cannot be deduced as particular cases of functions in (10) and (11) as the Cauchy integrals in this case will have singularities on the line of integration. They are given as follows

$$\Phi_1(z) = -\left\{ \frac{(\mu_1-\mu_2)(1+k_1)}{2(\mu_1+\mu_2 k_1)} + 1 \right\} \frac{C_1}{z-b}, \quad z \in S^+ \tag{15}$$

$$\Phi_2(z) = -\left\{ \frac{1}{\alpha} - \frac{(\mu_1-\mu_2)(1+k_1)}{2(\mu_2+\mu_1 k_2)} \right\} \frac{C_1}{z-b}, \quad z \in S^-$$

$$\Omega_1(z) = \frac{k_1 C_1}{z-b} - \frac{(\mu_1-\mu_2)(1+k_1)}{2(\mu_1+\mu_2 k_1)} \frac{C_1}{z-b}, \quad z \in S^+ \tag{16}$$

$$\Omega_2(z) = \frac{(1+k_1)C_1}{z-b} - \frac{C_1}{\alpha(z-b)} + \frac{(\mu_1-\mu_2)(1+k_1)C_1}{2(\mu_2+\mu_1 k_2)(z-b)}, \quad z \in S^-$$

$\Psi(z)$ can be easily calculated from (15), (16) and (3). Across L, stresses and displacements are given by

$$Y_y^+ - iX_y^+ = Y_y^- - iX_y^- = \left\{ \frac{(\mu_1-\mu_2)(1+k_1)}{2(\mu_1+\mu_2 k_1)} \right.$$

$$\left. - \frac{(\mu_1-\mu_2)(1+k_1)}{2(\mu_2+\mu_1 k_2)} + \frac{(1-\alpha k_1)}{\alpha} \right\} \frac{C_1}{b-x} \quad \text{on } L \tag{17}$$

$$u^+ + iv^+ = u^- + iv^- = \left\{ -\frac{k_1(\mu_1-\mu_2)(1+k_1)}{4\mu_1(\mu_1+\mu_2 k_1)} - \frac{(1+2k_1)}{2\mu_1} \right.$$

$$\left. + \frac{\mu_1+\mu_2 k_1}{2\mu_1(\mu_2+\mu_1 k_1)} - \frac{(\mu_1-\mu_2)(1+k_1)}{4\mu_1(\mu_2+\mu_1 k_2)} \right\} \frac{C_1}{x-b} \quad \text{on } L. \tag{18}$$

2. *Concentrated Force in Dissimilar Media Containing Crack*

Although the following analysis can be adapted for dissimilar media containing more than one crack, we shall give results for a concentrated force in dissimilar media containing only one crack.

Consider two dissimilar elastic half planes bonded together along the x-axis except over the region $-a \leqslant x \leqslant a$ where there is a line cracks Let a concentrated force $X + iY$ act at a point $z = z_0$ $(z_0 = x_0 + iy_0)$ in the interior of S^+.

We shall follow the notations of section 1. Following conditions should be satisfied

(1 *b*) $Y_y^+ - iX_y^+ = Y_y^- - iX_y^-$ on $y = 0$, $|x| > a$

(2 *b*) $u^+ + iv^+ = u^- + iv^-$ on $y = 0$, $|x| > a$

(3 *b*) $Y_y^+ - iX_y^+ = Y_y^- - iX_y^- = 0$ on $y = 0$, $|x| < a$

(4 *b*) $\Phi(z)$ and $\Psi(z)$ have required singularities which arise due to concentrated force at z_0.

(5 *b*) For large $|z|$, $\Phi(z)$ and $\Psi(z)$ are $0(z^{-1})$.

We construct each of the functions $\Phi(z)$ and $\Psi(z)$ as the sum of two functions

$$\Phi(z) = \Phi^*(z) + \Phi_0(z) \quad \text{and} \quad \Psi(z) = \Psi^*(z) + \Psi_0(z). \tag{19}$$

$\Phi^*(z)$ and $\Psi^*(z)$ are such that all the conditions from (1 *b*) to (5 *b*) except (3 *b*) are satisfied; $\Phi_0(z)$ and $\Psi_0(z)$ are obtained in such a way that the conditions (1 *b*), (2 *b*) and (5 *b*) are satisfied and the normal and shearing stresses calculated from them on $y = 0$, $|x| < a$ are equal and opposite to those given by $\Phi^*(z)$ and $\Psi^*(z)$ on $y = 0$, $|x| < a$.

We take $\Phi^*(z)$ and $\Psi^*(z)$ as the functions $\Phi(z)$ and $\Psi(z)$ given in (10) and (12). On the crack surfaces, they give normal and shearing stresses given in (13).

To find $\Phi_0(z)$ and $\Psi_0(z)$ we proceed as follows: Let $\Phi_0(z) = \Phi_{01}(z)$ and $\Psi_0(z) = \Psi_{01}(z)$ for $z \epsilon S^+$ and $\Phi_0(z) = \Phi_{02}(z)$ and $\Psi_0(z) = \Psi_{02}(z)$ for $z \epsilon S^-$. $\Phi_0(z)$ and $\Psi_0(z)$ are assumed to be holomorphic in the regions $y > 0$ and $y < 0$ and for large $|z|$ they have the forms given in (4). We denote $\lim\limits_{y \to 0+} \Phi_0(z) = \Phi_{01}^+(x)$ and $\lim\limits_{y \to 0-} \Phi_0(z) = \Phi_{02}^-(X)$ and similarly for

$\Psi_0(z)$. Since displacements and stresses calculated from $\Phi_0(z)$ and $\Psi_0(z)$ are to be continuous across $y = 0$, $|x| > a$, following conditions should be satisfied

$$\mu_2 k_1 \phi_{01}^{+}(x) + \mu_1 x \bar{\Phi}_{02}^{+}(x) + \mu_1 \bar{\psi}_{02}^{+}(x)$$
$$= \mu_1 k_2 \phi_{02}^{-}(x) + \mu_2 x \bar{\Phi}_{01}^{-}(x) + \mu_2 \bar{\psi}_{01}^{-}(x),$$
$$\text{on } y = 0, \quad |x| > a \tag{20}$$

and

$$\Phi_{01}^{+}(x) + \bar{\Phi}_{01}^{-}(x) + x \bar{\Phi}_{01}'^{-}(x) + \bar{\Psi}_{01}^{-}(x)$$
$$= \Phi_{02}^{-}(x) + \bar{\Phi}_{02}^{+}(x) + x \bar{\Phi}_{02}'^{+}(x) + \bar{\Psi}_{02}^{+}(x),$$
$$\text{on } y = 0, \quad |x| > a \tag{21}$$

where $\Phi_{01}(z) = \phi_{01}'(z)$ and $\Psi_{01}(z) = \psi_{01}'(z)$ and similarly for other functions.

Continuity of the derivatives of the displacements across L would require the following condition to be satisfied

$$\mu_2 k_1 \Phi_{01}^{+}(x) + \mu_1 x \Phi_{02}'^{+}(x) + \mu_1 \bar{\Phi}_{02}^{+}(x) + \mu_1 \bar{\Psi}_{02}^{+}(x)$$
$$= \mu_1 k_2 \Phi_{02}^{-}(x) + \mu_2 x \bar{\Phi}_{01}'^{-}(x) + \mu_2 \bar{\Phi}_{01}^{-}(x) + \mu_2 \bar{\Psi}_{01}^{-}(x). \tag{22}$$

We define functions $\theta(z)$ and $\omega(z)$ as follows

$$\theta(z) = \mu_2 k_1 \phi_{01}(z) + \mu_1 z \bar{\Phi}_{02}(z) + \mu_1 \bar{\psi}_{02}(z), \quad z \in S^{+}$$
$$= \mu_1 k_2 \phi_{02}(z) + \mu_2 z \bar{\Phi}_{01}(z) + \mu_2 \bar{\psi}_{01}(z), \quad z \in S^{-} \tag{23}$$

and

$$\omega(z) = \phi_{01}(z) - z \bar{\Phi}_{02}(z) - \bar{\psi}_{02}(z), \quad z \in S^{+}$$
$$= \phi_{02}(z) - z \bar{\Phi}_{01}(z) - \bar{\psi}_{01}(z), \quad z \in S^{-}. \tag{24}$$

It can be easily seen that

$$(\mu_1 + \mu_2 k_1) \phi_{01}(z) = \mu_1 \omega(z) + \theta(z), \quad z \in S^{+}$$
$$(\mu_2 + \mu_1 k_2) \phi_{02}(z) = \mu_2 \omega(z) + \theta(z), \quad z \in S^{-} \tag{25}$$

on the crack surfaces, following conditions should hold

$$\frac{\mu_1}{\mu_1+\mu_2 k_1}\,\omega'^{+}(x)-\frac{\mu_1 k_2}{\mu_2+\mu_1 k_2}\,\omega'^{-}x+\frac{1}{\mu_1+\mu_2 k_1}\,\theta'^{+}x$$

$$+\frac{1}{\mu_2+\mu_1 k_2}\,\theta'^{-}(x)=f\,(x) \tag{26}$$

$$\frac{\mu_2 k_1}{\mu_1+\mu_2 k_1}\,\omega'^{+}(x)+\frac{\mu_2}{\mu_2+\mu_1 k_2}\,\omega'^{-}(x)$$

$$+\frac{1}{\mu_1+\mu_2 k_1}\,\theta'^{+}(x)+\frac{1}{\mu_2+\mu_1 k_2}\,\theta'^{-}(x)=f\,(x) \tag{27}$$

where

$$f\,(x)=\frac{\mu_2\,(1+k_1)}{(\mu_2+\mu_1 k_2)}\,\frac{C_1}{x-z_0}-\frac{\mu_2\,(1+k_1)}{\mu_1+\mu_2 k_1}$$

$$\times\left\{\frac{k_1 C_1}{x-z_0}-\frac{2iy_0 C_1}{(x-\bar{z}_0)^2}\right\} \tag{28}$$

Subtracting (27) from (28) and using (1 *b*) we find $\omega'^{+}(x)-\omega'^{-}(x)=0$ on L and so $\omega'(z)$ is holomorphic in the entire plane. For large $|z|$, $\omega'(z)=0\,(1/z^2)$ and therefore $\omega'(z)=0$. Substituting $\omega'(z)=0$ in (26), we get

$$\theta'^{+}(x)+\alpha_0\theta'^{-}(x)=(\mu_1+\mu_2 k_1)\,f\,(x);\quad y=0,\quad |x|<a \tag{29}$$

where

$$\alpha_0=(\mu_1+\mu_2\,k_1)\,/\,(\mu_2+\mu_1\,k_2).$$

The solution of the Hilbert problem (29) can be written as

$$\theta'(z)=\frac{(\mu_1+\mu_2 k_1)\,X_0(z)}{2\pi i}\int_{-a}^{a}\frac{f\,(x)\,dx}{X_0^{+}(x)\,(x-z)}+X_0(z)\,P(z) \tag{30}$$

where

$$X_0(z)=(z+a)^{i\gamma-\frac{1}{2}}\,(z-a)^{-i\gamma-\frac{1}{2}},\quad \overline{2\pi\gamma}=\log|\alpha_0|,$$

$P(z)$ is an arbitrary polynomial in z. The function $X_0(z)$ refers to that branch which is holomorphic in the whole plane cut along $(-a, a)$ and which is such that $zX_0(z)\to 1$ as $|z|\to\infty$. Since $\theta'(z)=0\,(1/z^2)$ for large $|z|$, $P(z)=0$. The line integral in (30) can be evaluated with the help of contour integrals the details of which will be given for a more general case when we deal with the problem of crack opened by a point force at the origin. Evaluating integral, we get

$$\theta'(z) = \frac{(\mu_1 + \mu_2 k_1)}{(1 + \alpha_0)} X_0(z) \Big[\frac{\mu_2 (1 + k_1) C_1}{(\mu_2 + \mu_1 k_2)} \Big\{ \frac{1}{(z - z_0) X_0(z)} - \frac{1}{(z - z_0) X_0(z_0)} - 1 \Big\} - \frac{\mu_2 (1 + k_1) k_1 C_1}{(\mu_1 + \mu_2 k_1)} \times \Big\{ \frac{1}{(z - \bar{z}_0) X_0(z)} - \frac{1}{(z - \bar{z}_0) X_0(\bar{z}_0)} - 1 \Big\} + \frac{2 i y_0 \bar{C}_1 u_2 (1 + k_1)}{(\mu_1 + \mu_2 k_1)} \Big\{ \frac{1}{(z - \bar{z}_0)^2 X_0(z)} - \frac{1}{(z - \bar{z}_0)^2 X_0(\bar{z}_0)} + \frac{1}{(z - \bar{z}_0) X_0^2(\bar{z}_0)} \Big\} \qquad (31)$$

Indefinite integration of $\theta'(z)$ will involve integrals like $\int^{z} \frac{x_0(z)\, dz}{z - z_0}$ which can be evaluated numerically only and hence $\theta(z)$ cannot be obtained explicitly from $\theta'(z)$. $\Phi_0(z)$ and $\Psi_0(z)$ can be calculated from (25) and (24) and $\Phi(z)$ and $\Psi(z)$ from (19).

It is of interest to know normal and shearing stresses and displacements across the line L.

$$Y_y^+ - iX_y^+ = Y_y^- - iX_y^- = \Phi_{01}^+(x) + \Phi_{02}^-(x) - \frac{\mu_2 (1 + k_1)}{(\mu_2 + u_1 k_1)} \frac{C_1}{x - z_0} + \frac{\mu_2 (1 + k_1)}{(\mu_1 + \mu_2 k_1)} \Big\{ \frac{k_1 C_1}{x - \bar{z}_0} - \frac{2 i y_0 \bar{C}_1}{(x - \bar{z}_0)^2} \Big\}$$

$$= 0 \text{ on } y = 0, \quad |x| < a$$

$$= \Big\{ - \frac{\mu_2 (1 + k_1) C_1}{(\mu_2 + \mu_1 k_2)} \Big(1 + \frac{1}{(x - z_0) X_0(z_0)}\Big) + \frac{\mu_2 (1 + k_1) k_1 C_1}{(\mu_1 + \mu_2 k_1)} \times \Big(1 + \frac{1}{(x - \bar{z}_0) X_0(\bar{z}_0)}\Big) + \frac{2 i y_0 \bar{C}_1 \mu_2 (1 + k_1)}{(\mu_1 + \mu_2 k_1)} \times \Big(- \frac{1}{(z - \bar{z}_0)^2 X_0(\bar{z}_0)} + \frac{1}{(z - \bar{z}_0) X_0^2(\bar{z}_0)} \Big) \Big\} \times (x^2 - a^2)^{-\frac{1}{2}} \Big\{ \cos\Big(\gamma \log \Big|\frac{x + a}{x - a}\Big|\Big) + i \sin\Big(\gamma \log \Big|\frac{x + a}{x - a}\Big|\Big) \Big\}$$

$$y = 0, \quad |x| > a. \qquad (32)$$

It is clear from (32) that near the tips of the crack sign of stresses changes infinitely often in addition to the singularity present there. When $\mu_1 = \mu_2$ and $k_1 = k_2$, stresses have singularities at the tips of the crack but are not of oscillatory nature. The oscillatory phenomenon is confined at the most to a δ-neighbourhood of tip of the crack where $\delta/a = 2 \cdot 52 \times 10^{-4}$.

Calculation of displacements across L requires the knowledge of $\theta(z)$. Since $\theta(z)$ cannot be obtained explicitly from $\theta'(z)$, behaviour of displacements on the crack surfaces cannot be discussed analytically.

It can be easily seen that the conditions (1 *b*) to (5 *b*) are satisfied for this solution.

We shall consider now the problem of concentrated force acting at the origin on upper surface of the crack. Formulation of this problem is the same as described above except the change in the function $f(x)$ given in (28). We calculate $\theta'(z)$ from (30) where now $f(x)$ has the value

$$f(x) = \left\{\frac{(\mu_1 - \mu_2)(1 + k_1)}{2(\mu_1 + \mu_2 k_1)} - \frac{(\mu_1 - \mu_2)(1 + k_1)}{2(\mu_2 + \mu_1 k_2)} + \frac{(1 - \alpha_0 k_1)}{\alpha_0}\right\} \frac{C_1}{x} \tag{33}$$

It may be noted that $f(x)$ has singularity at $x = 0$. The integral

$$\int_{-a}^{a} \frac{f(x)\,dx}{(x - z)X_0^+(x)}$$

can be evaluated as follows:

Let Λ be the contour in Fig. 1. If z lies outside the contour then

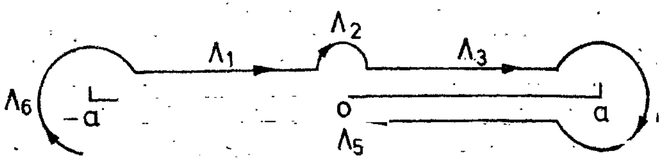
$$\int_{\Lambda} \frac{d\zeta}{\zeta(\zeta - z)X_0(\zeta)} = 2\pi i\left[\frac{1}{zX_0(z)} - 1\right]$$

Also

$$\int_{\Lambda} \frac{d\zeta}{\zeta(\zeta - z)X_0(\zeta)} = \int_{\Lambda_1 + \Lambda_2 + \cdots \Lambda_6} \frac{d\zeta}{\zeta(\zeta - z)X_0(\zeta)}$$

When Λ is shrunk to the arc $y = 0$, $|x| < a$, contour integrals over Λ_4 and Λ_6 tend to zero,

$$\int_{\Lambda_1 + \Lambda_3 + \Lambda_5} \frac{d\zeta}{\zeta(\zeta - z)X_0(\zeta)} \to (1 + \alpha_0)\int_{-a}^{a} \frac{f(x)\,dx}{x(x - z)X_0^+(x)}$$



$$\Lambda = \Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4 + \Lambda_5 + \Lambda_6$$

FIG. 1

and

$$\int_{\Lambda_2} \frac{d\zeta}{\zeta(\zeta - z)\, X_0(\zeta)} \to - \frac{\pi a}{\sqrt{\alpha_0}\, z}. \tag{34}$$

Bounded convergence theorem [13] has been used to obtain (34). Finally

$$\begin{aligned}\theta'(z) = \frac{(\mu_1 + \mu_2 k_1)\, C_1}{(1 + \alpha_0)} X_0(z) &\left\{ \frac{(\mu_1 - \mu_2)(1 + k_1)}{2(\mu_1 + \mu_2 k_1)} \right. \\ &\left. - \frac{(\mu_1 - \mu_2)(1 + k_1)}{2(\mu_2 + \mu_1 k_2)} + \frac{(1 - \alpha_0 k_1)}{\alpha_0} \right\} \\ &\times \left\{ \frac{1}{z\, X_0(z)} - 1 - \frac{ia}{2\sqrt{\alpha_0 z}} \right\}. \end{aligned} \tag{35}$$

Normal and shearing stresses on $y = 0$, $|x| > a$ are given by

$$\begin{aligned} & Y_y{}^+ - iX_y{}^+ = Y_y{}^- - iX_y{}^- \\ &= - \left\{ \frac{(\mu_1 - \mu_2)(1 + k_1)}{2(\mu_1 + \mu_2 k_1)} - \frac{(\mu_1 - \mu_2)(1 + k_1)}{2(\mu_2 + \mu_1 k_2)} \right. \\ &\quad \left. + \frac{(1 - \alpha_0 k_1)}{\alpha_0} \right\} C_1 . \\ &(x^2 - a^2)^{-1/2} \left(1 + \frac{ia}{2\sqrt{\alpha_0 x}}\right) \left\{ \cos\left(\gamma \log\left|\frac{x+a}{x-a}\right|\right) + i \sin \right. \\ &\quad \left. \left(\gamma \log\left|\frac{x+a}{x-a}\right|\right) \right\}. \end{aligned} \tag{36}$$

The conclusions drawn from (32) hold good for (36) also.

3. *Crack in Dissimilar Media Opened by Rigid Inclusion*

Consider two dissimilar elastic half planes bonded together along the x axis except over the region $-a \leqslant x \leqslant a$ where there is a line crack. The crack is opened by a rigid inclusion whose profile gives rise to the following displacements of the crack surfaces

$$\left.\begin{aligned} (1\,c)\ u^+ + iv^+ &= \epsilon i\left(1 - \frac{x^2}{a^2}\right) \\ u^- + iv^- &= -\,\epsilon i\left(1 - \frac{x^2}{a^2}\right) \end{aligned}\right\} y = 0, \quad |x| < a.$$

Following boundary conditions are to be satisfied along with (1 c)

(2 c) $u^+ + iv^+ = u^- + iv^-$ on $y = 0$, $|x| > a$

(3 c) $Y_y{}^+ - iX_y{}^+ = Y_y{}^- - iX_y{}^-$ on $y = 0$, $|x| > a$

(4 c) Stresses vanish at infinity.

We shall follow the notations and definitions of various functions given in sections 1 and 2. The condition (1 c) is equivalent to the following conditions on the crack surfces, *i.e.*, on $y = 0$, $|x| < a$

$$\frac{k_1\mu_1}{\mu_1 + k_1\mu_2}\,\omega^+(x) + \frac{\mu_1 k_2}{\mu_2 + \mu_1 k_2}\,\omega^-(x) + \frac{k_1}{\mu_1 + k_1\mu_2}\,\theta^+(x) - \frac{\theta^-(x)}{\mu_2 + \mu_1 k_2} = 2\mu_1\epsilon i\left(1 - \frac{x^2}{a^2}\right). \tag{37}$$

$$\frac{k_1\,\mu_2}{\mu_1 + k_1\mu_2}\,\omega^+(x) + \frac{k_2\mu_2}{\mu_2 + \mu_1 k_2}\,\omega^-(x) - \frac{\theta^+(x)}{\mu_1 + k_1\mu_2} + \frac{k_2\theta^-(x)}{\mu_2 + \mu_1 k_2} = -\,2\mu_2\epsilon i\left(1 - \frac{x^2}{a^2}\right). \tag{38}$$

Dividing (37) by μ_1 and (38) by μ_2 and subtracting, we get,

$$\theta^+(x) - \theta^-(x) = 4\mu_1\mu_2\epsilon i\left(1 - \frac{x^2}{a^2}\right); \quad y = 0,\ |x| < a. \tag{39}$$

From (2 c) $\theta^+(x) = \theta^-(x)$ on $y = 0$, $|x| > a$ and hence $\theta(z)$ is holomorphic in the entire plane cut along $(-a, a)$. The solution of the Hilbert problem (39) can be written as

$$\theta(z) = -\,\frac{2\mu_1\mu_2\epsilon}{\pi a^2}\left\{2az + (a^2 - z^2)\log\frac{z + a}{z - a}\right\}. \tag{40}$$

The following equation for $\omega_1(z)$ can be easily obtained provided the derivatives of displacements are considered in (1 c)

$$\omega'^{+}(x) + \alpha_1 \omega'^{-}(x) = \frac{4(\mu_2 - \mu_1 k_2)(\mu_1 + k_1 \mu_2)\epsilon i}{k_1(\mu_2 + \mu_1 k_2) a^2} x$$

$$- \frac{(k_1 k_2 - 1)}{k_1(\mu_2 + \mu_1 k_2)} \theta'^{+}(x); \quad y = 0, \quad |x| < a. \tag{41}$$

Using (3 c) we observe that $\omega'(z)$ is holomorphic in the entire plane cut along $(-a, a)$ and hence the solution of Hilbert problem (41) can be written as

$$\omega'(z) = 2\delta_1 z - 2\delta_1 \{z^2 - 2i\gamma_1 a - (\tfrac{1}{2} + 2\gamma_1 2) a^2\} X_1(z)$$

$$+ 2\delta_2 \left\{ 2a + z \log \frac{z-a}{z+a} \right\}, \tag{42}$$

where

$$\delta_1 = \frac{2\epsilon i (\mu_1 + k_1\mu_2)\{k_2(k_1\mu_2 - \mu_1) + k_1(\mu_2 - \mu_1 k_2)\}}{k_1^2 a^2 (1 + \alpha_1)^2 (\mu_2 + \mu_1 k_2)},$$

$$\alpha_1 = k_2(\mu_1 + k_1\mu_2)/k_1(\mu_2 + \mu_1 k_2), \; 2\pi\gamma_1 = \log |\alpha_1|,$$

$$\delta_2 = \frac{2\mu_1\mu_2(k_1k_2 - 1)\epsilon}{\pi a^2(1 + \alpha_1) k_1(\mu_2 + \mu_1 k_2)}$$

and

$$X_1(z) = (z + a)^{i\gamma_1 - \frac{1}{2}} (z - a)^{-i\gamma_1 - \frac{1}{2}}.$$

The function $X_1(z)$ refers to that branch which is holomorphic in the whole plane cut along $(-a, a)$ and which is such that $zX_1(z) \to 1$ as $|z| \to \infty$. Integrating $\omega'(z)$, we get,

$$\omega(z) = \delta_1 z^2 - \delta_1 (z^2 - a^2)(z - 2i\gamma_1 a) X_1(z)$$

$$+ \delta_2 \left\{ 2az - (a^2 - z^2) \log \frac{z-a}{z+a} \right\}$$

$$- \frac{2\epsilon i \{k_2(k_1\mu_2 - \mu_1) + k_1(\mu_2 - \mu_1 k_2)\}}{k_1(1 + \alpha_1)(\mu_2 + \mu_1 k_2)(\mu_1 + \mu_2 k_1)}. \tag{43}$$

The constant of integration in (43) has been chosen in such a way that the condition (1 c) is satisfied.

For $y = 0, \quad |x| < a$

$$Y_y^+ - iX_y^+ = \{2\mu_1\mu_2 (1 - k_1k_2)\, \delta_1/(\mu_1 + \mu_2k_1)(\mu_2 + \mu_1k_2)\}\, x$$

$$- \{2\delta_1\mu_1 (1 + k_1)/(\mu_1 + \mu_2k_1)\} \{x^2 - 2\, i\gamma_1\, ax - (\tfrac{1}{2} + 2\gamma_1^2)\, a^2\}\, X_1^+(x)$$

$$- \frac{4\mu_1\mu_2\epsilon}{\pi a^2 (\mu_1 + \mu_2k_1)} \left\{2a + x \log \left|\frac{x - a}{x + a}\right|\right\} .$$

$$\cdot \left\{(k_2 + \alpha_1k_1)/k_2 + \frac{\mu_1\mu_2 (1 - k_1k_2)^2}{k_1 (1 + \alpha_1)(\mu_2 + \mu_1k_2)^2}\right\} \tag{44}$$

$$- \frac{4\mu_1\mu_2\epsilon ix}{a^2 (\mu_1 + \mu_2k_1)} \left\{(k_2 - \alpha_1k_1)/k_2 - \frac{\mu_1 (k_1k_2 - 1)(1 + \alpha_1k_1)}{k_1 (1 + \alpha_1)(\mu_2 + \mu_1k_2)}\right\}$$

$$= \beta_1 + \beta_2 + \beta_3 + \beta_4 \qquad \text{(say)}$$

$$Y_y^- - iX_y^- = \beta_1 - \frac{\mu_2k_1 (1 + k_2)}{k_2\mu_1 (1 + k_1)} \beta_2 + \beta_3$$

$$- \frac{4\mu_1\mu_2\epsilon ix}{a^2k_2 (\mu_1 + \mu_2k_1)} \left\{k_2 - \alpha_1k_1 + \frac{\mu_2 (k_2 + \alpha_1)(k_1k_2 - 1)}{(1 + \alpha_1)(\mu_2 + \mu_1k_2)}\right\} . \tag{45}$$

$Y_y^+ - iX_y^+$ and $Y_y^- - iX_y^-$ for $y = 0$, $|x| > a$ are given by the sum of first three terms in (44) and the sum of first three terms in (45), respectively.

It may be noted that the stresses at the tips of the crack have singularities of the type $(x^2 - a^2)^{-\frac{1}{2}}$ but it is not possible analytically to say whether the stresses are oscillatory near the tips or not.

For $y = 0, \quad |x| > a$

$$u^+ + iv^+ = u^- + iv^- = \frac{\delta_1k_1 (1 + \alpha_1)}{(\mu_1 + \mu_2k_1)} \Big[x^2 - (x^2 - a^2)^{\frac{1}{2}} (x - 2i\gamma_1a).$$

$$\times \left\{\cos\left(\gamma_1 \log\left|\frac{x - a}{x + a}\right|\right) - i \sin\left(\gamma_1 \log\left|\frac{x - a}{x + a}\right|\right)\right\}$$

$$- \frac{k_1a^2 (1 + \alpha_1)}{\mu_1 + k_1\mu_2}\Big] \tag{46}$$

As

$$|x| \to \infty,\ u^+ + iv^+ \to - \frac{2\epsilon i \{k_2 (k_1\mu_2 - \mu_1) + k_1 (\mu_2 - \mu_1k_2)\}}{(\mu_1 + k_1\mu_2)(\mu_2 + \mu_1k_2)}$$

$$- \frac{\epsilon i \{k_2 (k_1\mu_2 - \mu_1) + k_1 (\mu_2 - \mu_1k_2)\}}{k_1 (1 + \alpha_1)(\mu_2 + \mu_1k_2)} (4\gamma_1^2 - 1)$$

For $\mu_1 = \mu_2 = \mu$ and $k_1 = k_2 = k$

$$Y_y^+ - iX_y^+ = -\frac{2\mu\epsilon(1+k)}{\pi a^2 k}\left\{2a + x\log\left|\frac{x-a}{x+a}\right|\right\} + \frac{2\mu\epsilon i(k-1)x}{a^2 k},$$

$$y = 0, \ |x| < a \tag{47}$$

$$Y_y^- - iX_y^- = -\frac{2\mu\epsilon(1+k)}{\pi a^2 k}\left\{2a + x\log\left|\frac{x-a}{x+a}\right|\right\} - \frac{2\mu\epsilon i(k-1)x}{a^2 k},$$

$$y = 0, \ |x| < a \tag{48}$$

$$Y_y^+ - iX_y^+ = Y_y^- - iX_y^- = -\frac{2\mu\epsilon(1+k)}{\pi a^2 k} \times \left\{2a + x\log\left|\frac{x-a}{x+a}\right|\right\}, \quad y = 0, \ |x| > a \tag{49}$$

$$u^+ + iv^+ = u^- + iv^- = 0, \quad y = 0, \quad |x| > a. \tag{50}$$

The problem of line crack is an elastic medium ($\mu_1 = \mu_2$ and $k_1 = k_2$) with displacements prescribed on the crack surfaces as in (1 c) is reported in [1] where it has been solved by using transform technique. The results given in (47)–(50) of this paper do not agree with the results given in [1]. The disagreement in results is due to the fact that in [1] it has been assumed that shearing stress is zero on L which amounts to saying that the crack is opened by a particular type of loading. Complex variable technique does not need this assumption for the solution of the problem. However, if one makes the assumption that shearing stress is zero on L then from (21) and (23) it can be seen that $\omega'(z) = 0$. By taking $\omega'(z) = 0$, we get the solution in [1].

4. Lastly, we discuss the problem of a line crack in dissimilar media opened by a rigid inclusion as in Section 3; with a concentrated force $X + iY$ acting at $z = z_0$ in the interior of S^+. The solution can be obtained by the superposition of two displacement systems. The first system of displacements is given by (14). The second system of displacements is obtained by

considering a rigid inclusion in a line crack in dissimilar media whose profile gives rise to the following displacements of the crack surfaces

$$(1\ d)\ u^+ + iv^+ = \epsilon i\left(1 - \frac{x^2}{z^2}\right) + G(x)$$

$$\text{on } y = 0, \quad |x| < a$$

$$u^- + iv^- = -\epsilon i\left(1 - \frac{x^2}{a^2}\right) + G(x)$$

where

$$G(x) = \frac{k_2(1+k_1)}{2(\mu_2 + \mu_1 k_2)} C_1 \log(x - z_0) + \frac{(1+k_1)}{2(\mu_1 + \mu_2 k_1)}$$

$$\times \left\{ k_1 C_1 \log(x - \bar{z}_0) + \frac{2iy_0 \bar{C}_1}{(x - \bar{z}_0)} \right\}$$

The solution of the crack problem with the displacements of the crack surfaces prescribed by (1 *d*) can be found out using the theory developed in section 3. Using the notations of section 3 it can be seen that the functions $\theta(z)$ and $\omega'(z)$ are given as follows

$$\theta(z) = -\frac{2\mu_1\mu_2\epsilon}{\pi a^2} \left\{ 2az + (a^2 - z^2) \log \frac{z+a}{z-a} \right\} \tag{51}$$

and

$$\omega'(z) = H(z) + \frac{(1+k_2)(\mu_1 + k_1\mu_2) X_1(z)}{k_1(\mu_2 + \mu_1 k_2)(1+\alpha_1)}$$

$$\times \left[\frac{k_2(1+k_1)}{2(\mu_2 + \mu_1 k_2)} C_1 \left\{ \frac{1}{(z - z_0) X_1(z)} - \frac{1}{(z - z_0) X_1(z_0)} - 1 \right\} \right.$$

$$+ \frac{(1+k_1) k_1 C_1}{2(\mu_1 + \mu_2 k_1)} \left\{ \frac{1}{(z - \bar{z}_0) X_1(z)} - \frac{1}{(z - \bar{z}_0) X_1(\bar{z}_0)} - 1 \right\}$$

$$+ \frac{2(1+k_1) iy_0 C_1}{(\mu_1 + \mu_2 k_1)} \left\{ \frac{1}{(z - \bar{z}_0)^2 X_1(z)} - \frac{1}{(z - \bar{z}_0)^2 X_1(\bar{z}_0)} \right.$$

$$\left. \left. + \frac{1}{(z - \bar{z}_0) X_1^2(\bar{z}_0)} \right\} \right] \tag{52}$$

Where $H(z)$ is given by the righthand side of (42), $\alpha_1 = k_2(\mu_1 + \mu_2 k_1)/k_1(\mu_2 + \mu_1 k_2)$, $2\pi\gamma_1 = \log|\alpha_1|$ and $X_1(z) = (z+a)^{i\gamma_1 - \frac{1}{2}}(z-a)^{-i\gamma_1 - \frac{1}{2}}$. Other quantities of interest can be calculated with the help of (51) and (52).

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