A CONFORMAL MAPPING PROCEDURE FOR USE IN TWO-DIMENSIONAL POTENTIAL FLOW PROBLEMS

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ABSTRACT

An analytical cum *numerical procedure to conformally transform an orbitrary closed or open curve to a unit circle and* vice versa *is described in the present paper. The forward transformation to the near circle is obtained using the orthonormal polynomials defined over the near circle or the unit circle. The method can be used in studying the potential flow problems around two deimnsional bodies which one encounters in Aerodynamics of wings and bodies in subsonic flow.*

Keywords: Conformal mapping, numerical procedure.

1. INTRODUCTION

The conformal transformation technique is a powerful tool for solving 2-dimensional potential flow problems in aerodynamics. The extensive use of the technique has been made in the study of flow about 2-dimensional airfoils and the cross flow about bodies of revolution, wing body combinations and slender wings with leading edge separation. Usually the flow about the required geometric shape in the physical plane is transformed to that about a circular cylinder or any other convenient shape in a transformed plane through the use of one or more conformal transformations. Except in some special cases, (*e.g.*, Joukowski airfoils, flow about slender bodies with elliptic cross section, etc.) it is not always possible to get the required transformation functions analytically in closed form. Such situations arise, in practice, in the study of arbitrary airfoil shapes, wing body interference problems with non-circular body and nonplanar wing shapes in the cross flow plane and also in the case of nonplanar delta wings with

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leading edge separation. When the shape to be transformed is represented by an arbitrary open or closed curve, a numerical conformal mapping procedure with a consistent accuracy has to be resorted to. In general we are in need of a mapping function which will map the region exterior to the curve C in Z-plane to the region exterior of a unit circle, while the point at infinity is to be mapped on to itself. The conformal mapping of arbitrary regions into circles has been the subject of extensive study by Szego [1], Burgman [2] and Nehari [3]. The required mapping is closely related to the set of complex orthonormal polynomials defined on the curve C to be transformed to the unit circle. In the following, we give a numerical analytical procedure to get such conformal mapping functions.

2. CONFORMAL MAPPING THEORY

The generalised conformal mapping function $H(Z)$ which maps the region of the Z-plane exterior to the (open or closed) curve C into the region exterior to the unit circle in the H-plane, *viz.*, $|H| \geq 1$, is constructed as follows. Let us consider the set of analytic functions of the complex variable Z, *i.e.*, the set

$$1, Z, Z^2, \ldots, Z^n, \ldots \tag{1}$$

Using the functions from the above set, we construct a set of polynomials which are orthonormal in the sense of Szego inner product defined over the curve C. The Szego inner product of two analytic functions $g(Z)$ and $h(Z)$ over the curve C in Z-plane is defined by the line integral

$$(g, h) = \int_C g(Z)\, \overline{h(Z)}\, ds \tag{2}$$

where $\overline{h(Z)}$ is the complex conjugate of $h(Z)$ and ds an element of the curve C. We use the well known Gram-Schmidt orthogonalisation procedure to construct the set of complete orthonormal functions

$$B_i(Z),\ i = 1, 2, 3, \ldots \tag{3}$$

We note that the functions in the set $B_i(Z)$, which depend on the type of the curve C over which the Szego inner product is defined, by definition of orthonormality, should satisfy the condition

$$\begin{aligned} (B_i B_j) = \int_C B_i(Z)\, \overline{B_j(Z)}\, ds &= 0 \quad \text{if } i \neq j \\ &= 1 \quad \text{if } i = j. \end{aligned} \tag{4}$$

Using the orthonormal functions $B_i(Z)$ the required mapping function is obtained as

$$H(Z) = \underset{i\to\infty}{\mathrm{Lt}} \frac{B_{i+1}(Z)}{B_i(Z)} \tag{5}$$

3. Procedure to obtain B_i (Z)

Let the open or the closed curve C in Z plane be defined by

$$\operatorname{Im} Z = z = z(y) \qquad -1 \leq \operatorname{Re} Z = y \leq 1 \tag{6}$$

Let $A_i(Z)$ be the ith orthogonal function (a polynomial in Z of degree $i-1$) constructed from the set of analytic functions

$$1, Z, Z^2, \ldots, Z^n, \ldots, \tag{7}$$

so that

$$\int_C A_i(Z)\overline{A_j(Z)}\, ds = 0 \qquad \text{if } i \neq j$$

$$\neq 0 \qquad \text{if } i = j. \tag{8}$$

Let

$$N(A_i) = \left[\int_C |A_i(Z)|^2\, ds\right]^{\frac{1}{2}} \tag{9}$$

be the norm of $A_i(Z)$. Then the ith orthonormal function $B_i(Z)$ is given by

$$B_i(Z) = \frac{A_i(Z)}{N(A_i)}. \tag{10}$$

To get $B_i(Z)$ we use the well known Gram-Schmidt orthogonalisation procedure. In the sequence eq. (7) we start with function 1 as the first orthogonal function so that

$$A_1 = 1;\; B_1 = \frac{1}{N(1)} = \frac{1}{\left[\int_C ds\right]^{\frac{1}{2}}} \tag{11}$$

The second orthogonal function $A_2(Z)$ is expressed as a linear combination of Z and $B_1(Z)$ and is given by

$$A_2(Z) = Z + A_{21}B_1(Z) \tag{12}$$

where A_{21} is obtained from the orthogonality condition as

$$A_{21} = -(Z, B_1). \tag{13}$$

This process is continued further to generate the subsequent orthogonal functions by using the condition that the ith function $A_i(Z)$ be orthogonal to all the $(i-1)$ functions $A_j(Z)$, $j = 1, 2, \ldots, i-1$. In general we express the orthogonal function $A_i(Z)$ as

$$A_i(Z) = A_{ii} Z^{i-1} + A_{i\,i-1} B_{i-1} + \ldots + A_{ik} B_k + \ldots + A_i \;\; B_1 \tag{14}$$

with $A_{ii} = 1$ and $A_1(Z) = 1$.

The corresponding orthonormal function $B_i(Z)$ is given by

$$B_i(Z) = B_{ii} Z^{i-1} + B_{i\,i-1} Z^{i-2} + \ldots + B_{i1}. \tag{15}$$

From the orthogonality conditions, one can establish the following relations between A_{ij} and B_{ij} (*see* Appendix).

$$A_{ij} = - \sum_{k=1}^{i} \bar{B}_{jk} I_{ik} \quad \text{for} \quad \begin{array}{l} i = 2, 3, 4 \\ j = 1, 2, \ldots, (i-1) \end{array} \tag{16}$$

with

$$A_{ii} = 1, \quad i = 1, 2, \ldots$$

and

$$B_{ij} = \frac{1}{N(A_i)} \sum_{k=j}^{i-1} A_{ik} B_{kj} \quad \text{for} \quad \begin{array}{l} i = 2, 3, 4, \ldots \\ j = 2, 3, \ldots, \;(i-1) \end{array}$$

with

$$B_{ii} = 1/N(A_i), \qquad i = 1, 2, 3, \ldots. \tag{17}$$

The I_{ik} are the Szego inner products of the powers of Z and are given by

$$I_{ik} = (Z^{i-1}, Z^{k-1}) = \int_C Z^{(i-1)} \;\; \bar{Z}^{(k-1)} \, ds. \tag{18}$$

The orthonormalisation scheme given above can be used for machine computation to get the constants A_{ij} or B_{ij}. But the numerical procedure would become unstable because of inherent truncation and round off errors associated with fixed word length of the digital computers. We also observe that due to recursive nature of the scheme the error propagates rapidly and the orthogonal function $A_i(Z)$ becomes nonorthonormal beyond a certain value of i. An iterative correction scheme due to Davis and Rabinowitz [4] has been incorporated to improve the orthogonality of each function generated. According to the scheme, let us suppose a set of k functions $A_1, A_2, \ldots, A_{k-1}, A_k$ are generated such that

$$(A_i, A_j) = 0 \qquad i \neq j$$

$$i, j, = 1, 2, \ldots, (k-1).$$

$$= N(A_i)^2 \qquad i = j$$

Also

$$(A_k', A_k') = N(A_k')^2$$

but

$$(A_k', A_j) = \epsilon_j \qquad \text{for} \qquad j = 1, 2, \ldots, (k-1)$$

where ϵ_j is small. Then the improved orthogonal function A_k is given by

$$A_k = A_k' - \sum_{j=1}^{k-1} (A_k', A_j)\, A_j \tag{19}$$

The above correction scheme is applied to each function A_i as it is generated until the convergence is obtained.

Next, to get a workable form of the mapping function, we replace the lim $i \to \infty$ in eq. (5) by $i \to N$ where N is sufficiently large. The value of N is fixed such that for all values of $i > N$, the sequence $B_{i+1\, i+1}/B_{ii}$ converges to an asymptotic value. The actual value of N for which the above sequence converges depends on the shape of the curve C. If C is very nearly circular, then even with $N = 8$ the sequence converges and we get a useful transformation function with finite number of terms. Fixing N sufficiently large, the transformation function can now be written as

$$H(Z) = \frac{\sum_{i=1}^{N+1} B_{N+1\, i}\, Z^{i-1}}{\sum_{i=1}^{N} B_{Ni}\, Z^{i-1}} . \tag{20}$$

For the flow velocities to be unchanged both in magnitude and direction at large distances from the body we require

$$(a)\ H \to \infty \qquad \text{as} \quad Z \to \infty \text{ and}$$

$$(b)\ dH/dZ \to 1 \quad \text{as} \quad Z \to \infty. \tag{21}$$

From eq. (20) it is easily seen that (*a*) is satisfied automatically for all N, while (*b*) will be satisfied provided N is sufficiently large. Now writing only the coefficients of leading power of Z in dH/dZ, we have

$$\frac{dH}{dZ}=\frac{\{[B_{NN}Z^{N-1}+O(Z^{N-2})][NB_{N+1\,N+1}Z^{N-1}+O(Z^{N-2})]-[B_{N+1\,N+1}Z^{N}+O(Z^{N-1})][B_{NN}(N-1)Z^{N-2}+O(Z^{N-3})]\}}{B_{NN}^{2}Z^{2N-2}+O(Z^{2N-3})}. \tag{22}$$

Writing only the coefficients of the leading power of Z, *i.e.*, Z^{2N-2}, we have

$$\frac{dH}{dZ}=\frac{B_{N+1\,N+1}B_{NN}Z^{2N-2}+O(Z^{2N-3})}{B_{NN}^{2}Z^{2N-2}+O(Z^{2N-3})}$$

$$=\frac{B_{N+1\,N+1}B_{NN}+O(1/Z)}{B_{NN}^{2}+O(1/Z)}. \tag{23}$$

Hence

$$\frac{dH}{dZ}{}_{Z\to\infty}\to\frac{B_{N+1\,N+1}}{B_{NN}}\to 1 \text{ as } N\to\infty.$$

Hence the transformation, eq. (20), satisfies both the conditions (*a*) and (*b*) at infinity,

4. INVERSE TRANSFORMATION

The inverse transformation $Z=Z(H)$ is again obtained by a numerical procedure. The mapping function $Z=Z(H)$ is expanded in terms of complete orthonormal function in the region $|H|\geq 1$ as

$$Z(H)=\frac{1}{\sqrt{2\pi}}\sum_{k=1}^{M}C_k\,H^{(2-k)}. \tag{24}$$

The complex constants C_k are given by the following integrals defined on the boundary of the unit circle $|H|=1$, *viz.*,

$$C_k=\frac{1}{\sqrt{2\pi}}\int_{|H|=1}Z(H)\,\bar{H}^{(2-k)}\,ds.$$

Separating real and imaginary parts of C_k, we get

$$C_k=\frac{1}{\sqrt{2\pi}}\int_0^{2\pi}[X\cos(k-2)\theta-Y\sin(k-2)\theta]\,d\theta$$

$$+\frac{i}{\sqrt{2\pi}}\int_0^{2\pi}[X\sin(k-2)\theta+Y\cos(k-2)\theta]\,d\theta \tag{25}$$

with

$$s = r\theta, \quad r = 1, \; Z(\theta) = X(\theta) + iY(\theta).$$

The boundary values of $Z = Z(\theta)$ are obtained from the forward transformation eq. (20).

In machine computation the inversion formula eq. (24) leads to values of Z which are in error by upto 5 per cent in the modulus even for values of $M = 12$, and situation is not improved by increasing M even upto 20. Hence the approximate value of Z corresponding to a given H is calculated with $M = 12$ and it is improved upon by using complex Newton-Raphson technique using the relation

$$Z_{n+1} = Z_n - \frac{H(Z_n) - H}{H'(Z_n)} \tag{26}$$

iteratively until the sequence Z_n converges to Z within a preassigned tolerance. The initial approximation to Z_0 is obtained from eq. (24).

4.1. *An Example—Flat Airfoil to Circle*

To construct a mapping function $H(Z)$ which maps the region exterior to the slit

$$-1 \leqslant \text{Re}(Z) = y \leqslant 1; \qquad \text{Im}(Z) = 0 \tag{27}$$

into the region exterior to unit circle $/ H(Z) / \geqslant 1$, we must obtain a set of analytic functions orthonormal in the Szego sense over the slit (eq. 27). It can be shown that the normalized Legendre polynomials of the first kind and order n form the required set of orthonormal functions, since

$$(P_n, P_m) = \int_{-1}^{1} P_m(Z)\, \overline{P_m(Z)}\, ds = 0 \qquad n \neq m$$

$$= 1 \qquad n = m. \tag{28}$$

Further, the functions $P_m(Z)$ also obey the recurrence relation

$$(n+1) P_{n+1}(Z) - (2n+1) Z P_n(Z) + n P_{n-1}(Z) = 0. \tag{29}$$

In the limit $n \to \infty$, we have

$$H(Z) - 2Z + \frac{1}{H(Z)} = 0 \tag{30}$$

where

$$H(Z) = \underset{n \to \infty}{\text{Lt}} \frac{P_{n+1}(Z)}{P_n(Z)} \tag{31}$$

Solving eq. (30) for $H(Z)$ we have

$$H(Z) = Z \pm \sqrt{Z^2 - 1} \tag{32}$$

which is the well known inverse of the Joukowski transformation

$$Z = \tfrac{1}{2}\left(H + \frac{1}{H}\right)$$

which maps a unit circle in H-plane to a slit $(-1, 1)$ in Z plane.

4.2. *Numerical Example—A Reflex Camber Shape to a Unit Circle*

Eventhough the transformation in eq. (20) is most general and can be used for mapping any curve to a unit circle, in practice the value of N will be very large if the curve is very much dissimilar to the unit circle as is in the case of arbitrary mean camber lines or thin wing sections. In such cases one can transform the open or closed contour in the physical Z plane to a near circle in an intermediate G plane using the transformation in

TABLE I

Camber distribution of reflex curvature profile

y	z
0	0.024
0.1	0.02465316
0.2	0.02663658
0.3	0.02991798
0.4	0.03410647
0.5	0.03811599
0.6	0.040000
0.7	0.03727182
0.8	0.02800269
0.9	0.01309692
1.0	0

Note.—Equation to the camber line

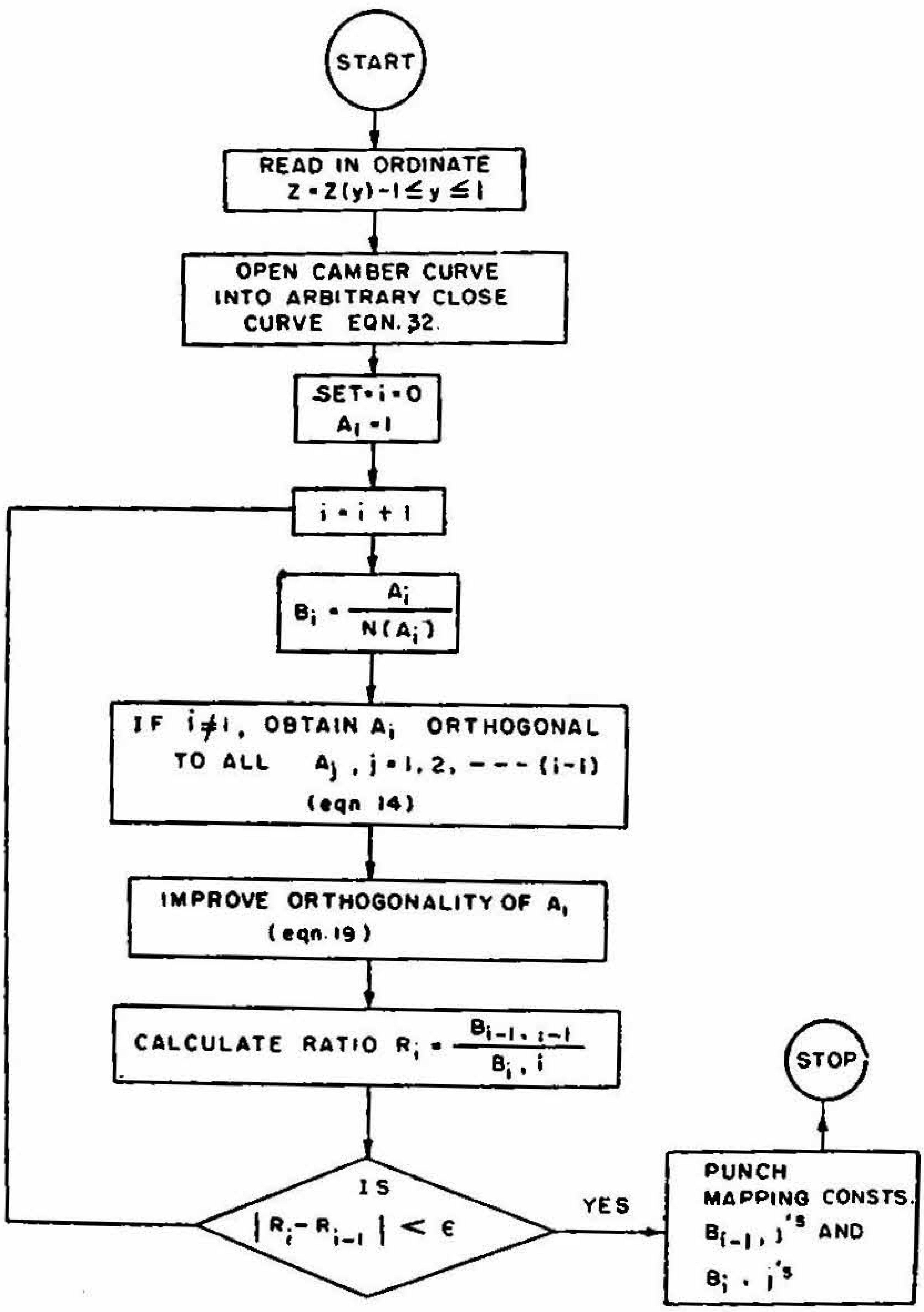
$z = z(y) \qquad -1 \leq y \leq 1.$

y and z are nondimensionalised with respect to semi-chord.

eq. (32). The resulting near circle then can be transformed to a unit circle in the H plane using the transformation

$$H(G) = \frac{\sum_{i=1}^{N+1} B_{N+1\,i}\, G^{i-1}}{\sum_{i=1}^{N} B_{Ni}\, G^{i-1}} \tag{33}$$

which can be obtained as described earlier. The successive stages of transformation of a reflex mean camber shape (ordinates given in Table I) to a unit circle is shown in Fig. 2. The numerical values of the constants $B_{N+1\,i}$ and B_{Ni} are given in Table II for value of $N = 8$ and 9, 19 and 20. Convergence of the sequence $B_{nn}/B_{n+1\,n+1}$ is shown in Table III.



1. FIG. Flow chart to get mapping constants.

TABLE II

Transformation constant B_{Ni}

B_{Ni} = (*Real part*) *Imaginary part*

2.*A* *N* 8, = *i* = 1 to 8

B_{88} =	(0.393568,	0)
B_{87} =	(0·415879 × 10^{-9};	—0·14443)
B_{86} =	(—0.265081 × 10^{-1};	—0.173414 × 10^{-9}):
B_{85} =	(0.330779 × 10^{-9};	—0.437361 × 10^{-1}):
B_{81} =	(—0.917124 × 10^{-2};	—0.19048 × 10^{-9});
B_{83} =	(0.268734 × 10^{-9};	0.189901 × 10^{-1});
B_{82} =	(0.335643 × 10^{-2}:	—0.141788 × 10^{-9});
B_{81} =	(0.22669 × 10^{-9};	0.431727 × 10^{-2}).

2.*B*. *N* = 9. *i* = 1 to 9

B_{99} =	(0.39292;	0)
B_{98} =	(0.46616 × 10^{-9}:	—0.162075)
B_{97} =	(—0.33511 × 10^{-1};	—0.21534 × 10^{-9});
B_{96} =	(0.366084 × 10^{-9};	—0.489963 × 10^{-1})
B_{95} =	(—0.127626 × 10^{-1};	—0.238803 × 10^{-9});
B_{94} =	(0.3674461 × 10^{-9};	0.226618 × 10^{-1});
B_{93} =	(0.453153 × 10^{-2};	—0.18646 × 10^{-9});
B_{92} =	(0.268063 × 10^{-9};	0.509114 × 10^{-2});
B_{91} =	(0.222081 × 10^{-2};	—0.15634 × 10^{-9});

2.*C*. *N* = 19. *i* = 1 to 9

$B_{19,\,19}$ =	(0.386506;	0)
$B_{19,\,18}$ =	(0.970264 × 10^{-9};	—0.336514)
$B_{19,\,17}$ =	(—0.146940;	—0.900319 × 10^{-9})
$B_{19,\,16}$ =	(0.502919 × 10^{-9};	—0.757683 × 10^{-1})
$B_{19,\,15}$ =	(—0.742102 × 10^{-1};	—0.101665 × 10^{-8})
$B_{19,\,14}$ =	(0.283490 × 10^{-9};	0.782413 × 10^{-1})
$B_{19,\,13}$ =	(0.309992 × 10^{-1};	—0.743438 × 10^{-9})
$B_{19,\,12}$ =	(0.365740 × 10^{-9};	0.712649 × 10^{-2})
$B_{19,\,11}$ =	(0.198421 × 10^{-1};	—0.688141 × 10^{-9})
$B_{19,\,10}$ =	(0.395855 × 10^{-9};	—0.117334 × 10^{-1})
$B_{19,\,9}$ =	(—0.337596 × 10^{-2};	—0.677176 × 10^{-9})
$B_{19,\,8}$ =	(0.357077 × 10^{-9};	—0.818452 × 10^{-3})
$B_{19,\,7}$ =	(—0.236386 × 10^{-2};	—0.625543 × 10^{-9})
$B_{19,\,6}$ =	(0.331963 × 10^{-9};	0.125763 × 10^{-2})
$B_{19,\,5}$ =	(0.442614 × 10^{-3};	—0.569697 × 10^{-9})
$B_{19,\,4}$ =	(0.317904 × 10^{-9};	0.404055 × 10^{-3})
$B_{19,\,3}$ =	(0.543735 × 10^{-3};	—0.537616 × 10^{-9})
$B_{19,\,2}$ =	(0.279396 × 10^{-9};	0.136603 × 10^{-3})
$B_{19,\,1}$ =	(0.263708 × 10^{-3};	—0.494671 × 10^{-9})

2.D $N = 20, \quad i = 1$ to 20

$B_{20,20} = (0.38587; \quad 0)$
$B_{20,19} = (0.101968 \times 10^{-8}; \quad -0.353637)$
$B_{20,18} = (-0.162528; \quad -0.993019 \times 10^{-9})$
$B_{20,17} = (0.486661 \times 10^{-9}; \quad -0.751129 \times 10^{-1})$
$B_{20,16} = (-0.823139 \times 10^{-1}; \quad -0.111181 \times 10^{-8})$
$B_{20,15} = (0.238183 \times 10^{-9}); \quad 0.864025 \times 10^{-1})$
$B_{20,14} = (0.356364 \times 10^{-1}; \quad -0.802611 \times 10^{-9})$
$B_{20,13} = (0.342805 \times 10^{-9}; \quad 0.657772 \times 10^{-2})$
$B_{20,12} = (0.224044 \times 10^{-1}; \quad -0.748464 \times 10^{-9})$
$B_{20,11} = (0.385194 \times 10^{-9}; \quad -0.138840 \times 10^{-1})$
$B_{20,10} = (-0.422812 \times 10^{-2}; \quad -0.744221 \times 10^{-9})$
$B_{20,9} = (0.346014 \times 10^{-9}; \quad -0.860377 \times 10^{-3})$
$B_{20,8} = (-0.279066 \times 10^{-2}; \quad -0.693609 \times 10^{-9})$
$B_{20,7} = (0.322675 \times 10^{-9}; \quad 0.159635 \times 10^{-2})$
$B_{20,6} = (0.566518 \times 10^{-3}; \quad -0.638154 \times 10^{-9})$
$B_{20,5} = (0.314462 \times 10^{-9}; \quad 0.445160 \times 10^{-3})$
$B_{20,4} = (0.636249 \times 10^{-3}; \quad -0.590516 \times 10^{-9})$
$B_{20,3} = (0.270043 \times 10^{-9}); \quad 0.996174 \times 10^{-4})$
$B_{20,2} = (0.29391 \times 10^{-3}; \quad -0.548116 \times 10^{-9})$
$B_{20,1} = (0.235249 \times 10^{-9}; \quad 0.547836 \times 10^{-4})$

TABLE III

Convergence of $B_{NN}/B_{N+1\ N+1}$

N	B_{NN}	$B_{NN}/B_{N+1\ N+1}$
1	0.397532	0.999949
2	0.397534	1.0017816
3	0.396827	1.0016609
4	0.396169	1.001732
5	0.395485	1.0015777
6	0.394861	1.0016564
7	0.394208	1.0016261
8	0.393568	1.0016491
9	0.392920	1.0016468
10	0.392274	1.0016469
11	0.391629	1.0016471
12	0.390985	1.0016472
13	0.390342	1.0016474
14	0.389700	1.0016474
15	0.389059	1.0016477
16	0.388419	1.0016478
17	0.387780	1.0016478
18	0.387142	1.0016455
19	0.386506	1.0016482
20	0.38587	

A suitable numerical integration scheme is needed to evaluate the contour integrals in eqs. 9, 18, and 25. The trapezium rule has been found to yield good results in the numerical evaluation of the contour integrals of analytic functions [5]. Hence these integrals are computed using trapezoidal formula taking 100 points on the appropriate contours. The flow chart for evaluating the constants B_{ij} of the transformation function is shown in Fig. 1. A computer programme in FORETRAN IV language has been written up, and the numerical results quoted were obtained using the IBM 360/44 computer at the Institute of Science.

The procedure described above has been used by the authors in their study on nonplanar slender delta wings with leading edge separation [6]. Further, the method is being made use of in the study of two dimensional thick symmetric section to find the changes in pressure distribution due to changes in the contour shape and *vice versa*. The work in this connection

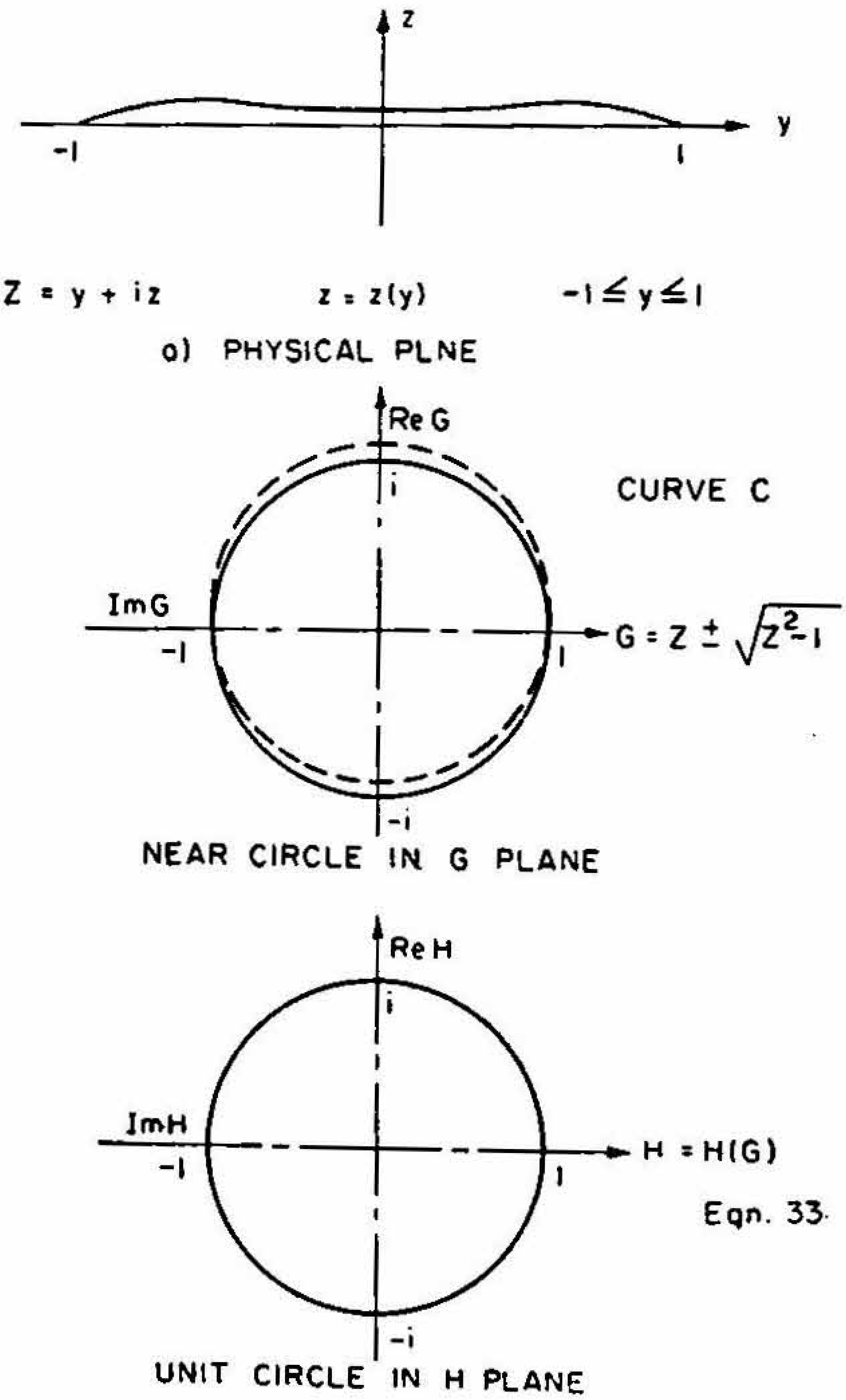


FIG. 2. Transformation of reflex camber to a unit circle.

is in progress. The method can be used in the study of nonplanar slender wing body interference problems using slender body theory.

APPENDIX

The expressions for A_{ij} and B_{ij} in eqs. (16) and (17) can be derived as follows.

As was mentioned in the text, the orthogonal polynomial $A_i(Z)$ is constructed as a linear combination of all the $(i-1)$ orthonormal polynomials $B_j(Z)$ ($j = 1$ to $i-1$) together with the leading term as Z^{i-1}, so that we have

$$A_i(Z) = A_{ii}\, Z^{i-1} + A_{i\, i-1}\, B_{i-1} + A_{i\, i-2}\, B_{i-2} + A_{ij}\, B_j + \ldots + A_{i1}\, B_1 \qquad \text{(A–1)}$$

with $A_{ii} = 1$.

Now to find the coefficients A_{ij} we make use of the orthogonality conditions. Multiplying by $\bar{B}_j = \overline{A_j(Z)}/N(A_j)$ on both sides of (A–1) and integrating over the curve C we have (due to orthogonality)

$$\int_C A_i(Z) \frac{\overline{A_j(Z)}}{N(A_j)}\, ds = 0 = \int_C Z^{i-1}\, \bar{B}_j\, ds + A_{ij} \int_C B_j\, \bar{B}_j\, ds \qquad \text{(A–2)}$$

since all the other integrals are zero. So we have

$$A_{ij} = -\int_C Z^{i-1}\, \overline{\sum_{k=1}^{j} B_{jk}\, Z^{k-1}}\, ds$$

$$= -\sum_{k=1}^{j} \bar{B}_{jk}\, I_{i,k} \qquad \text{for} \qquad i = 2, 3, 4, \ldots$$

$$j = 1 \text{ to } (i-1) \qquad \text{(A–3)}$$

where

$$I_{i,k} = \int_C Z^{i-1}\, \bar{Z}^{k-1}\, ds. \qquad \text{(A–4)}$$

Again to get B_{ij} of $B_i(Z)$ in terms of A_{jk} and B_{jk} of $A_j(Z)$ and $B_j(Z)$ ($j = 1, 2, \ldots, i-1$) we proceed as follows. We have

$$B_i(Z) = B_{ii}\, Z^{i-1} + B_{i\, i-1}\, Z^{i-1} + \ldots + B_{ij}\, Z^{j-} + \ldots + B_i\,. \qquad \text{(A–5)}$$

Now by definition,

$$B_i(Z) = \frac{1}{N(A_i)}[1 \, . \, Z^{i-1} + A_{i\,i-1} B_{i-1} + \ldots + A_{ij} B_j + \ldots + A_{i1} B_1] \tag{A-6}$$

$$= \frac{1}{N(A_i)}[Z^{i-1} + A_{i\,i-1}(B_{i-1\,i-1} Z^{i-2} + \ldots + B_{i-1\,1}) + \ldots + A_{ir}(B_{rr} Z^{r-1} + B_{r\,r-1} Z^{r-2} + \ldots + B_r) + A_{12}(B_{22}Z + B_{21}) + A_{i1} B_{11})]. \tag{A-7}$$

Collecting the coefficients of Z^{j-1} in eq. (7), we have

$$B_{ij} = \frac{1}{N(A_i)} \sum_{k=j}^{i-1} A_{ik} \, B_{kj} \qquad i = 2, 3, 4, \ldots \quad j = 1 \text{ to } i-1$$

with

$$B_{ii} = \frac{1}{N(A_i)} \, i = 1, 2, 3, \ldots \tag{A-8}$$

Also writing $A_i(Z)$ in powers of Z we have

$$A_i(Z) = A_{ii} Z^{i-1} + A_{i\,i-1} Z^{i-2} + \ldots + A_{ij} Z^{j-1} + \ldots + A_{i1} \tag{A-9}$$

where

$$A_{ii} = 1, \quad i = 1, 2, 3, \ldots \text{ and}$$

$$A_{ij} = \sum_{k=j}^{i-1} A_{ik} \, B_{kj}, \qquad i = 2, 3, 4, \ldots \quad j = 1 \text{ to } (i-1)$$

Now all the A_{ij} $(j \neq i)$ in $A_i(Z)$ can be written in terms of the coefficients A_{ik} and B_{kj} of the orthogonal polynomials of order upto $(i-1)$, and since $\bar{A}_{ii} = 1 = A_{ii}$ we can find $N(A_i)$ and hence B_{ij} using eq. (A–8). These relations can be used to compute the coefficients of the orthonormal polynomials $B_{N+1}(Z)$ and $B_N(Z)$ upto any required value of N, until a good convergence is obtained for the ratio $B_{N+1\,N+1}/B_{NN}$.

NOMENCLATURE

$A_i(Z)$: orthogonal polynomials in complex variable Z

A_{ij} : complex constants in $A_i(Z)$

$B_i(Z)$: orthonormal polynomials $= A_i(Z)/N(A_i)$

B_{ij} : complex constants in $B_i(Z)$

C_k : complex constants of inverse transformation function. *See* eq. (24)

ds : incremental arcual length of any curve C

(g, h) : $\int_C g(Z)\, h(Z)\, ds$—the Szego inner product of complex analytic function g and h defined over the curve C.

I_{ik} : the inner product integrals defined in eq. (10)

M, N : real constants

$Z = y + iz$: complex variable in the physical Z plane y and z are non-dimensional with respect to reference length

G, H : complex variable in the near circle plane G and the unit circle plane H

Re ( ), Im ( ): real and imaginary part of the complex function

$\overline{(\ )}$: denotes conjugate of the complex function.

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